On constants in nonoscillation criteria for half-linear differential equations

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$$u'' + c(x)u = 0$$

$$C(x) = \frac{1}{x} \int_1^x \int_1^s c(t)\,\mathrm{d}t\ \mathrm{d}s$$

P. Hartman: If

$$-\infty < \liminf_{x\to\infty} C(x) < \limsup_{x\to\infty} C(x) \le \infty,$$

or

$$\lim_{x\to\infty} C(x) = \infty,$$

then the equation is oscillatory.

Observation:

If $\displaystyle\lim_{x\to\infty} \int^x c(t)\,\mathrm{d}t = \infty$, then $\displaystyle\lim_{x\to\infty} C(x) = \infty$.

$$g(x) = x \int_x^\infty c(t)\,\mathrm{d}t$$

$$g_* = \liminf_{t\to\infty} x \int_x^\infty c(t)\,\mathrm{d}t$$

$$g^* = \limsup_{t\to\infty} x \int_x^\infty c(t)\,\mathrm{d}t$$

E. Hille: $c(x) \ge 0$

If $g^* < \dfrac{1}{4}$, then the equation is nonoscillatory.

If $g_* > \dfrac{1}{4}$, then the equation is oscillatory.

If $g^* > 1$, then the equation is oscillatory.

$$u'' + c(x)u = 0$$

$$C_0 := \lim_{x\to\infty} C(x) = \lim_{x\to\infty} \frac{1}{x}\int_1^x \int_1^s c(t)\,\mathrm{d}t\ \mathrm{d}s$$

$$Q(x) = x\left(C_0 - \int_1^x c(t)\,\mathrm{d}t\right)$$

$$= \cdots \text{if } \lim_{x\to\infty}\int_1^x c(t)\,\mathrm{d}t \text{ exists} \cdots = g(x)$$

$$H(x) = \frac{1}{x}\int_1^x t^2 c(t)\,\mathrm{d}t$$

$$Q_* = \liminf_{x\to\infty} Q(x) \qquad H_* = \liminf_{x\to\infty} H(x)$$

$$Q^* = \limsup_{x\to\infty} Q(x) \qquad H^* = \limsup_{x\to\infty} H(x)$$

Lomtatidze et. al: a collection of oscillatory and nonoscillatory criteria in terms of the numbers Q_*, Q^*, H_*, H^*

Sufficient conditions for oscillation:

- $Q_* > \frac{1}{4}$
- $H_* > \frac{1}{4}$
- $0 \le Q_* \le \frac{1}{4}$ and $H^* > \frac{1}{2}(1 + \sqrt{1 - 4Q_*})$
- $0 \le H_* \le \frac{1}{4}$ and $Q^* > \frac{1}{2}(1 + \sqrt{1 - 4H_*})$

Sufficient conditions for nonoscillation:

- $-\frac{3}{4} < Q_*$ and $Q^* < \frac{1}{4}$
- $-\infty < Q_* \le -\frac{3}{4}$ and $Q^* < Q_* - 1 + \sqrt{1 - 4Q_*}$

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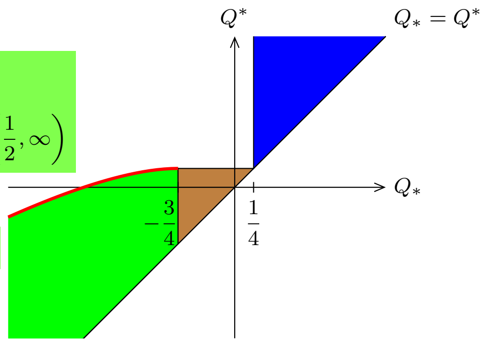
$$Q_* = \frac{1}{2}\left(-t - \sqrt{2t}\right)$$
$$Q^* = \frac{1}{2}\left(-t + \sqrt{2t}\right), t \in \left(\frac{1}{2}, \infty\right)$$

$$Q^* = Q_* - 1 + \sqrt{1 - 4Q_*}$$



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Proofs:

The function

$$y(x) = \int_x^\infty c(t)\,\mathrm{d}t + \frac{1}{4x}$$

satisfies

$$|y(x)| \le \frac{1}{\sqrt{4x^2}}$$

and the Riccati inequality

$$y' + y^2 + c(x) \le 0$$

$$v'' = -\frac{1}{x^2}\Big(Q^2(x) + 2\alpha Q(x) + \alpha(\alpha - 1)\Big)v - \frac{2}{x}\big(\alpha + Q(x)\big)v'$$

$$u(x) = x^\alpha v(x) \exp\left[\int_1^x \frac{Q(t)}{t}\,\mathrm{d}t\right]$$

$$Q^2(x) + 2\alpha Q(x) + \alpha(\alpha - 1) < 0$$

$$L[x] := \big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \tag{1}$$

$$\widetilde{L}[x] := \big(r(t)\Phi(x')\big)' + \widetilde{c}(t)\Phi(x) = 0 \tag{2}$$

Theorem A (O. Došlý). *Let $h \in C^1$ be a positive function such that $h'(t) > 0$ for large t, say $t > T$,*

$$F_1(t) = \int_T^t \frac{\mathrm{d}s}{r(s)h^2(s)(h'(s))^{p-2}}.$$

Suppose that

$$\lim_{t\to\infty} F_1(t)r(t)h(t)\Phi(h'(t)) = \infty \tag{3}$$

and

$$\lim_{t\to\infty} F_1^2(t)r(t)h^3(t)(h'(t))^{p-2}\widetilde{L}[h](t) = 0. \tag{4}$$

If the integral $\displaystyle\int^\infty \big(c(s) - \widetilde{c}(s)\big)h^p(s)\,\mathrm{d}s$ *is convergent,*

$$\limsup_{t\to\infty} F_1(t)\int_t^\infty \big(c(s) - \widetilde{c}(s)\big)h^p(s)\,\mathrm{d}s < \frac{1}{2q}$$

and

$$\liminf_{t\to\infty} F_1(t)\int_t^\infty \big(c(s) - \widetilde{c}(s)\big)h^p(s)\,\mathrm{d}s > -\frac{3}{2q},$$

then (1) is nonoscillatory.

Theorem B (O. Došlý, J. Řezníčková). *Let $h \in C^1$ be a positive function such that $h'(t) > 0$ for large t, say $t > T$, and denote*

$$F_2(t) = \int_t^\infty \frac{\mathrm{d}s}{r(s)h^2(s)(h'(s))^{p-2}} < \infty.$$

Suppose that (3) and (4) with F_1 replaced by F_2 hold. If

$$\limsup_{t\to\infty} F_2(t) \int_T^t \bigl(c(s) - \widetilde{c}(s)\bigr) h^p(s)\,\mathrm{d}s < \frac{1}{2q}$$

and

$$\liminf_{t\to\infty} F_2(t) \int_T^t \bigl(c(s) - \widetilde{c}(s)\bigr) h^p(s)\,\mathrm{d}s > -\frac{3}{2q}$$

for some $T \in \mathbb{R}$ sufficiently large, then (1) is nonoscillatory.

Theorem 1. *Let h be a function such that $h(t) > 0$ and $h'(t) \neq 0$, both for large t. Suppose that the following conditions hold*

$$\begin{cases} \displaystyle\int^{\infty} \frac{\mathrm{d}t}{r(t)h^2(t)|h'(t)|^{p-2}} < \infty, \\ \displaystyle\lim_{t\to\infty} r(t)h(t)|\Phi(h'(t))| \int_t^{\infty} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} = \infty. \end{cases}$$

- *If*

$$\limsup_{t\to\infty} r(t)h^3(t)|h'|^{p-2}(t)\widetilde{L}\left[h(t)\right] \left(\int_t^{\infty} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}}\right)^2 = 0$$

$$\limsup_{t\to\infty} \int_t^{\infty} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int^t \Big(c(s) - \widetilde{c}(s)\Big) h^p(s)\,\mathrm{d}s < \frac{1}{q}\left(-\alpha + \sqrt{2\alpha}\right),$$

$$\liminf_{t\to\infty} \int_t^{\infty} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int^t \Big(c(s) - \widetilde{c}(s)\Big) h^p(s)\,\mathrm{d}s > \frac{1}{q}\left(-\alpha - \sqrt{2\alpha}\right)$$

for some $\alpha > 0$, then equation (1) is nonoscillatory.

- *If*

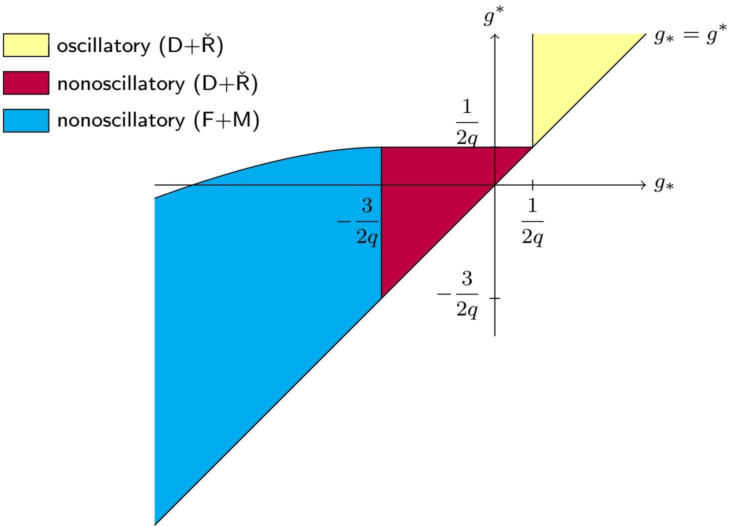
$$\limsup_{t\to\infty} \int_t^{\infty} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int^t h(s)L[h](s)\,\mathrm{d}s < \frac{1}{q}\left(-\alpha + \sqrt{2\alpha}\right),$$

$$\liminf_{t\to\infty} \int_t^{\infty} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int^t h(s)L[h](s)\,\mathrm{d}s > \frac{1}{q}\left(-\alpha - \sqrt{2\alpha}\right)$$

for some $\alpha > 0$, then equation (1) is nonoscillatory.

$$g(t)=\int_t^\infty \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}}\int^t \Big(c(s)-\widetilde{c}(s)\Big)h^p(s)\,\mathrm{d}s$$

$$g_* = \liminf_{x\to\infty} g(t), \qquad g^* = \limsup_{x\to\infty} g(t),$$



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Theorem 2. *Let h be a function such that $h(t) > 0$ and $h'(t) \neq 0$, both for large t. Suppose that*

$$\int^{\infty} h(t)L[h](t)\,\mathrm{d}t \text{ is convergent,}$$

$$\lim_{t\to\infty} r(t)h(t)|\Phi(h'(t))| \int^{t} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} = \infty.$$

If

$$\limsup_{t\to\infty} \int^{t} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int_t^{\infty} h(s)L[h](s)\,\mathrm{d}s < \frac{1}{q}\left(-\alpha + \sqrt{2\alpha}\right),$$

$$\liminf_{t\to\infty} \int^{t} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int_t^{\infty} h(s)L[h](s)\,\mathrm{d}s > \frac{1}{q}\left(-\alpha - \sqrt{2\alpha}\right)$$

for some $\alpha > 0$, then equation (1) is nonoscillatory.

Theorem 3. *Let h be a function such that $h(t) > 0$ and $h'(t) \neq 0$, both for large t. Suppose that the following conditions hold*

$$\int^{\infty} \Big(c(t) - \widetilde{c}(t)\Big) h^p(t)\,\mathrm{d}t \text{ is convergent},$$

$$\lim_{t\to\infty} r(t)h(t)|\Phi(h'(t))| \int^t \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} = \infty,$$

$$\limsup_{t\to\infty} r(t)h^3(t)|h'(t)|^{p-2}\widetilde{L}\big[h(t)\big] \left(\int^t \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}}\right)^2 = 0.$$

If

$$\limsup_{t\to\infty} \int^t \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int_t^{\infty} \Big(c(s) - \widetilde{c}(s)\Big) h^p(s)\,\mathrm{d}s < \frac{1}{q}\left(-\alpha + \sqrt{2\alpha}\right),$$

$$\liminf_{t\to\infty} \int^t \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int_t^{\infty} \Big(c(s) - \widetilde{c}(s)\Big) h^p(s)\,\mathrm{d}s > \frac{1}{q}\left(-\alpha - \sqrt{2\alpha}\right)$$

for some $\alpha > 0$, then equation (1) is nonoscillatory.

Theorem 4. *Let the following conditions hold:*

$$\int^{\infty} r^{1-q}(t)\,\mathrm{d}t < \infty \tag{5}$$

and for some $\alpha > 0$

$$\begin{aligned}
\limsup_{t\to\infty} \left(\int_t^{\infty} r^{1-q}(s)\,\mathrm{d}s\right)^{p-1} \int^t c(s)\,\mathrm{d}s < -\alpha + \alpha^{1/q},\\
\liminf_{t\to\infty} \left(\int_t^{\infty} r^{1-q}(s)\,\mathrm{d}s\right)^{p-1} \int^t c(s)\,\mathrm{d}s > -\alpha - \alpha^{1/q}.
\end{aligned} \tag{6}$$

Then equation (1) is nonoscillatory.