Riccati differential equation and it's generalizations

A real treasure in oscillation and comparison theory of second order linear and half-linear differential equations

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REAL STORY

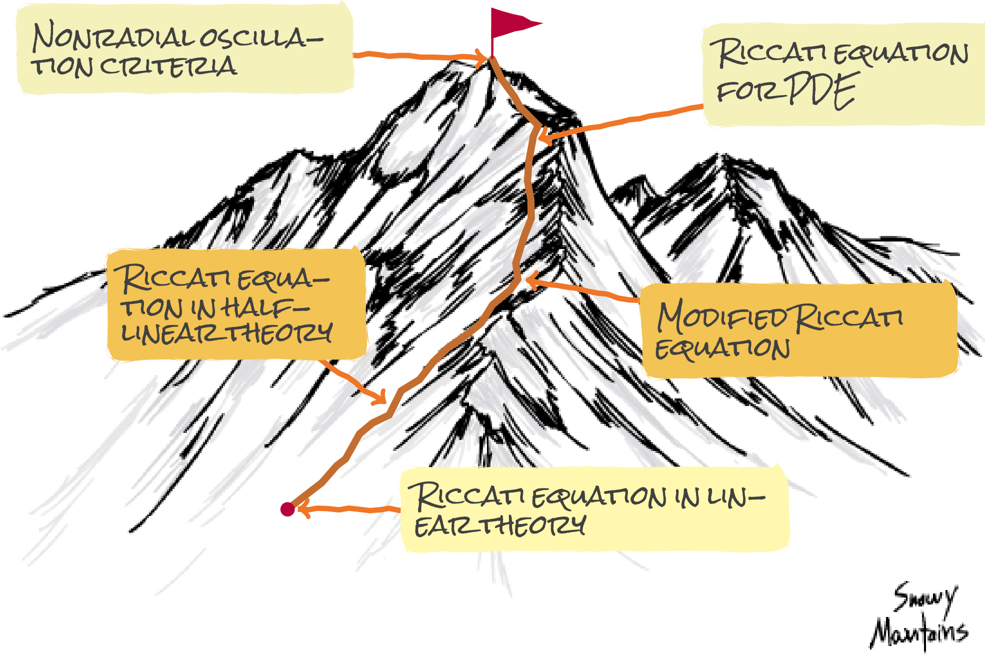
A talk with the man at a copy center where I ordered to make hardcopies of my master thesis.

I have seen many Bessel functions in your thesis. I am also a mathematician. I wrote my thesis about Riccati equation. It is a kind of useless differential equation. There are no applications.

THE AIM OF THE TALK

The aim of the talk is to convince the audience that the worker from the copy center was wrong.

Outline of the talk



Nonradial oscillation criteria

Riccati equation for PDE

Riccati equation in half-linear theory

Modified Riccati equation

Riccati equation in linear theory

Part 1

Second order linear differential equation

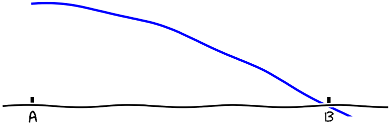
• Riccati equation method for second order linear ODE • Important pairs of points
• Zero interlacing theorem • Sturm majorant theorem • Oscillation criteria
• Principal solutions

Riccati equation method for second order linear ODE

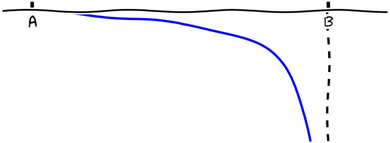
$$\begin{tikzcd} \boxed{x''+c(t)x=0} \arrow[r, shift left, "w=x'/x"] & \boxed{w'+c(t)+w^2=0} \arrow[l, shift left, "x=\exp(\int w)"] \end{tikzcd}$$

Important pairs of points

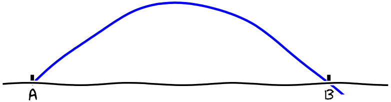
- b is the first focal point to $t=a$



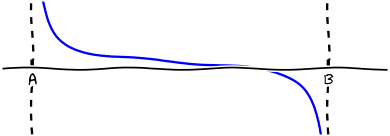
- solution given by $w(a)=0$ exists on (a,b) and $w(b-)=-\infty$



- b is the first conjugate point to $t=a$



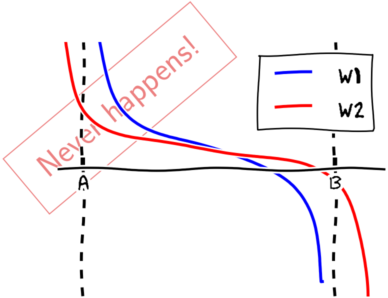
- solution given by $w(a+)=\infty$ exists on (a,b) and $w(b-)=-\infty$



Zero interlacing theorem

The following cannot occur due to unique solvability of IVP for Riccati equation

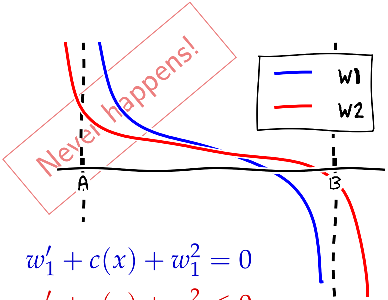
- x_1, x_2 solutions of the same equation
- $x_1 > 0$ on (a, b), $x(a) = 0 = x(b)$
- $x_2 > 0$ on $[a, b]$



Sturm majorant theorem

The following cannot occur due to comparison theorem for first order differential inequalities

- x_1 solution of $x'' + c(t)x = 0$
- $x_1 > 0$ on (a, b), $x(a) = 0 = x(b)$
- x_2 solution of $x'' + C(t)x = 0$, $C(t) \geq c(t)$
- $x_2 > 0$ on $[a, b]$



$$w_1' + c(x) + w_1^2 = 0$$

$$w_2' + c(x) + w_2^2 \leq 0$$

$$\left(w_2' + C(x) + w_2^2 = 0\right)$$

Oscillation theory

$$x'' + c(t)x = 0$$

- Oscillatory if some solution (and thus all solutions) have zeros in every neighborhood of ∞.
- Nonoscillatory otherwise (there exists a solution positive in some neighborhood of ∞).

(Non-)oscillation in terms of Riccati equation

$x'' + c(t)x = 0$	$w' + c(t) + w^2 = 0$
• equation is nonoscillatory	• equation has solution on $[T, \infty)$ for large T
• equation is oscillatory	• equation has no solution on $[T, \infty)$, no matter how large T is

OSCILLATION CRITERIA

Oscillation of

$$x'' + c(t)x = 0$$

is ensured if $c(t)$ is large enough. For example in one of the following sense

pointwise criteria simplest possibility, from comparison with suitable equation,

e.g.: $x'' + kt^{-2}x = 0$ is nonoscillatory iff $k \leq \dfrac{1}{4}$

$x'' + c(t)x = 0$ is oscillatory if $c(t) \geq \left(\dfrac{1}{4} + \varepsilon\right) t^{-2}$ for large t

and nonoscillatory if $c(t) \leq \dfrac{1}{4} t^{-2}$ for large t

integral criteria can be used if $c(t)$ is not large enough for every t but its mean value is large

e.g.: $\int^{\infty} c(t)\,\mathrm{d}t = \infty$ is sufficient for oscillation

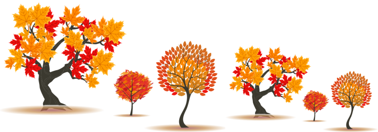
series of conjugacy criteria can be used if the mean value of $c(t)$ is small, but there is a series of subintervals where $c(t)$ is large enough to bend every solution to zero. Certain lower bound for mean value of $c(t)$ is required if we wish to eliminate this possibility how the equation can be turned to oscillation, e.g. in nonoscillation criteria.

PRINCIPAL SOLUTIONS

$$x'' + c(t)x = 0$$

Leighton, Morse: Let u and v be nontrivial solutions of nonoscillatory equation. The following conditions are equivalent and each can be used to determine principal solution u (unique up to a constant multiple) and nonprincipal solution v (linearly independent on principal solution).

- $\lim\limits_{t\to\infty} \dfrac{u(t)}{v(t)} = 0$
- $\dfrac{u'(t)}{u(t)} < \dfrac{v'(t)}{v(t)}$ in a neighborhood of ∞
- $\displaystyle\int^{\infty} \frac{1}{u^2(t)}\,\mathrm{d}t = \infty, \int^{\infty} \frac{1}{v^2(t)}\,\mathrm{d}t < \infty$



Part 2

Half-linear ODE

- Half-linear ODE
- Generalized Riccati equation
- Use of linear transformation theory
- Linear transformation theory in the terms of Riccati equation
- Half-linear substitute for missing transformation theory
- Modified Riccati equation
- Available estimates for H function
- Theorem samples (principal solution)
- Theorem samples (perturbed Euler equation)
- Theorem samples (Nehari-type nonoscillation criterion)

Half-linear ODE

$$\Big(r(t)\Phi(x')\Big)' + c(t)\Phi(x) = 0, \qquad (*)$$

where

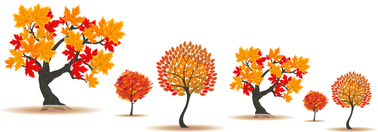
$$\Phi_p(x) = |x|^{p-2}x, \quad p > 1$$

and if the subscript is missing, we assume it to be p, i.e. $\Phi(x) := \Phi_p(x)$.

Equation (*) has been introduced in 70's by Mirzov and Elbert. Equation (*) and related equations have been later studied by many authors (including Došlá, Došlý, Drábek, Fišnarová, Hasil, Hata, Jaroš, Kong, Kusano, Li, Lomtatidze, Manojlovič, Marič, Marini, Matsumura, Matucci, Naito, Ogata, Onitsuka, Řehák, Sugie, Sun, Tanigawa, Veselý, Wang, Usami, Xu, Yamaoka, Yoshida)

Generalized Riccati equation

$$\begin{tikzcd} \boxed{(r(t)\Phi(x'))' + c(t)\Phi(x) = 0} \arrow[r, shift left, "w = r\Phi(x'/x)"] & \boxed{\begin{array}{c} w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0 \\ q = \frac{p}{p-1} \end{array}} \arrow[l, shift left, "x = \mathrm{e}^{\int \Phi^{-1}(w/r)}"] \end{tikzcd}$$



(i.e. essentially the same as in the linear case)

USE OF LINEAR TRANSFORMATION THEORY

Kneser: $x'' + c(t)x = 0$ $\begin{cases} c(t) = \dfrac{1}{4t^2} \text{ is critical case} \\ \text{oscillatory if } \liminf\limits_{t\to\infty} t^2 c(t) > \dfrac{1}{4} \\ \text{nonoscillatory if } \limsup\limits_{t\to\infty} t^2 c(t) < \dfrac{1}{4} \end{cases}$

Refinement (e.g. Hartman's book):

$$\begin{tikzcd}
\boxed{x'' + \left[\dfrac{1}{4t^2} + d(t)\right] x = 0} \arrow[r, bend right, "x = \sqrt{t}y"'] & \boxed{(ty')' + td(t)y = 0} \arrow[r, bend left, "s = \ln(t),\ y(s) = x(t)"] & \boxed{\ddot{y} + t^2 d(t) y(s) = 0}
\end{tikzcd}$$

Critical case: $t^2 d(t) = \dfrac{1}{4s^2} \iff d(t) = \dfrac{1}{4t^2 \ln^2 t}$

$$x'' + \left[\frac{1}{4t^2} + d(t)\right] x = 0 \quad \begin{cases} \text{oscillatory if } \liminf\limits_{t\to\infty} t^2 \ln^2(t) d(t) > \dfrac{1}{4} \\ \text{nonoscillatory if } \limsup\limits_{t\to\infty} t^2 \ln^2(t) d(t) < \dfrac{1}{4} \\ d(t) = \dfrac{1}{4t^2 \ln^2(t)} \text{ is critical case} \implies \text{another transformation} \dots \end{cases}$$

This approach does not have known extension to half-linear ODE's (lack of transformation formula).

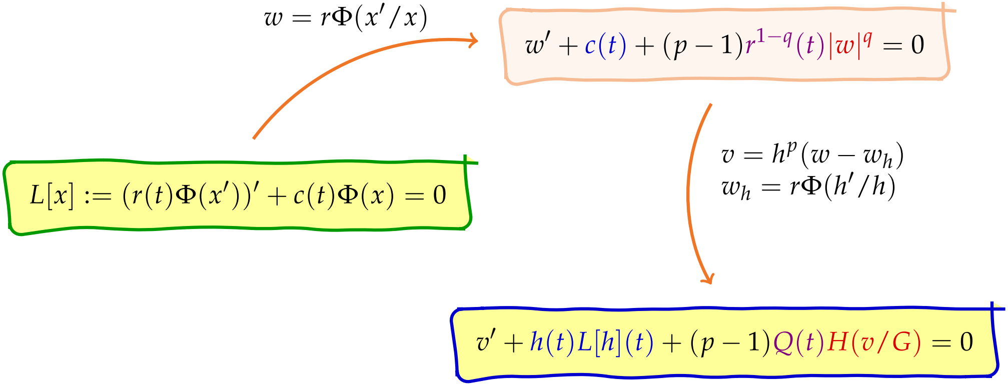
Linear transformation theory in the terms of Riccati equation

We use two different languages to describe one thing (dependent variable transformation).

$$\begin{tikzcd}
\boxed{L[x] := (r(t)x')' + c(t)x = 0} \arrow[rr, "w = r\frac{x'}{x}"] \arrow[d] & & \boxed{w' + c(t) + r^{-1}(t)w^2 = 0} \arrow[d] \\
x(t) = h(t)y(t) \arrow[rr, "\text{logaritmic derivative}", "\text{and new notation}"'] \arrow[d] & & v = h^2(w - w_h),\ w_h = r\frac{h'}{h} \arrow[d] \\
\boxed{\begin{array}{l} \left(R(t)y'\right)' + C(t)y = 0 \\ R(t) = r(t)h^2(t) \\ C(t) = h(t)L[h](t) \end{array}} \arrow[rr, "v = R\frac{y'}{y}"] & & \boxed{v' + C(t) + R^{-1}(t)v^2 = 0}
\end{tikzcd}$$

$$x = h(t)y \implies \frac{x'}{x} = \frac{h'}{h} + \frac{y'}{y} \implies rh^2\frac{y'}{y} = h^2\left(r\frac{x'}{x} - r\frac{h'}{h}\right) \implies R\frac{y'}{y} = h^2\left(r\frac{x'}{x} - r\frac{h'}{h}\right)$$

Half-linear substitute for missing transformation theory



$$L[x] := (r(t)\Phi(x'))' + c(t)\Phi(x) = 0$$

$$w = r\Phi(x'/x)$$

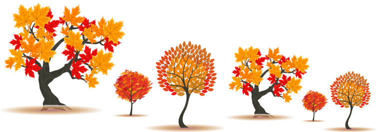
$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0$$

$$v = h^p(w - w_h)$$
$$w_h = r\Phi(h'/h)$$

$$v' + h(t)L[h](t) + (p-1)Q(t)H(v/G) = 0$$

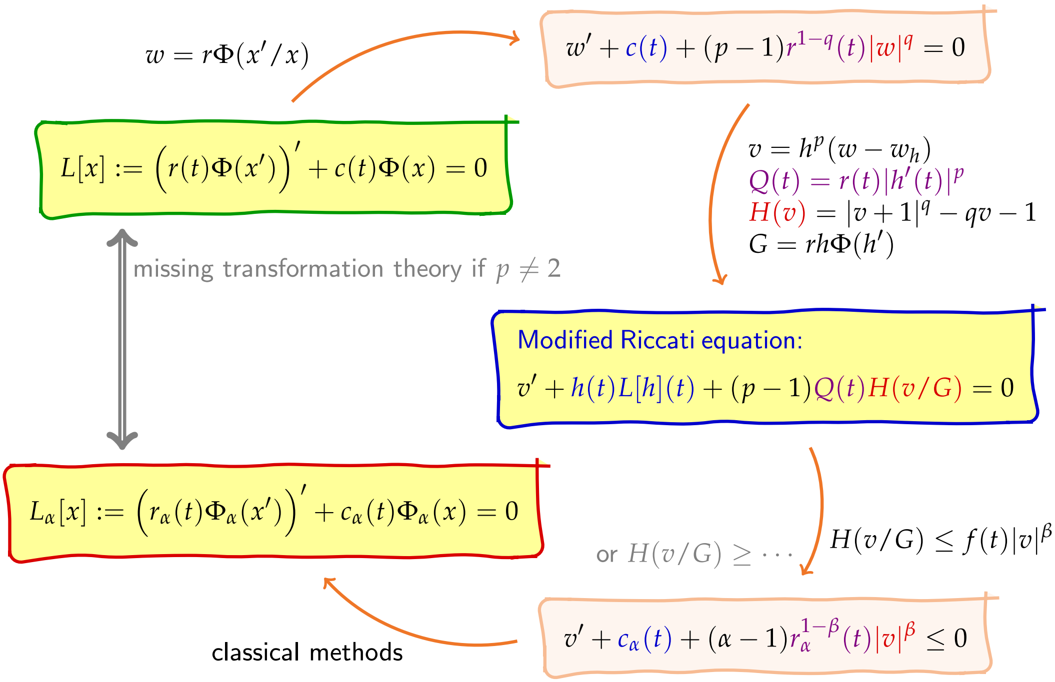
$$Q(t) = r(t)|h'(t)|^p$$

$$H(v) = |v+1|^q - qv - 1 = q\left(\frac{|v+1|^q}{q} - (v+1) + \frac{1}{p}\right) \geq 0,\ G(t) = r(t)h(t)\Phi(h'(t))$$



(note slightly different notation than usually found in papers)

Modified Riccati equation



$w = r\Phi(x'/x)$

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0$$

$$L[x] := \Big(r(t)\Phi(x')\Big)' + c(t)\Phi(x) = 0$$

$v = h^p(w - w_h)$

$Q(t) = r(t)|h'(t)|^p$

$H(v) = |v+1|^q - qv - 1$

$G = rh\Phi(h')$

missing transformation theory if $p \neq 2$

Modified Riccati equation:

$$v' + h(t)L[h](t) + (p-1)Q(t)H(v/G) = 0$$

$H(v/G) \leq f(t)|v|^\beta$

or $H(v/G) \geq \cdots$

$$L_\alpha[x] := \Big(r_\alpha(t)\Phi_\alpha(x')\Big)' + c_\alpha(t)\Phi_\alpha(x) = 0$$

$$v' + c_\alpha(t) + (\alpha-1)r_\alpha^{1-\beta}(t)|v|^\beta \leq 0$$

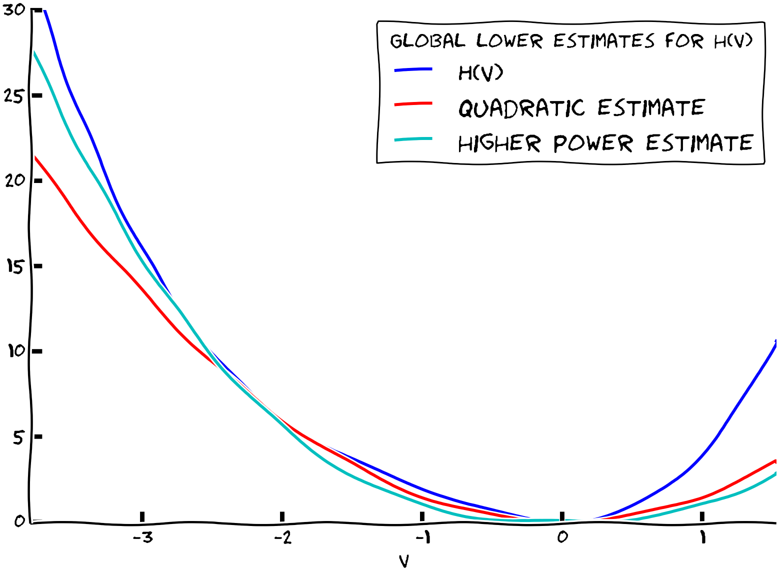
classical methods

Existence of positive solution of $L[x] = 0$ implies existence of positive solution $L_\alpha[x] = 0$.

AVAILABLE ESTIMATES FOR H FUNCTION

$$H(v) = |v+1|^q - qv - 1 = q\left(\frac{|v+1|^q}{q} - (v+1) + \frac{1}{p}\right) \geq 0$$

$H(v) \approx \frac{q(q-1)}{2}v^2$ for small v, $H(v) \approx |v|^q$ for large v



Global estimates

$$H(v) = |v+1|^q - qv - 1 \\ = q\left(\frac{|v+1|^q}{q} - (v+1) + \frac{1}{p}\right) \geq 0$$

- $H(v) \leq \frac{q}{2}v^2$ (for $p \leq 2$)
- $H(v) \geq \frac{q}{2}v^2$ (for $p \geq 2$)
- $H(v) \leq \beta v^\alpha$ (for $p \leq 2$, $\alpha \in [2, q]$, suitable β, missing sharp effective formula for β)
- $H(v) \geq \beta v^\alpha$ (for $p \geq 2$, $\alpha \in [q, 2]$, suitable β, missing sharp effective formula for β)
- $\frac{|t|^\beta}{\beta} - t + \frac{1}{\alpha} \leq \frac{|t|^q}{q} - t + \frac{1}{p}$ (if $\beta \leq q$)

Local estimates

- $K_1 v^2 \leq H(v) \leq K_2 v^2$ (for suitable K_1, K_2 and on compact interval)
- $H(v) = \frac{q(q-1)}{2}v^2(1 + o(1))$ (as $v \to 0$)
- $\frac{|t|^\beta}{\beta} - t + \frac{1}{\alpha} \leq \frac{\beta - 1}{q - 1}\left[\frac{|t|^q}{q} - t + \frac{1}{p}\right]$(if either $\beta \leq q$ and $t \geq 1$ or $\beta \geq q$ and $t \in (1 - \varepsilon, 1]$)

THEOREM SAMPLES (PRINCIPAL SOLUTION)

Došlý and Elbert via global quadratic estimate of H:

$$p \geq 2: \quad u \text{ is principal} \implies \int^{\infty} \frac{1}{r(t)u^2(t)|u'(t)|^{p-2}}\, \mathrm{d}t = \infty$$

$$p \in (1,2]: \quad \int^{\infty} \frac{1}{r(t)u^2(t)|u'(t)|^{p-2}}\, \mathrm{d}t = \infty \implies u \text{ is principal}$$

Fišnarová and Mařík via global power-like estimate of H:

The integral $\int^{\infty} \frac{1}{r(t)u^2(t)|u'(t)|^{p-2}}\, \mathrm{d}t$ can be replaced by

$$I_\alpha := \int^{\infty} \frac{\mathrm{d}t}{r^{\alpha-1}(t)u^{\alpha}(t)|u'(t)|^{(p-1)(\alpha-q)}},$$

where $\alpha \in [q,2]$ if $p \geq 2$ and $\alpha \in [2,q]$ if $p \leq 2$. (q is a conjugate number to p)

Example: The latter condition (F-M) allows to detect $u(t) = 1 - 1/t^9$ as principal solution of

$$\left(\Phi(x')\right)' + \frac{15t^{-3/2}}{(t^9-1)^{1/2}}\Phi(x) = 0, \quad p = 3/2, \quad t > 1,$$

the former (D-E) fails.

Theorem samples (perturbed Euler equation)

Došlý, Fišnarová, Mařík (J. Math. Anal. Appl. 2013): Let $h(t) > 0$, $h'(t) > 0$, $h(t)L_p(h)(t) \geq 0$ and either

$$\limsup_{t\to\infty} r(t)h(t)\Phi(h'(t)) < \infty \text{ and } \int^{\infty} r^{1-q}(t)h^{-q}(t)\,\mathrm{d}t = \infty$$

or

$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) = \infty \text{ and } \int^{\infty} r^{-1}(t)h^{-2}(t)h'^{2-p}(t)\,\mathrm{d}t = \infty.$$

Denote

$$a(t) = \left(\frac{q}{\beta}\right)^{1-\alpha} r(t)h^{\alpha-q(\alpha-1)}(t)|h'(t)|^{-\alpha+p},\ b(t) = h^{q(1-\alpha)}(t)\left[\left(\frac{q}{\beta}-1\right)r(t)h'^{p}(t) + c(t)h^{p}(t)\right]$$

If $\alpha > p$ and $L_p[x] = 0$ is nonoscillatory, then $(\Phi_\alpha(a(t)x'))' + b(t)\Phi_\alpha(x) = 0$ is also nonoscillatory.

Sugie and Yamaoka (Acta Math. Hungar. 2006) – a special case of the previous result: If $\alpha > p$ and

$$(\Phi_p(x'))' + t^{-p}\left[\left(\frac{p-1}{p}\right)^{p} + \left(\frac{p-1}{p}\right)^{p-1}\delta(t)\right]\Phi_p(x) = 0$$

is nonoscillatory, then

$$(\Phi_\alpha(x'))' + t^{-\alpha}\left[\left(\frac{\alpha-1}{\alpha}\right)^{\alpha} + \left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\delta(t)\right]\Phi_\alpha(x) = 0$$

is also nonoscillatory.

Theorem samples (Nehari-type nonoscillation criterion)

$$L[x] := \big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0 \qquad \text{(under examination)} \qquad (*)$$

$$\widetilde{L}[x] := \big(r(t)\Phi(x')\big)' + \widetilde{c}(t)\Phi(x) = 0 \qquad \text{(nonoscillatory)}$$

Došlý (J. Math. Anal. Appl. 2006): Let $h \in C^1$ be a positive function such that $h'(t) > 0$ for large t, say $t > T$. Denote $G(t) := \displaystyle\int_T^t \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}}$ and suppose that

$$\lim_{t\to\infty} G(t)r(t)h(t)\Phi(h'(t)) = \infty, \qquad \text{and} \qquad \lim_{t\to\infty} G^2 r(t)h^3(t)(h'(t))^{p-2}\widetilde{L}[h](t) = 0.$$

If $\displaystyle\int^\infty [c(t) - \widetilde{c}(t)]h^p(t)\,\mathrm{d}t < \infty$ and

$$\limsup_{t\to\infty} \int_T^t \frac{\mathrm{d}s}{r(s)h^2(s)(h'(s))^{p-2}} \int_t^\infty \big(c(s) - \widetilde{c}(s)\big)h^p(s)\,\mathrm{d}s < \frac{1}{2q},$$

$$\liminf_{t\to\infty} \int_T^t \frac{\mathrm{d}s}{r(s)h^2(s)(h'(s))^{p-2}} \int_t^\infty \big(c(s) - \widetilde{c}(s)\big)h^p(s)\,\mathrm{d}s > -\frac{3}{2q},$$

then (*) is nonoscillatory.

Special cases of this theorem produce efficient and sharp oscillation criteria. The result of the effort to improve further the constants on right-hand sides will be presented (among others) in the talk at RIMS Kyoto, Nov 4, 2014.

Part 3

Half-linear PDE

• Elliptic half-linear PDE • Generalized Riccati equation • Concept of oscillation
• Known result – detection of oscillation from ODE
• Observation and curious question 1/2 • Answer 1/2 – nonradial oscillation criteria
• Known result – linear oscillation criterion • Observation and curious question 2/2
• Answer 2/2 – sublinear versus general

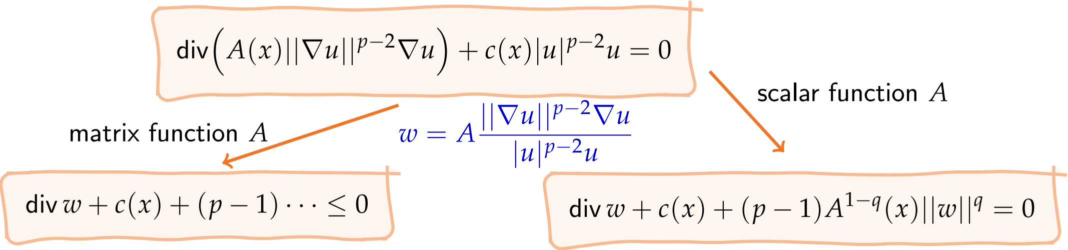
ELLIPTIC HALF-LINEAR PDE

$$\operatorname{div}\Big(A(x)||\nabla u||^{p-2}\nabla u\Big) + c(x)|u|^{p-2}u = 0 \qquad (*)$$

$A(x)$ is either scalar function or elliptic matrix with maximal and minimal eigenvalues $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$.

Alegretto, delPino, Drábek, Jaroš, Kusano, Mawhin, Naito, Usami, Xu, Yohsida

GENERALIZED RICCATI EQUATION



$$\boxed{\operatorname{div}\Big(A(x)||\nabla u||^{p-2}\nabla u\Big) + c(x)|u|^{p-2}u = 0}$$

matrix function A

$$w = A\frac{||\nabla u||^{p-2}\nabla u}{|u|^{p-2}u}$$

scalar function A

$$\boxed{\operatorname{div} w + c(x) + (p-1)\cdots \leq 0}$$

$$\boxed{\operatorname{div} w + c(x) + (p-1)A^{1-q}(x)||w||^{q} = 0}$$

- Can be used to derive oscillation criteria (the inequality if A is a matrix does not matter)
- We (almost) miss nonoscillation criteria – from the solvability of Riccati equation we cannot (without additional assumptions) deduce, that (*) has a positive solution.

CONCEPT OF OSCILLATION

Eq. (*) is said to be *oscillatory* if it possesses no solution $u(x)$ which is positive for large $||x||$.

KNOWN RESULT – DETECTION OF OSCILLATION FROM ODE

Došlý (2001): Equation

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0$$

is oscillatory, if the ordinary differential equation

$$\left(r^{n-1}|u'|^{p-2}u'\right)' + r^{n-1}\left(\frac{1}{\omega_n r^{n-1}}\int_{||x||=r} c(x)\ \mathrm{d}\sigma\right)|u|^{p-2}u = 0 \qquad (*)$$

is oscillatory. The number ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

- **Usami (1998)** proved essentialy equivalent version of this theorem formulating the conclusion in terms of associated Riccati equation rather than half-linear ODE (*).
- **Jaroš, Kusano and Yoshida (2000)** proved independently similar result (for $A(x) = a(\|x\|)I$ and differentiable $a(\cdot)$).

OBSERVATION AND CURIOUS QUESTION 1/2

Oscillation criteria depend in fact on the mean value of $c(x)$ over spheres centered in the origin. Is it possible to detect oscillation of

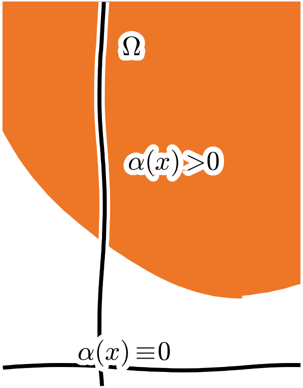
$$\operatorname{div}\Big(\boldsymbol{A(x)}\|\nabla u\|^{p-2}\nabla u\Big) + \boldsymbol{c(x)}|u|^{p-2}u = 0$$

in such an extreme case as

$$\int_{||x||=r} \boldsymbol{c(x)}\,\mathrm{d}\sigma = 0?$$

ANSWER 1/2 – NONRADIAL OSCILLATION CRITERIA

Let Ω be unbounded simply connected domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$.
Let $k \in (1, \infty)$ real number and α be nonnegative smooth function satisfying



(i) $\alpha(x) = 0$ iff $x \notin \Omega$,

(ii) $$\int_1^\infty \left(\int_{\substack{||x||=t \\ x\in\Omega}} \alpha(x)\, \mathrm{d}\sigma \right)^{1-q} \mathrm{d}t = \infty.$$

If

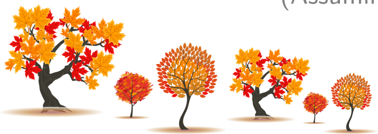
$$\lim_{t\to\infty} \int_{\substack{1\le ||x||\le t \\ x\in\Omega}} \alpha(x) \left(\boldsymbol{c(x)} - \frac{k}{(p\alpha(x))^p} \|\nabla\alpha(x)\|^p \right) \mathrm{d}x = \infty,$$

then the equation

$$\operatorname{div}\left(\|\nabla u\|^{p-2} \nabla u \right) + \boldsymbol{c(x)} |u|^{p-2} u = 0$$

is oscillatory.

(Assuming that the integrals are well defined.)



Corollary: oscillation in upper half-plane

For $n = 2$ consider the equation

$$\boxed{\Delta u + c(x)u = 0} \tag{*}$$

If

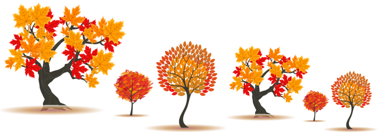
$$\boxed{\lim_{t\to\infty} \frac{1}{\ln t} \int_1^t r \int_0^\pi c(r,\varphi)\, \sin^2(\varphi)\, \mathrm{d}\varphi\, \mathrm{d}r > \frac{\pi}{2},} \tag{**}$$

where $c(r,\varphi)$ is $c(x)$ in polar coordinates, then (*) is oscillatory.

Example. The potential $c(r,\varphi) = \dfrac{A}{r^2}\sin\varphi$ satisfies

$$\int_0^{2\pi} \sin\varphi\, \mathrm{d}\varphi = 0 \qquad \text{and} \qquad \int_0^\pi \sin^3\varphi\, \mathrm{d}\varphi = \frac{4}{3} > 0$$

Thus the oscillation of (*) cannot be detected by any criterion based on the mean value of c over the whole sphere centered in the origin, but the oscillation is guaranteed by (**) for sufficiently large A.



Known result – linear oscillation criterion

Xu (2006): Let $\theta \in C^1([r_0,\infty],\mathbb{R}^+)$, $m > 1$, $\lambda \in C([r_0,\infty),\mathbb{R}^+)$. If

$$\lim_{r\to\infty}\int_{r_0\le||x||\le r}\left[\theta(\|x\|)\boldsymbol{c(x)} - \boldsymbol{\lambda(\|x\|)}\frac{m}{4}\frac{\theta'^2(\|x\|)}{\theta(\|x\|)}\right]\,\mathrm{d}x = \infty$$

and

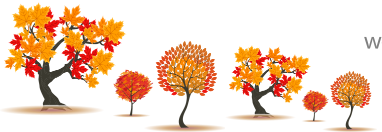
$$\lim_{r\to\infty}\int_{r_0\le||x||\le r}\frac{\boldsymbol{1}}{\boldsymbol{\theta(\|x\|)\lambda(\|x\|)}}\,\mathrm{d}x = \infty, \qquad \text{where } \boldsymbol{\lambda(r) \ge \max_{||x||=r} \lambda_{\mathrm{max}}(x)}$$

then $\boxed{\operatorname{div}\Big(\boldsymbol{A(x)}\nabla u\Big) + \boldsymbol{c(x)}u = 0}$ is oscillatory.

Observation and curious question 2/2

The oscillation is ensured if $\boldsymbol{c(x)}$ is large enough and $\boldsymbol{A(x)}$ small enough. The measure of this "smallness" in the linear case is $\boldsymbol{\lambda_{\mathrm{max}}}$. There is a bunch of similar half-linear oscilation theorems, where the measure for the "smallness" of A is $\dfrac{\boldsymbol{\lambda^p_{\mathrm{max}}}}{\boldsymbol{\lambda^{p-1}_{\mathrm{min}}}}$. Why such a discrepancy appears?

(Note that $\lambda_{\mathrm{max}} \le \dfrac{\lambda^p_{\mathrm{max}}}{\lambda^{p-1}_{\mathrm{min}}}$ and thus the half-linear approach is worse than linear even in the case when $p = 2$!)



Answer 2/2 – sublinear versus general

Let

$$b(r) = \int_{||x||=r} c(x)\,\mathrm{d}\sigma\,,$$

and either

$$1 < p \leq 2 \quad \text{and} \quad a(r) = \int_{||x||=r} \lambda_{\text{max}}(x)\,\mathrm{d}\sigma$$

or

$$1 < p \quad \text{and} \quad a(r) = \int_{||x||=r} \lambda_{\text{max}}^{p}(x)\lambda_{\text{min}}^{1-p}(x)\,\mathrm{d}\sigma\,.$$

If

$$\Big(a(r)\Phi(u')\Big)' + b(r)\Phi(u) = 0$$

is oscillatory, then

$$\text{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + c(x)|u|^{p-2}u = 0.$$

is also oscillatory.

Detailed comparison of sub/supper-linear cases

$$w(x) = A(x)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}, \qquad \|w\| \leq \lambda_{\mathsf{max}}\frac{\|\nabla u\|^{p-1}}{|u|^{p-1}}$$

$p > 1$ arbitrary:

$$\operatorname{div} w + c + (p-1)\frac{\left\langle A\|\nabla u\|^{p-2}\nabla u, \nabla u\right\rangle}{|u|^p} = 0$$

$$\operatorname{div} w + c + (p-1)\lambda_{\mathsf{min}}\frac{\|\nabla u\|^p}{|u|^p} \leq 0$$

$$\frac{\|\nabla u\|^p}{|u|^p} \geq \frac{\|w\|^q}{\lambda_{\mathsf{max}}^q}$$

$$\operatorname{div} w + c + (p-1)\lambda_{\mathsf{min}}\frac{1}{\lambda_{\mathsf{max}}^q}\|w\|^q \leq 0$$

$1 < p \leq 2$:

$$\operatorname{div} w + c + (p-1)\left\langle w, A^{-1}w\right\rangle\frac{\|\nabla u\|^{2-p}}{|u|^{2-p}} = 0$$

$$\left\langle w, A^{-1}w\right\rangle \geq \|w\|^2\frac{1}{\lambda_{\mathsf{max}}}$$

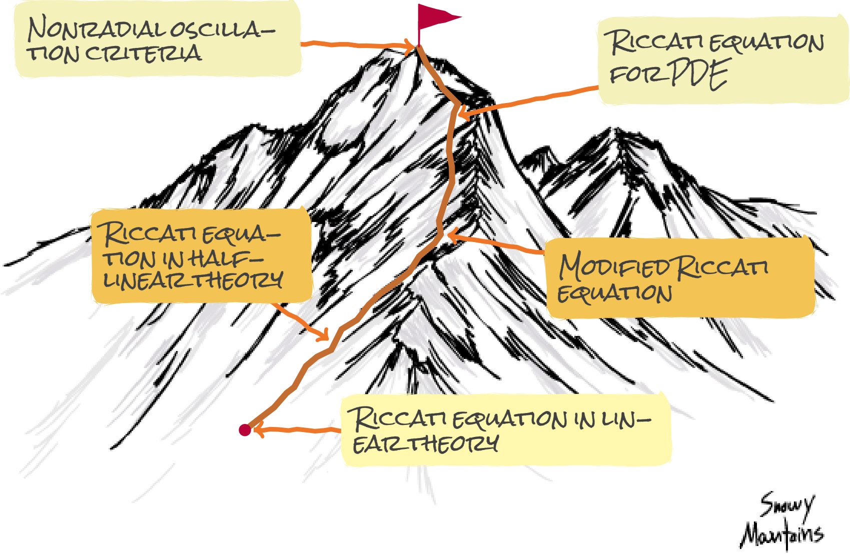
$$\frac{\|\nabla u\|^{2-p}}{|u|^{2-p}} \geq \frac{\|w\|^{(2-p)/(p-1)}}{\lambda_{\mathsf{max}}^{(2-p)/(p-1)}}$$

$$\operatorname{div} w + c + (p-1)\lambda_{\mathsf{max}}^{1-q}\|w\|^q \leq 0$$

Explanation of the difference: The inequality for $\|w\|$ is powered to $(2-p)/(p-1)$ if $p \in (1,2]$, which is not possible if $p > 2$. If $p > 2$, then $\|w\|$ is powered to $q > (2-p)/(p-1)$.

Part 4

Summary



Conclusion: Riccati equation provides tool for simple proof of **Sturmian comparison theorems** (even for half-linear equations), can be used as a half-linear **replacement for missing transformation theory** of half-linear equations, can be used to formulate **nonradial oscillation criteria** of half-linear PDE's.