Oscillation and nonoscillation criteria for half-linear differential equations

Exploring "terra incognita" and the hunt for better oscillation constants

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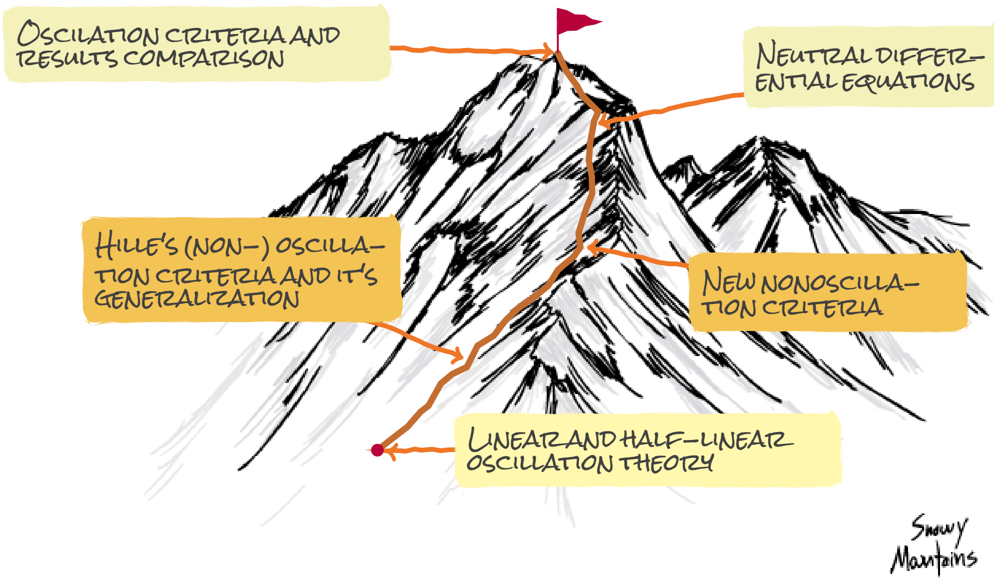
Oscilation criteria and results comparison
Neutral differ-ential equations
Hille's (non-) oscilla-tion criteria and it's generalization
New nonoscilla-tion criteria
Linear and half-linear oscillation theory
Snowy Mountains

Part 1

Half-linear ODE, oscillation theory

- Half-linear second order differential equation
- Oscillation theory

Half-linear second order differential equation

$$\Big(r(t)\Phi\big(x'(t)\big)\Big)' + c(t)\Phi\Big(x(t)\Big) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1 \qquad (*)$$

- The differential operator is scalar p-Laplacian (Allegretto, Drábek, Jaroš, Kusano, Manásevich, Mawhin, del Pino, Usami, Yoshida)
- Equation (*) has been introduced in 70's by Mirzov and Elbert. This and related equations have been later studied by many authors (including Došlá, Došlý, Drábek, Fišnarová, Hasil, Hata, Jaroš, Kong, Kusano, Li, Lomtatidze, Manojlovič, Marić, Marini, Matsumura, Matucci, Naito, Ogata, Onitsuka, Řehák, Sugie, Sun, Tanigawa, Usami, Veselý, Wang, Xu, Yamaoka, Yoshida).
- Equation (*) preserves many properties of linear equation ($p = 2$)
  - Some of the results known for linear equations can be smoothly extended to (*) (oscillation theory with zero-interlacing-like and majorant-like theorems).
  - Some of the "linear" methods have to be modified for (*) (there is no transformation theory).
  - Some of the "linear" results fail (Wronskian).

Oscillation theory

- Equation is oscillatory if some (and thus all) solutions have zeros in every neighborhood of ∞.
- Equation is nonoscillatory otherwise (there exists a solution positive in some neighborhood of ∞).

Oscillation of

$$x'' + c(t)x = 0$$

is ensured if $c(t)$ is large enough. For example in one of the following sense.

pointwise criteria simplest possibility, from comparison with suitable equation,

e.g.: $x'' + \frac{k}{t^2}x = 0$ is nonoscillatory iff $k \le \frac{1}{4}$,

$x'' + c(t)x = 0$ is oscillatory if $c(t) \ge \left(\frac{1}{4} + \varepsilon\right) t^{-2}$ for large t

and nonoscillatory if $c(t) \le \frac{1}{4}t^{-2}$ for large t

integral criteria can be used if $c(t)$ is not large enough for every t but its mean value is large

e.g.: $\int^{\infty} c(t)\,\mathrm{d}t = \infty$ is sufficient for oscillation

series of conjugacy criteria can be used if the mean value of $c(t)$ is small, but there is a series of subintervals where $c(t)$ is large enough to bend every solution to zero. Certain lower bound for mean value of $c(t)$ is required if we wish to eliminate this possibility how the equation can be turned to oscillation, e.g. in nonoscillation criteria.

Basic oscillation/nonoscillation criteria

$$x'' + c(t)x = 0$$

$$C(t) = \frac{1}{t}\int_1^t \int_1^s c(\xi)\,\mathrm{d}\xi\ \mathrm{d}s$$

Hartman, Wintner: If

$$-\infty < \liminf_{t\to\infty} C(t) < \limsup_{t\to\infty} C(t) \le \infty,$$

or

$$\lim_{t\to\infty} C(t) = \infty,$$

then the equation is oscillatory.

Observation:

$$\lim_{t\to\infty} \int^t c(s)\,\mathrm{d}s = \infty \implies \lim_{t\to\infty} C(t) = \infty$$

$$g(t) = t\int_t^\infty c(s)\,\mathrm{d}s$$

$$g_* = \liminf_{t\to\infty} t\int_t^\infty c(s)\,\mathrm{d}s$$

$$g^* = \limsup_{t\to\infty} t\int_t^\infty c(s)\,\mathrm{d}s$$

Hille: $c(t) \ge 0$

If $g_* > \frac{1}{4}$, then the equation is oscillatory.

If $g^* > 1$, then the equation is oscillatory.

If $g^* < \frac{1}{4}$, then the equation is nonoscillatory.

Further authors: Leighton, Nehari, Kamenev, Philos

Part 2

Nonoscillation criteria for half-linear ODE

• Lomtatidze's extension of Hille and Nehari criteria • Half-linear Riccati equation
• Nonoscillatory criteria (guessing solution)

Lomtatidze's extension of Hille and Nehari criteria

$$x'' + c(t)x = 0$$

(no sign restrictions on $c(t)$, half-linear version also exists)

$$C_0 := \lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{1}{t}\int_1^t \int_1^s c(\xi)\,\mathrm{d}\xi\ \mathrm{d}s$$

$$Q(t) = t\left(C_0 - \int_1^t c(s)\,\mathrm{d}s\right)$$

$$= \left(\text{if } \lim_{t\to\infty}\int_1^t c(s)\,\mathrm{d}s \text{ exists}\right) = g(t)$$

$$H(x) = \frac{1}{t}\int_1^t s^2 c(s)\,\mathrm{d}s$$

$$Q_* = \liminf_{t\to\infty} Q(t) \qquad H_* = \liminf_{t\to\infty} H(t)$$

$$Q^* = \limsup_{t\to\infty} Q(t) \qquad H^* = \limsup_{t\to\infty} H(t)$$

Sufficient conditions for oscillation:

- $Q_* > \frac{1}{4}$
- $H_* > \frac{1}{4}$
- $0 \le Q_* \le \frac{1}{4}$ and $H^* > \frac{1}{2}(1 + \sqrt{1-4Q_*})$
- $0 \le H_* \le \frac{1}{4}$ and $Q^* > \frac{1}{2}(1 + \sqrt{1-4H_*})$

Sufficient conditions for nonoscillation:

- $-\frac{3}{4} < Q_*$ and $Q^* < \frac{1}{4}$
- $-\infty < Q_* \le -\frac{3}{4}$ and $Q^* < Q_* - 1 + \sqrt{1-4Q_*}$

  (i.e. $-\frac{3}{4} < Q_*$ can be broken, but the upper bound for Q^* must be decreased)

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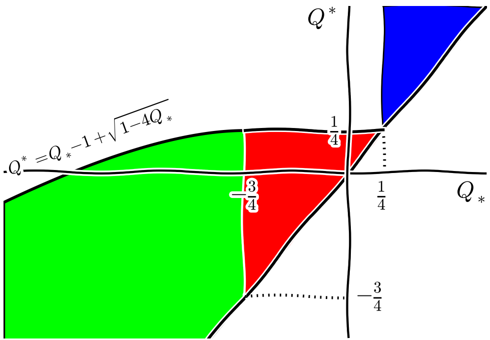
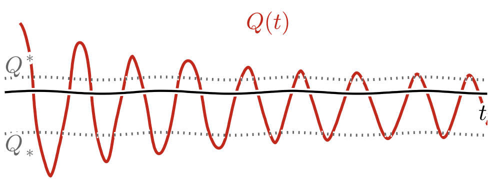


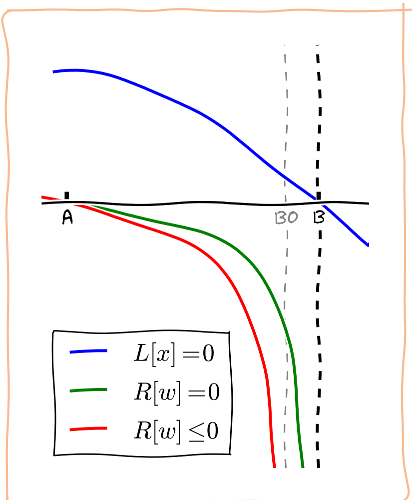


$$L[x] := (r(t)\Phi(x'))' + c(t)\Phi(x) = 0$$

$$w = r\Phi(x'/x)$$

$$x = \exp(\textstyle\int \Phi^{-1}(w/r))$$

$$R[w] := w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0$$
$$q = \frac{p}{p-1}$$



Nonoscillation criteria: Show that

$$R[w] \leq 0$$

has a solution in a neighborhood of infinity.

Oscillation criteria: Show that

$$R[w] = 0$$

does not have solution defined in a neighborhood of infinity.

In order to prove nonoscillation of

$$x'' + c(t)x = 0 \tag{*}$$

it is sufficient to prove that the Riccati *inequality*

$$w' + c(t) + w^2 \leq 0 \tag{**}$$

has a solution on the interval $[t_0, \infty)$ for some t_0.

Example (proof of Hille's criterion):

Consider

$$w(t) = \int_t^\infty c(s)\,\mathrm{d}s + \frac{1}{4t}, \qquad w' = -c(t) - \frac{1}{4t^2}.$$

The function w satisfies (**) iff $w^2 \leq \dfrac{1}{4t^2}$ (direct substitution to (**)), i.e. iff $|w| \leq \dfrac{1}{2t}$.

To ensure this condition it is sufficient to suppose that

$$-\frac{3}{4} < \underbrace{\liminf_{t\to\infty} t \int_t^\infty c(s)\,\mathrm{d}s}_{g_*} \leq \underbrace{\limsup_{t\to\infty} t \int_t^\infty c(s)\,\mathrm{d}s}_{g^*} < \frac{1}{4}.$$

(Really, just put the definition of $w(t)$ into $|w| \leq \dfrac{1}{2t}$, multiply by t and subtract $\dfrac{1}{4}$.)

Nonoscillation criteria (perturbation of nonoscillatory equation)

$$L[x] := \big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0 \qquad \text{(under examination)} \qquad (*)$$

$$\widetilde{L}[x] := \big(r(t)\Phi(x')\big)' + \widetilde{c}(t)\Phi(x) = 0 \qquad \text{(nonoscillatory)}$$

Theorem A (Došlý, Řezníčková). *Let $h \in C^1$ be a positive function such that $h'(t) > 0$ for large t, say $t > T$. Suppose ... (assumptions on h: the function h is close to certain solution of $\widetilde{L}[x] = 0$). If*

$$\limsup_{t\to\infty} \int_t^\infty \frac{\mathrm{d}s}{r(s)h^2(s)(h'(s))^{p-2}} \int_T^t \big(c(s) - \widetilde{c}(s)\big)h^p(s)\,\mathrm{d}s < \frac{1}{2q},$$

$$\liminf_{t\to\infty} \int_t^\infty \frac{\mathrm{d}s}{r(s)h^2(s)(h'(s))^{p-2}} \int_T^t \big(c(s) - \widetilde{c}(s)\big)h^p(s)\,\mathrm{d}s > -\frac{3}{2q}$$

for some $T \in \mathbb{R}$ sufficiently large, then (*) is nonoscillatory.

Observation: note bounds for $c(t) - \widetilde{c}(t)$ rather that for $c(t)$, as in the linear case. Due transformation, there is no loos of generality in the linear case to consider $\widetilde{c}(t) = 0$.

Proof: Denote $R := rh^2|h'|^{p-2}$, $w = w_h + h^{-p}v$, where $w_h = r\dfrac{\Phi(h')}{\Phi(h)}$ and

$$v(t) = -\frac{1}{2q}\left(\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right)^{-1} - \int^t \big(c(s) - \widetilde{c}(s)\big)h^p(s)\,\mathrm{d}s.$$

The function w is solution of the inequality which arises from the associated Riccati equation by replacing "$=$" with "$\le$".

Corollary. If

$$\limsup_{t\to\infty} \frac{1}{\ln t}\int^t c(s)s^{p-1}\ln^2 s\,\mathrm{d}s < \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$$

$$\liminf_{t\to\infty} \frac{1}{\ln t}\int^t c(s)s^{p-1}\ln^2 s\,\mathrm{d}s > -\frac{3}{2}\left(\frac{p-1}{p}\right)^{p-1}$$

then

$$\left(\Phi(x')\right)' + \left[\left(\frac{p-1}{p}\right)^p t^{-p} + c(t)\right]\Phi(x) = 0$$

is nonoscillatory.

Proof. Special case of the previous theorem for

$$\left(\Phi(x')\right)' + \left(\frac{p-1}{p}\right)^p t^{-p}\Phi(x) = 0 \quad \text{and} \quad h(t) = t^{(p-1)/p}\ln^{2/p} t.$$

Why $\widetilde{c}(t) = \left(\frac{p-1}{p}\right)^p t^{-p}$?

$$\left(\Phi(x')\right)' + kt^{-p}\Phi(x) = 0$$

is oscillatory iff $k > \left(\frac{p-1}{p}\right)^p$ and this equation is on the border line between oscillation and nonoscillation.

$$L[x] := \big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0 \qquad \text{(under examination)} \qquad (*)$$

$$\widetilde{L}[x] := \big(r(t)\Phi(x')\big)' + \widetilde{c}(t)\Phi(x) = 0 \qquad \text{(nonoscillatory)}$$

Theorem 1. *Let h be a function such that $h(t) > 0$ and $h'(t) \neq 0$, both for large t. Suppose ... (technical assumptions on h).*
If

$$\limsup_{t\to\infty} \int_t^\infty \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int^t \Big(c(s) - \widetilde{c}(s)\Big) h^p(s)\,\mathrm{d}s < \frac{1}{q}\left(-\alpha + \sqrt{2\alpha}\right),$$

$$\liminf_{t\to\infty} \int_t^\infty \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int^t \Big(c(s) - \widetilde{c}(s)\Big) h^p(s)\,\mathrm{d}s > \frac{1}{q}\left(-\alpha - \sqrt{2\alpha}\right)$$

for some $\alpha > 0$, then equation (*) is nonoscillatory.

Remark: For $\alpha = \frac{1}{2}$ we have Theorem by Došlý and Řezníčková.

Proof. Denote $R := rh^2|h'|^{p-2}$, $w = w_h + h^{-p}v$, where $w_h = r\dfrac{\Phi(h')}{\Phi(h)}$ and

$$v(t) = -\frac{\alpha}{q}\left(\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right)^{-1} - \int^t \big(c(s) - \widetilde{c}(s)\big)h^p(s)\,\mathrm{d}s.$$

The function w is solution of the inequality which arises from the associated Riccati equation by replacing "$=$" with "$\leq$".

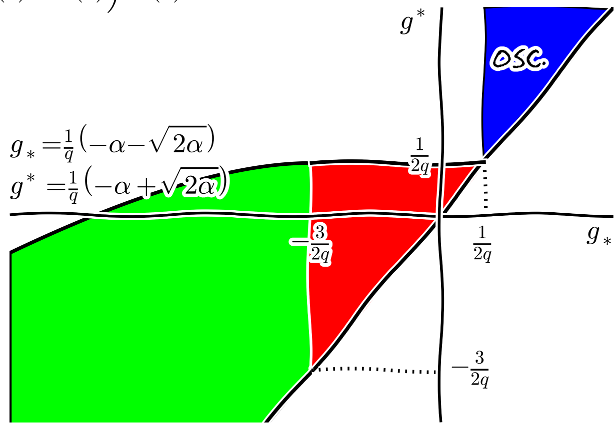
The value $\alpha = \frac{1}{2}$ has been used by D+Ř, since it produces maximum in the expression involving $\limsup$. Is it reasonable to choose general $\alpha \neq \frac{1}{2}$?

Yes, we obtain similar extension, as extension of Hille's criteria obtained by Lomtatidze et al.

$$g(t) = \int_t^\infty \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int^t \Big(c(s) - \tilde{c}(s)\Big) h^p(s)\,\mathrm{d}s$$

$$g_* = \liminf_{x\to\infty} g(t)$$

$$g^* = \limsup_{x\to\infty} g(t)$$



Regions of nonoscillation:

D+Ř: $\begin{cases} g^* < \dfrac{1}{2q} \\ g_* > -\dfrac{3}{2q} \end{cases}$

F+M: $\begin{cases} g^* < \dfrac{1}{q}(-\alpha + \sqrt{2\alpha}) \\ g_* > \dfrac{1}{q}(-\alpha - \sqrt{2\alpha}) \end{cases}$

Summary of nonoscillation criteria

- Replacing a fixed constant in the proof of nonoscillation criteria by a parameter we succeeded to find a parametric curve which forms a boundary of the region where nonoscillation is ensured.
- The results correspond to the Lomtatidze's extension of Hille nonoscillation criteria (we just obtained parametric curve rather than analytic formula).
- We have several modification of the expression which is tested on $\liminf$ and $\limsup$, depending on the convergence/divergence of some integrals arising from r and h and depending on the fact whether we use $c-\widetilde{c}$ or $L[h]$ in the expression to be tested. (Some of the results were new even in the linear case.)

Part 3

Neutral half-linear differential equations

• Half-linear Euler equation • Motivation (a known result to be examined in details) • Neutral differential equation • Comparison method • Riccati transformation • Suggested enhancements to both methods • Main results (comparison method) • Main results (Riccati method) • Comparison of available methods

Half-linear Euler equation

$$\left[\Phi\Big(x'(t)\Big)\right]' + \frac{\beta}{t^p}\Phi\Big(x(t)\Big) = 0$$

The equation is oscillatory if and only if

$$\beta > \left(\frac{p-1}{p}\right)^p =: \Gamma_p.$$

If $p = 2$, then $\beta > 1/4$ is necessary and sufficient for oscillation of $x'' + \dfrac{\beta}{t^2}x = 0$.

Motivation (a known result to be examined in details)

(Sun, Li, Han, Li; 2012)

$$\left[\Phi\Big(\big(x(t) + b(t)x(\lambda_1 t)\big)'\Big)\right]' + \frac{\beta}{t^p}\Phi\big(x(\lambda_2 t)\big) = 0 \qquad (*)$$

with $0 \le b(t) \le b_0 < \infty$, $p \ge 2$, $\beta > 0$, $\lambda_1 \in (0,1)$.

- If $0 < \lambda_2 \le \lambda_1$, then (*) is oscillatory if $\beta > \Gamma_p \dfrac{2^{p-2}}{\lambda_2^{p-1}}\left(1 + \dfrac{b_0^{p-1}}{\lambda_1}\right)$.
- If $\lambda_2 \in [\lambda_1, \infty)$, then (*) is oscillatory if $\beta > \Gamma_p \dfrac{2^{p-2}}{\lambda_1^{p-1}}\left(1 + \dfrac{b_0^{p-1}}{\lambda_1}\right)$.
- If formally $\lambda_1 = \lambda_2 = 1$ and $b(t) \equiv 0$, the equation becomes Euler equation, but the oscillation constant is worse by a multiplicative factor 2^{p-2}. Brief sketch of literature reveled that this factor appears frequently in the oscillation criteria for (*).

$$\boxed{\left(r(t)\Phi(z'(t))\right)' + c(t)\Phi\Big(x\big(\sigma(t)\big)\Big) = 0}$$

$$z(t) = x(t) + b(t)x(\tau(t))$$

Assumptions:

- $\sigma(t) \leq \tau(t) \leq t,\ \lim\limits_{t\to\infty} \sigma(t) = \infty.$
- $\sigma(\tau(t)) = \tau(\sigma(t))$
- $r(t) > 0,\ \int^{\infty} r^{1-q}(t)\,\mathrm{d}t = \infty,\ c(t) \geq 0$
- $\tau'(t) \geq \tau_0 > 0,\ b(t) \leq b_0$
- $p \geq 2$ (i.e. $\Phi(x)$ is a convex function)

Terminology:

- Solution = Classical solution which is not eventually constant
- Oscillatory equation = All solutions are oscillatory (no eventually positive solution exist)

(Baculíková, Džurina, Rogovchenko)

- Consider the original equation and the equation shifted from t to $\tau(t)$ and multiplied

$$\Big[r(t)\Phi(z'(t))\Big]' + c(t)\,\Phi(x(\sigma(t))) = 0$$

$$\frac{b_0^{p-1}}{\tau'(t)}\Big[r(\tau(t))\Phi(z'(\tau(t)))\Big]' + c(\tau(t))\, b_0^{p-1}\Phi(x(\sigma(\tau(t)))) = 0$$

- Consider new variable $y(t) = r(t)z'(t) + \dfrac{b_0^{p-1}}{\tau_0} r(\tau(t))z'(\tau(t))$. This variable satisfies

$$y'(t) + \min\{c(t), c(\tau(t))\}\left[\Phi\Big(x(\sigma(t))\Big) + \Phi\Big(b_0 x(\sigma(\tau(t)))\Big)\right] \leq 0$$

- Under appropriate assumptions we get

$$y' + \min\{c(t), c(\tau(t))\}\, \mathbf{2^{2-p}}\Phi(z(\sigma(t))) \leq 0$$

$$y' + \min\{c(t), c(\tau(t))\}\, \mathbf{2^{2-p}} \left[\int_{t_0}^{\sigma(t)} r^{1-q}(s)\,\mathrm{d}s\right]^{p-1} \frac{\tau_0}{\tau_0 + b_0^{p-1}} y(\tau^{-1}(\sigma(t))) \leq 0$$

- The previous steps can be performed also for quasilinear equation

$$\Big(r(t)\Phi_\alpha(z'(t))\Big)' + c(t)\Phi_\beta\Big(x(\sigma(t))\Big) = 0$$

but we end up with nonlinear inequality.

First order delay differential inequality

(Chanturia, Kitamura, Koplatadze, Kusano)

Let $q(t) \geq 0$.

(i) If $\sigma(t) < t$ and

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\,\mathrm{d}s > \frac{1}{e},$$

then

$$y'(t) + q(t)y(\sigma(t)) \leq 0$$

has no eventually positive solution.

(ii) If $\sigma(t) > t$ and

$$\liminf_{t\to\infty} \int_{t}^{\sigma(t)} q(s)\,\mathrm{d}s > \frac{1}{e},$$

then

$$y'(t) - q(t)y(\sigma(t)) \geq 0$$

has no eventually positive solution.

(iii) Let $\sigma(t) < t$, $\alpha \in (0,1)$. If

$$\int_{t_0}^{\infty} q(s)\,\mathrm{d}s = \infty,$$

then

$$y'(t) + q(t)y^{\alpha}(\sigma(t)) \leq 0$$

has no eventually positive solution.

(iv) Let $\sigma(t) > t$, $\alpha \in (1,\infty)$. If

$$\int_{t_0}^{\infty} q(s)\,\mathrm{d}s = \infty,$$

then

$$y'(t) - q(t)y^{\alpha}(\sigma(t)) \geq 0$$

has no eventually positive solution.

Riccati transformation

(Bohner, Džurina, Rogovchenko, Stavroulakis, Li)

- Classical Riccati transformation:

  If $(rx')' + qx = 0$, then $w = \rho \dfrac{rx'}{x}$ satisfies $w' = \dfrac{\rho'}{\rho} w - \rho q - \dfrac{1}{\rho r} w^2$.

- For $\omega(t) = \rho(t) \dfrac{r(t)\Phi(z'(t))}{\Phi(z(\sigma(t)))}$ and $v(t) = \rho(t) \dfrac{r(\tau(t))\Phi(z'(\tau(t)))}{\Phi(z(\sigma(t)))}$ we obtain Riccati-like inequalities.

- We combine them like in the comparison method. Among others, we use similar inequalities and estimates.

- Proceed like in Riccati method for ordinary differential equation.

Observation common to both methods

The oscillation criteria are expressed in terms of $\min\{c(t), c(\tau(t))\}$ and contain constant 2^{2-p} (which has no analogy in the case without delay).

Idea 1 (power of 2)

Inequality

$$x_1^{p-1} + x_2^{p-1} \geq 2^{2-p}(x_1 + x_2)^{p-1}$$

causes undesired constant 2^{p-2}. The constant 2^{2-p} is optimal in this inequality, however writing the inequality in the form

$$\frac{1}{2}x_1^{p-1} + \frac{1}{2}x_2^{p-1} \geq \left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right)^{p-1}$$

we see that it is just the immediate consequence of convexity of x^{p-1}.

> **Suggestion**: What about to use general convex linear combination with coefficients $\frac{1}{l}$, $\frac{1}{l^*}$ rather than $\frac{1}{2}$?
>
> **Expectation**: We hope that the constant 2^{2-p} will be replaced by something more convenient when choosing optimal l and l^*.

Idea 2 (minimum)

To write the term

$$c(t)x^{p-1}(t) + c(\tau(t))x^{p-1}(\tau(t))$$

in the form of product

$$(\text{"factor"})\left[x^{p-1}(t) + x^{p-1}(\tau(t))\right]$$

we have to introduce common multiplicative factor, i.e. we have to replace both $c(t)$ and $c(\tau(t))$ by $\min\Big\{c(t), c(\tau(t))\Big\}$. Here we loose.

> **Suggestion**: What about to try not to loose so much and arrange thing so that $\min\Big\{c(t), \varphi c(\tau(t))\Big\}$ appears instead of $\min\Big\{c(t), c(\tau(t))\Big\}$?
>
> **Expectation**: We hope that the factor φ allows to make $c(t)$ closer to $\varphi c(\tau(t))$ than to $c(\tau(t))$.

Classical approach:

$$y' + \min\{c(t), c(\tau(t))\} \left[\int_{t_1}^{\sigma(t)} r^{1-q}(s)\,\mathrm{d}s\right]^{p-1} 2^{2-p} \frac{\tau_0}{\tau_0 + p_0^{p-1}}\, y(\tau^{-1}(\sigma(t))) \le 0$$

Improved approach: for $\eta(t) \le \sigma(t)$ we have

$$y' + \min\{c(t), \varphi c(\tau(t))\} \left[\int_{t_1}^{\eta(t)} r^{1-q}(s)\,\mathrm{d}s\right]^{p-1} \left(1 + (\varphi/\tau_0)^{1/\alpha} p_0\right)^{1-p} y(\tau^{-1}(\eta(t))) \le 0$$

Theorem 2. *Suppose that there exists number* $\varphi > 0$ *and a function* $\eta(t)$ *satisfying* $\eta(t) \le \sigma(t)$ *and* $\lim_{t\to\infty} \eta(t) = \infty$ *such that* $\eta(t) < \tau(t) \le t$ *and for every* T *there exists* $t_1 > T$ *such that*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\eta(t))}^{t} C_\eta(s; \varphi, t_1)\,\mathrm{d}s > \frac{1}{e},$$

where

$$C_\eta(t; \varphi, t_1) := \min\{c(t), \varphi c(\tau(t))\} \left[\int_{t_1}^{\eta(t)} r^{1-q}(s)\,\mathrm{d}s\right]^{p-1} \left(1 + (\varphi/\tau_0)^{1/\alpha} p_0\right)^{1-p}.$$

Then $\boxed{\left(r(t)\Phi(z'(t))\right)' + c(t)\Phi\left(x(\sigma(t))\right) = 0}$ *is oscillatory.*

Theorem 3. *If there exist positive mutually conjugate numbers* l, l^* *and positive functions* $\rho(t)$, $\varphi(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \rho(s)C(s) - \frac{1}{p^p}\frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{p-1}}$$
$$\left[l^{p-2}\left(\frac{\rho'_+(s)}{\rho(s)}\right)^p + (l^*)^{p-2}\frac{b_0^{p-1}\varphi(s)}{\tau_0}\left(\frac{\rho'(s)}{\rho(s)} + \left(\frac{b_0^{p-1}\varphi(s)}{\tau_0}\right)'\frac{\tau_0}{b_0^{p-1}\varphi(s)}\right)_+^p\right]\mathrm{d}s = \infty,$$

where

$$C(t) = \min\Big\{c(t), \varphi(t)c(\tau(t))\Big\},$$

then

$$\boxed{\Big(r(t)\Phi(z'(t))\Big)' + c(t)\Phi\Big(x(\sigma(t))\Big) = 0}$$

is oscillatory.

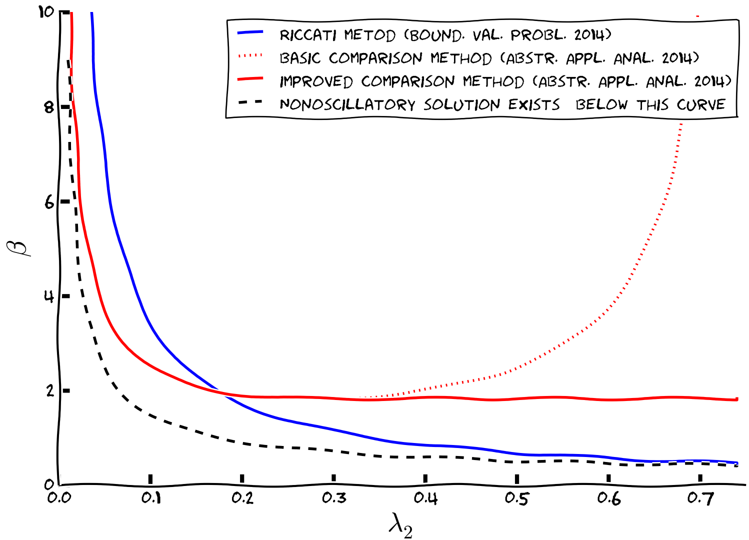
$$\left[\Phi\Big(\big(x(t)+b_0x(\lambda_1 t)\big)'\Big)\right]' + \frac{\beta}{t^p}\Phi\big(x(\lambda_2 t)\big) = 0$$

$0 \le b_0 < \infty$, $p \ge 2$, $\beta > 0$, $0 < \lambda_2 \le \lambda_1 < 1$, $\Gamma_p := \left(\dfrac{p}{p-1}\right)^p$

Lower bound for oscillation

Sun, Li, Han, Li (2012)		$\beta > \Gamma_p \dfrac{1}{\lambda_2^{p-1}} 2^{p-2} \left(1 + \dfrac{b_0^{p-1}}{\lambda_1}\right)$
Fišnarová, Mařík (2014)	Riccati method	$\beta > \Gamma_p \dfrac{1}{\lambda_2^{p-1}} (1 + b_0\lambda_1)^{p-1}$
	comparison method	$\beta > \dfrac{1}{e} \dfrac{1}{\lambda_2^{p-1}} \dfrac{(1+b_0\lambda_1)^{p-1}}{\log(\lambda_1/\lambda_2)}$

$$\left[x(t) + 0.5x(0.75t)\right]'' + \frac{\beta}{t^p}x(\lambda_2 t) = 0$$

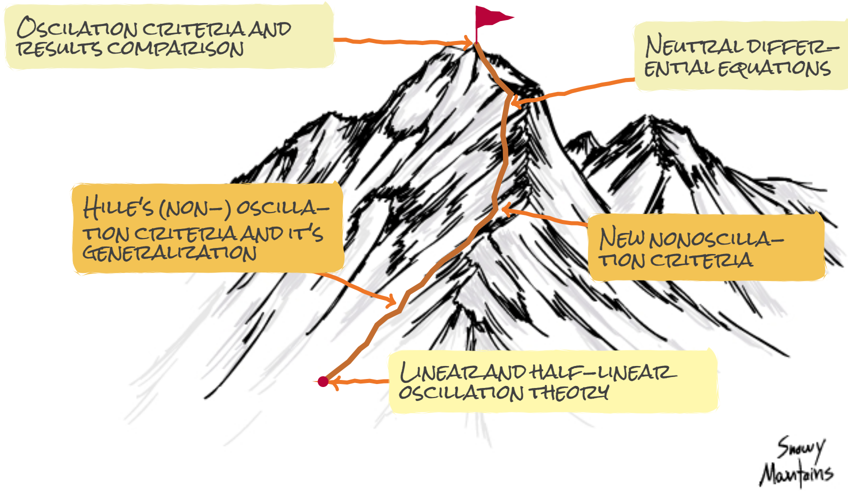


Summary on neutral equation

- Introducing parameters in the existing oscillation criteria for neutral half-linear and quasilinear equation we generalized existing results. We have shown that this extension is not empty, but significant.
- The method removes undesired expressions heavily used in the oscillation criteria in the literature (2^{p-2} and $\min\{c(t), c(\tau(t))\}$), improves the estimates used in the proofs and naturally produces better results.
- The improvement is in basic steps used in most oscillation criteria and thus a vast number of results can be improved in this direction.

Part 4

Summary



Two success stories based on examination of widely used inequalities and using them in another way than usual have been presented.