# TANGENT, LINEAR APPROXIMATION, TAYLOR APPROXIMATION

Robert Mařík

January 21, 2006

## Contents

# 1 Tangent

The derivative of the function $f(x)$ at a point $x_0$ is the slope of the tangent to the graph of the funciton $f$ at the point $x_0$. Having the slope $f'(x_0)$ and one point of the line (the point $[x_0, f(x_0)]$), it is easy to write the point-slope form of the tangent as

$$y = f'(x_0)(x - x_0) + f(x_0).$$

Find tangent to the graph of $y = \dfrac{1}{\sqrt{1-x}}$ at the point $x_0 = 0$.

Find tangent to the graph of $y = \dfrac{1}{\sqrt{1-x}}$ at the point $x_0 = 0$.

general formula: $y = y'(x_0) \cdot (x - x_0) + y(x_0)$

We start with the general formula for the tangent.

Find tangent to the graph of $y = \frac{1}{\sqrt{1-x}}$ at the point $x_0 = 0$.

general formula: $y = y'(x_0) \cdot (x - x_0) + y(x_0)$

$$y(x_0) = \frac{1}{\sqrt{1-0}} = 1$$

We evaluate the function at the point undder consideration.

Find tangent to the graph of $y = \dfrac{1}{\sqrt{1-x}}$ at the point $x_0 = 0$.

general formula: $y = y'(x_0) \cdot (x - x_0) + y(x_0)$

$$y(x_0) = \frac{1}{\sqrt{1-0}} = 1$$

$$y'(x) = \left((1-x)^{-\frac{1}{2}}\right)' = -\frac{1}{2}(1-x)^{-\frac{3}{2}}(-1)$$

We differentiate.

Find tangent to the graph of $y = \frac{1}{\sqrt{1-x}}$ at the point $x_0 = 0$.

general formula: $y = y'(x_0)\cdot(x-x_0)+y(x_0)$

$$y(x_0) = \frac{1}{\sqrt{1-0}} = 1$$

$$y'(x) = \left((1-x)^{-\frac{1}{2}}\right)' = -\frac{1}{2}(1-x)^{-\frac{3}{2}}(-1)\Big|_{x=x_0} = \frac{1}{2}$$

We evaluate the derivative at the point under consideration.

Find tangent to the graph of $y = \frac{1}{\sqrt{1-x}}$ at the point $x_0 = 0$.

general formula: $y = y'(x_0) \cdot (x - x_0) + y(x_0)$

$$y(x_0) = \frac{1}{\sqrt{1-0}} = 1$$

$$y'(x) = \left((1-x)^{-\frac{1}{2}}\right)' = -\frac{1}{2}(1-x)^{-\frac{3}{2}}(-1)\Big|_{x=x_0} = \frac{1}{2}$$

tangent: $y = \frac{1}{2} \cdot (x - 0) + 1 = 1 + \frac{1}{2}x$

The best linear approximation for small $x$: $\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$

We use the general formula and find the tangent, which is the best local linear approxiamtion for the function.

# 2 Linear approximation

The tangent is the best linear approximation to the function. This approximation can be used to replace some complicated and inconvenient formulas by simpler ones.

$$\frac{1}{\sqrt{1-x}} \approx 1+\frac{1}{2}x$$

The function can be replaced by its tangent for small $x$.

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$$

$$E = mc^2$$

We start with Einstein formula for total energy of a moving body.

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$$

$$E = mc^2 = m_0 c^2 \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

The mass is an increasing function of the velocity, as the formula shows.

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$$

$$E = mc^2 = m_0 c^2 \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$
$$\approx m_0 c^2 \left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right)$$

The formula contains our function for $x = \left(\frac{v}{c}\right)^2$. We replace it by its tangent. This can be done if the velocity is many times smaller than $c$, the velocity of the light in vacuum..

$$\frac{1}{\sqrt{1-x}} \approx 1+\frac{1}{2}x$$

$$\begin{aligned}E = mc^2 = m_0c^2 \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\\ \approx m_0c^2\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2\right) = \overbrace{m_0c^2}^{\text{rest mass energy}} + \underbrace{\frac{1}{2}m_0v^2}_{\text{kinetic energy}}\end{aligned}$$

After some algebraic modifications we get the part of the rest mass energy (not connected with any motion) and the part which includes the velocity and thus connected with the motion. This part corresponds with the formula for kinetic energy known from Newtonian mechanics.

# 3 Higer order approximation

The linear approximation is usually limited to a narrow neigborhood only. If the range of variables is greater than this neighborhood and the linear approximation does not give satisfactory results, it is possible to use higher order approximation by higher order Taylor polynomial.

$$y = \frac{1}{\sqrt{1-x}}$$
$$y' = \frac{1}{2}(1-x)^{-3/2}$$
$$y'' = \frac{3}{4}(1-x)^{-5/2}$$
$$y''' = \frac{15}{8}(1-x)^{-7/2}$$

We find higher order approximation for the Einstein's formula from the preceeding chapter. We differentiate up to third order derivative.

$$y = \frac{1}{\sqrt{1-x}} \qquad y(0) = 1$$

$$y' = \frac{1}{2}(1-x)^{-3/2} \qquad y'(0) = \frac{1}{2}$$

$$y'' = \frac{3}{4}(1-x)^{-5/2} \qquad y''(0) = \frac{3}{4}$$

$$y''' = \frac{15}{8}(1-x)^{-7/2} \qquad y'''(0) = \frac{15}{8}$$

We evaluate the function and its derivatives at zero.

$$y = \frac{1}{\sqrt{1-x}} \qquad y(0) = 1$$

$$y' = \frac{1}{2}(1-x)^{-3/2} \qquad y'(0) = \frac{1}{2}$$

$$y'' = \frac{3}{4}(1-x)^{-5/2} \qquad y''(0) = \frac{3}{4}$$

$$y''' = \frac{15}{8}(1-x)^{-7/2} \qquad y'''(0) = \frac{15}{8}$$

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x + \frac{3}{4}\frac{1}{2!}x^2 + \frac{15}{8}\frac{1}{3!}x^3 + \cdots$$

We write the Taylor polynomial.

$$y = \frac{1}{\sqrt{1-x}} \qquad y(0) = 1$$

$$y' = \frac{1}{2}(1-x)^{-3/2} \qquad y'(0) = \frac{1}{2}$$

$$y'' = \frac{3}{4}(1-x)^{-5/2} \qquad y''(0) = \frac{3}{4}$$

$$y''' = \frac{15}{8}(1-x)^{-7/2} \qquad y'''(0) = \frac{15}{8}$$

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x + \frac{3}{4}\frac{1}{2!}x^2 + \frac{15}{8}\frac{1}{3!}x^3 + \cdots$$

$$\approx 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$$

We simplify.

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$$

Higher order approximation for our function.

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$$

$$E = m_0 c^2 \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \approx \overbrace{m_0 c^2}^{\text{rest mass energy}} + \underbrace{\frac{1}{2} m_0 v^2}_{\text{classical kinetic energy}} +$$

$$+ \underbrace{\frac{3}{8} m_0 \frac{v^4}{c^2}}_{\text{the 1-st relativistic correction}} + \underbrace{\frac{5}{16} m_0 \frac{v^6}{c^4}}_{\text{the 2-nd relativistic correction}} + \cdots$$

Higher order approximation of the Einstein's formula for energy of moving object.

# Futher reading

- http://en.wikipedia.org/wiki/Kinetic_energy
- http://www.phys.unsw.edu.au/einsteinlight/jw/module5_dynamics.htm