Principal solutions of half-linear differential equation: yet another integral characterization

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(joint work with Simona Fišnarová)

$$\boxed{L[x] := \Bigl(r(t)\Phi_p(x')\Bigr)' + c(t)\Phi_p(x) = 0,} \tag{1}$$

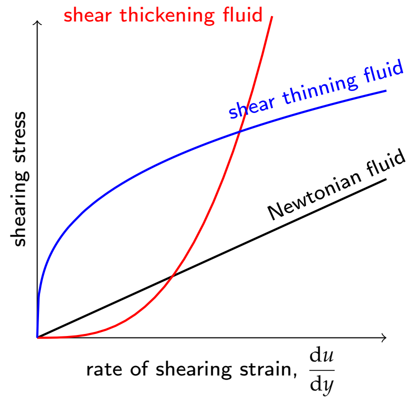
- $\Phi_p(x) = |x|^{p-2}x$, $p > 1$, and r, c are continuous functions, $r(t) > 0$ on the interval $[t_0, \infty)$
- $\Bigl(r(t)\Phi_p(x')\Bigr)'$ is scalar p-Laplacian
- constant multiple of every solution is also a solution
- zeros of nontrivial solution cannot accumulate in finite time
- if t_1, t_2 are consecutive zeros of nontrivial solution, then every other independent solution has zero between t_1 and t_2
- equations can be classified as
  - oscillatory (every solution has infinitely many zeros in a neighborhood of ∞)
  - nonoscillatory (the opposite case, i.e. there exists a solution which is positive in a neighborhood of ∞)

The p-Laplacian is used to describe non-Newtonian fluids.

- Viscosity is a physical characteristic of fluids which measures internal resistance to flow (fluid friction).

  small viscosity: water, ethanol
  large viscosity: glycerol, honey, molten chocolate, molten glass, deep cooled vodka
  extremely large viscosity: solids with amorphous structure, glass

- Viscosity is the slope of the curves in the diagram.



- Suspension of cornstarch in water is shear thickening (dilatant).
- Simply mix cornstarch into the water and continue adding cornstarch until the mixture becomes non-Newtonian (it will be difficult to mix fast, at the end mix slowly and use your bare hands).
- There are many instructions over the Internet (the ratio of water and cornstarch is usually not correct, we had to use more Dr. Oetker's cornstarch).
- There are many funny videos with pool filled with cornstarch on Youtube.

$$L_2[x] := \Big(r(t)x'\Big)' + c(t)x = 0,$$

$$R_2[w] := w' + c(t) + r^{-1}(t)|w|^2 = 0$$

$$\left(u > 0 \text{ and } w = r\frac{u'}{u}\right) \implies R_2[w] = \frac{L_2[u]}{u}$$

Principal solution (at ∞). [Leighton, Morse]
Let u and v be solutions of nonoscillatory equation, positive in neighborhood of infinity. The following conditions are equivalent and each can be used to determine principal solution u (unique up to a constant multiple) and nonprincipal solution v.

(i) $\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0$

(ii) $w_u(t) < w_v(t)$ in a neighborhood of ∞

(iii) $\int^\infty \frac{1}{r(t)u^2(t)}\,\mathrm{d}t = \infty$, $\int^\infty \frac{1}{r(t)v^2(t)}\,\mathrm{d}t < \infty$

$$L[x] := \left(r(t)\Phi_p(x')\right)' + c(t)\Phi_p(x) = 0,$$
$$R[w] := w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0$$

$$\left(u > 0 \text{ and } w = r\Phi\left(\frac{u'}{u}\right)\right) \implies R[w] = \frac{L[u]}{\Phi(u)}$$

Principal solution (at ∞). [Mirzov]

- Wronskian identity which links all three properties in the linear case is not available (Elbert).
- Principal solution: a solution which generates minimal solution of Riccati equation at ∞.

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We are interested in the integral characterization (allows to deduce principality without the independent solution).

- [Mirzov] There exist well defined numbers m_* and m^* (depend on p, $m^* \geq m_*$) such that

$$\int^\infty \frac{1}{r^{q-1}(t)|u(t)|^{m_*}} = \infty \implies u \text{ is principal}$$

$$u \text{ is principal} \implies \int^\infty \frac{1}{r^{q-1}(t)|u(t)|^{m^*}} = \infty$$

- [Došlý, Elbert]

$$p \geq 2: \quad u \text{ is principal} \implies \int^\infty \frac{1}{r(t)u^2(t)|u'(t)|^{p-2}}\,\mathrm{d}t = \infty$$

$$p \in (1,2]: \quad \int^\infty \frac{1}{r(t)u^2(t)|u'(t)|^{p-2}}\,\mathrm{d}t = \infty \implies u \text{ is principal}$$

- [Došlá, Došlý]

$$\int^\infty \frac{1}{r^{q-1}(t)|u(t)|^p} \frac{\Phi_p^{-1}(w_u(t)) - \Phi_p^{-1}(w_v(t))}{w_u(t) - w_v(t)}\,\mathrm{d}t = \infty \iff u \text{ is principal}$$

- [Došlý, Elbert]

$$p \geq 2: \quad u \text{ is principal} \implies \int^{\infty} \frac{1}{r(t)u^2(t)|u'(t)|^{p-2}}\,\mathrm{d}t = \infty$$

$$p \in (1,2]: \quad \int^{\infty} \frac{1}{r(t)u^2(t)|u'(t)|^{p-2}}\,\mathrm{d}t = \infty \implies u \text{ is principal}$$

- [Fišnarová, Mařík] The integral $\int^{\infty} \frac{1}{r(t)u^2(t)|u'(t)|^{p-2}}\,\mathrm{d}t$ can be replaced by

$$I_\alpha := \int^{\infty} \frac{\mathrm{d}t}{r^{\alpha-1}(t)h^{\alpha}(t)|h'(t)|^{(p-1)(\alpha-q)}},$$

where $\alpha \in [q,2]$ if $p \geq 2$ and $\alpha \in [2,q]$ if $p \leq 2$. (q is a conjugate number to p)

Remarks

- For $\alpha = 2$ we get the result of Došlý & Elbert.
- Depending on the value of the limit $\lim_{t\to\infty} r(t)h(t)|h'(t)|^{p-1}$ the convergenece of I_α and the convergence of the integral from Došlý & Elbert may difffer.
- To appear in Nonlinear Analysis TMA.

Consider equation [Došlá & Došlý]

$$\left(\Phi_{3/2}(x')\right)' + \frac{15t^{-3/2}}{(t^9-1)^{1/2}}\Phi_{3/2}(x) = 0, \quad t > 1$$

- If $x > 0$, then $\Phi_{3/2}(x) = \sqrt{x}$.
- The equation is nonoscillatory, every solution x satisfies

$$\int^{\infty} \frac{1}{r(t)x^2(t)|x'(t)|^{p-2}}\,\mathrm{d}t < \infty.$$

- There is no solution u which satisfies $\int^{\infty} \frac{1}{r(t)u^2(t)|u'(t)|^{p-2}}\,\mathrm{d}t = \infty$. Thus the principal solution cannot be detected by the result of Došlý & Elbert.
- $q = 3$
- For the solution $h(t) = 1 - 1/t^9$ we have

$$\int^{\infty} \frac{\mathrm{d}t}{r^{\alpha-1}(t)h^{\alpha}(t)|h'(t)|^{(p-1)(\alpha-q)}} = \int^{\infty} \frac{\mathrm{d}t}{3^{\alpha-3}(1-t^{-9})^{\alpha}t^{15-5\alpha}} \geq \int^{\infty} \frac{\mathrm{d}t}{3^{\alpha-3}t^{15-5\alpha}}$$

and the integral diverges if $\alpha \in [14/5, 3]$. The solution $h(t)$ is principal.

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