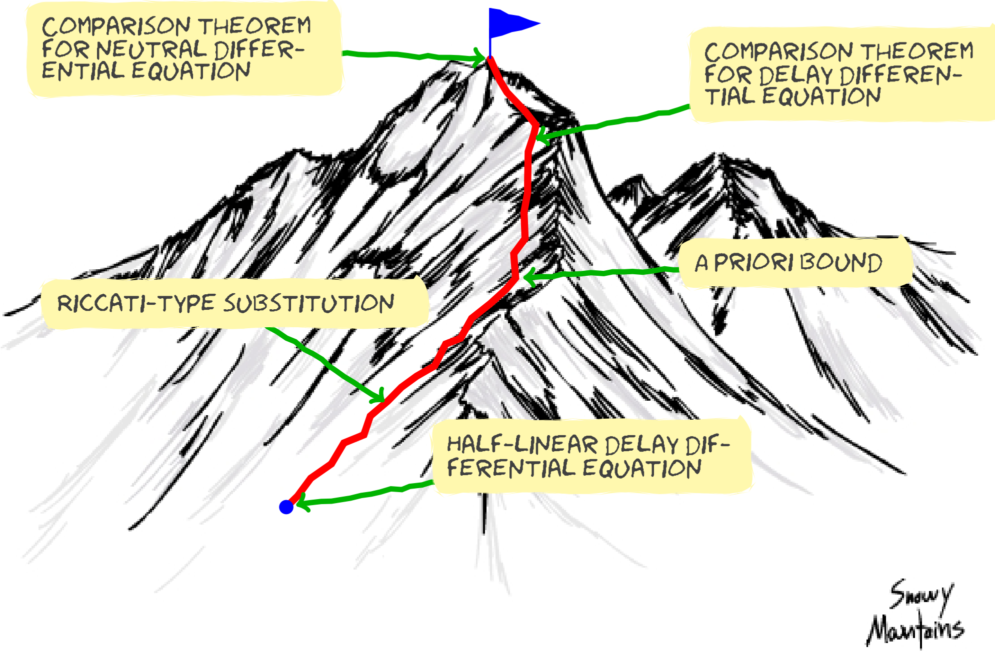
Riccati technique for half-linear delay differential equation

Robert Mařík

Mendel University in Brno
Czech Republic

(joint work with Simona Fišnarová)

Faculty of Forestry and Wood Technology



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Half-linear second order delay differential equation

$$\boxed{\Big(r(t)\Phi\big(x'(t)\big)\Big)' + c(t)\Phi\Big(x\big(\tau(t)\big)\Big) = 0}, \quad \Phi(x) := |x|^{p-2}x, \ p > 1 \tag{1}$$

Assumptions:

- r, c, τ are continuous
- sign restrictions: $r(t) > 0$, $c(t) \geq 0$ for large t,
- delay: $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$

Notation: q is the conjugate number to the number p, i.e. $q = \dfrac{p}{p-1}$.

Solutions, oscillatory solutions

- Under the solution of (1) we understand any differentiable function $x(t)$ which does not identically equal zero eventually, such that $r(t)\Phi(x'(t))$ is differentiable and (1) holds for large t.
- The solution of equation (1) is said to be oscillatory if it has infinitely many zeros tending to infinity.
- Equation (1) is said to be oscillatory if all its solutions are oscillatory. In the opposite case, i.e., if there exists an eventually positive solution of (1), equation (1) is said to be nonoscillatory.

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$$x''(t) + c(t)x(\tau(t)) = 0 \tag{2}$$

- Case $\int^{\infty} t\, c(t)\, \mathrm{d}t < \infty$: (2) is nonoscillatory
- Case $\int^{\infty} t\, c(t)\, \mathrm{d}t = \infty$: Riccati equation technique (but unlike for ordinary differential equations, for delay equation we get inequality instead of equation)
  - Substitution with delay in denominator: $w(t) = \dfrac{x'(t)}{x(\tau(t))}$
    - Riccati inequality is simpler (the term $x(\tau(t))$ cancels).
    - The method gives oscillation results under stronger conditions on τ.
  - Classical Riccati substitution: $w(t) = \dfrac{x'(t)}{x(t)}$
    - Since $x(\tau(t))$ does not cancel, we have to find a priori bound for $\dfrac{x(\tau(t))}{x(t)}$.
    - Simply inequalities which hold for every positive concave-down function are used typically to find these a priori bounds in most papers. This procedure has been improved significantly by Opluštil and Šremr .

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$$x''(t) + c(t)x(\tau(t)) = 0 \tag{2}$$

- Case $\int^{\infty} t\,c(t)\,\mathrm{d}t = \infty$, Substitution $w(t) = \dfrac{x'(t)}{x(t)}$
  - We have to find a priori bound for $\dfrac{x(\tau(t))}{x(t)}$.
  - Typically inequalities which hold for every positive concave-down function, namely
  $$\frac{x(\tau(t))}{x(t)} \geq \frac{\tau(t)}{t}\frac{\tau(t)-T}{t-T} \qquad \text{or} \qquad \frac{x(\tau(t))}{x(t)} \geq k\frac{\tau(t)}{t}, \quad k \in (0,1)$$
  are used.
  - Opluštil, Šremr: $\dfrac{x(\tau(t))}{x(t)} \geq \cancel{k}\,\dfrac{\tau(t)}{t}$ (and several stronger conditions under additional assumptions).

Our aim: to extend Opluštil's and Šremr's result to half-linear equation (1)

Our final result:
- the problem appeared not so easy as we originally supposed
- our extension does not include the linear result as a special case

Assumptions

- $\displaystyle\int^{\infty} r^{1-q}(t)\,\mathrm{d}t = \infty,\ r'(t) \geq 0$ .... ensures that positive solutions are increasing
- $\displaystyle\int^{\infty} c(t)\tau^{p-1}(t)\,\mathrm{d}t = \infty$ .... a version of the condition used in the linear case

Conclusion

$$\boxed{\frac{x(\tau(t))}{x(t)} \geq \frac{\tau(t)}{t}}$$

No $k \in (0,1)$ and no $\dfrac{\tau(t)-T}{t-T}$ anymore.

What is the difference to methods used by other authors? We used not only the fact that x is concave down, but used also the fact that the function is a solution of (1).

What it the relationship of the linear version of this result and the known result in the linear case? If $p=2$, then the condition on c reads

$$\int^{\infty} c(t)\tau(t)\,\mathrm{d}t = \infty.$$

However Opluštil and Šremr succeeded to use in the linear case weaker condition

$$\int^{\infty} c(t)t\,\mathrm{d}t = \infty.$$

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Oscillation theorem

Deriving Riccati type inequality for positive solutions of the delay equation we can prove oscillation theorem in terms of nonexistence of the solution if the Riccati-type inequality. This is equivalent to oscillation of the associated second order differential equation without delay.

Theorem 1. *Suppose that conditions*

$$\int^{\infty} r^{1-q}(t)\,\mathrm{d}t = \infty, \quad \int^{\infty} c(t)\tau^{p-1}(t)\,\mathrm{d}t = \infty, \quad \text{and } r'(t) \geq 0 \text{ for large } t,$$

hold. If the ordinary differential equation

$$\Big(r(t)\Phi\big(x'(t)\big)\Big)' + c(t)\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi\Big(x(t)\Big) = 0$$

is oscillatory, then

$$\Big(r(t)\Phi\big(x'(t)\big)\Big)' + c(t)\Phi\Big(x\big(\tau(t)\big)\Big) = 0$$

is also oscillatory.

Summary (delay equation)

$$\Big(r(t)\Phi\big(x'(t)\big)\Big)' + c(t)\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi\Big(x\big(\cancel{\tau}(t)\big)\Big) = 0$$

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$$\boxed{\Big(r(t)\Phi(z'(t))\Big)' + c(t)\Phi\Big(x(\tau(t))\Big) = 0} \tag{3}$$

$$\boxed{z(t) = x(t) + a(t)x(\theta(t))}$$

Assumptions:

- $0 \leq a(t) < 1$, $r(t) > 0$, $c(t) \geq 0$,
- $\tau(t) \leq t$, $\theta(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \theta(t) = \infty$.

Riccati substitution:

- $w(t) = r(t)\dfrac{\Phi(z'(t))}{\Phi(z(t))}$
- Equation (3) is transformed into equation involving w and $\dfrac{x(\tau(t))}{z(t)}$

$$\frac{x(\tau(t))}{z(t)} = \overbrace{\frac{x(\tau(t))}{z(\tau(t))}}^{\text{The term can be estimated by an expression involving } a(t) \text{ if } a(t)<1.} \underbrace{\frac{z(\tau(t))}{z(t)}}_{\text{The term can be estimated similarly like the corresponding expression for the delay equation.}}$$

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Theorem 2. *Suppose that*

$$\int^{\infty} c(s)\Big(1-a\big(\tau(s)\big)\Big)^{p-1}\tau^{p-1}(s)\,\mathrm{d}s = \infty$$

and

$$\int^{\infty} r^{1-q}(t)\,\mathrm{d}t = \infty, \quad r'(t) \geq 0 \text{ for large } t,$$

hold. If the ordinary half-linear differential equation

$$\Big(r(t)\Phi\big(x'(t)\big)\Big)' + c(t)\Big[1-a\big(\tau(t)\big)\Big]^{p-1}\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi\Big(x(t)\Big) = 0$$

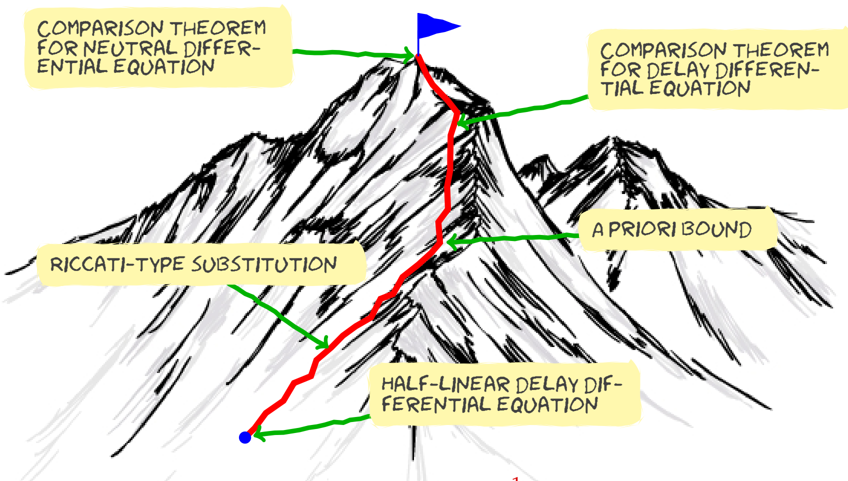
is oscillatory, then

$$\Big(r(t)\Phi\big(z'(t)\big)\Big)' + c(t)\Phi\Big(x\big(\tau(t)\big)\Big) = 0, \qquad z(t) = x(t) + a(t)x(\theta(t))$$

is also oscillatory.

Summary (neutral equation)

$$\Big(r(t)\Phi\big(\overset{x'}{\cancel{z'}}(t)\big)\Big)' + c(t)\Big[1-a\big(\tau(t)\big)\Big]^{p-1}\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi\Big(x\big(\cancel{\tau}(t)\big)\Big) = 0$$



$$\Big(r(t)\Phi\big(x'(t)\big)\Big)' + c(t)\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi\Big(x\big(\cancel{\tau}(t)\big)\Big) = 0$$

$$\Big(r(t)\Phi\big(\cancelto{x'}{z'}(t)\big)\Big)' + c(t)\Big[1 - a\big(\tau(t)\big)\Big]^{p-1}\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi\Big(x\big(\cancel{\tau}(t)\big)\Big) = 0$$