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Picone type inequality for half-linear differential operators with anisotropic p-Laplacian

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Half-linear PDE with isotropic p-Laplacian

$$l(u) := \operatorname{div}\Big(a(x)\,\|\nabla u\|^{p-2}\,\nabla u\Big) + c(x)|u|^{p-2}u = 0 \tag{1}$$

$$L(u) := \operatorname{div}\Big(A(x)\,\|\nabla u\|^{p-2}\,\nabla u\Big) + C(x)|u|^{p-2}u = 0 \tag{2}$$

- $\Omega \in \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ for which the Gauss–Ostrogradskii divergence theorem holds,
- $a \in C^1(\overline{\Omega}, \mathbb{R}^+)$ and $A \in C^1(\overline{\Omega}, \mathbb{R}^+)$ are *scalar functions*
- the domain $D_l(\Omega)$ of operator l is the set of all functions $u(x) \in C^1(\overline{\Omega})$ such that $a(x)\,\|\nabla u\|^{p-2}\,\nabla u \in C^1(\Omega) \cap C(\overline{\Omega})$. In a similar way we define domain $D_L(\Omega)$ of the operator L.

Picone idenity for isotropic p-Laplacian

Picone identity can be used to derive important results in comparison and oscillation theory of related differential equations, but can be also used to get uniqueness or nonexistence results, monotonicity of eigenvalue in domain, results for various eigenvalue problems and inequalities and other results. The following version of Picone identity is due to *J. Jaroš, T. Kusano, N. Yoshida, A Picone–type identity and Sturmian comparison and oscillation theorems for a class of half–linear partial differential equations of second order, Nonlinear Anal.* ***40*** *(2000).*

$$\begin{aligned}
&\operatorname{div}\left(\frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}v\,a(x)\,\|\nabla u\|^{p-2}\,\nabla u - |u|^{p-2}u\,A(x)\,\|\nabla v\|^{p-2}\,\nabla v\Big]\right)\\
&\quad= \Big[a(x) - A(x)\Big]\|\nabla u\|^p + \Big[C(x) - c(x)\Big]|u|^p + A(x)Y(u,v)\\
&\qquad+ \frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}v\,l(u) - |u|^{p-2}u\,L(v)\Big],
\end{aligned}$$

where

$$Y(u,v) = \|\nabla u\|^p + (p-1)\left\|\frac{u}{v}\nabla v\right\|^p - p\left\|\frac{u}{v}\nabla v\right\|^{p-2}\left\langle \nabla u, \frac{u}{v}\nabla v\right\rangle \geq 0,$$

and

$$Y(u,v) = 0 \text{ iff } u \text{ is a constant multiple of } v.$$

Half-linear PDE with anisotropic p-Laplacian

- Essentialy like (1) and (2), but a and A are *elliptic matrices*.
- By $\Lambda_{\max}(x)$ and $\Lambda_{\min}(x)$ we denote the maximal and minimal eigenvalues of the matrix $A(x)$ and similarly $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ denote the maximal and minimal eigenvalues of the matrix $a(x)$.

Picone inequality for anisotropic p-Laplacian

In the proof of the Picone identity we are required to draw out the coefficients a and A from norms and scalar products. This is easy with scalar functions but impossible with matrix product. Is there any chance to derive suitable repacement for Picone identity in this case?
The answer is yes – but we have to lighten the right hand side by replacing terms involving a and A in the isotropic version with expressions involving eigenvalues of the matrices a and A. Skipping technical details, we get the following result.

Theorem 1. *Let $u \in D_l(\Omega)$ and $v \in D_L(\Omega)$, $v \neq 0$ on Ω. Denote*

$$K(x) = \begin{cases} \left(\dfrac{\Lambda_{\max}(x)}{\Lambda_{\min}(x)}\right)^{p-1}\Lambda_{\max}(x) & \text{for } p > 2\\ \Lambda_{\max}(x) & \text{for } 1 < p \leq 2. \end{cases}$$

The inequality

$$\begin{aligned}
&\operatorname{div}\left(\frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}v\,a(x)\,\|\nabla u\|^{p-2}\,\nabla u - |u|^{p-2}u\,A(x)\,\|\nabla v\|^{p-2}\,\nabla v\Big]\right)\\
&\quad\geq \Big[\lambda_{\min}(x) - K(x)\Big]\|\nabla u\|^p + \Big[C(x) - c(x)\Big]|u|^p\\
&\qquad+ \frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}v\,l(u) - |u|^{p-2}u\,L(v)\Big]
\end{aligned}$$

holds for every $x \in \Omega$. The inequality can be replaced by equality if and only if the following conditions hold

(i) $\nabla u(x)$ is an eigenvector of the matrix $a(x)$ associated with the eigenvalue $\lambda_{\min}(x)$,

(ii) $\nabla v(x)$ is an eigenvector of the matrix $A(x)$ associated with the eigenvalue $\Lambda_{\max}(x)$,

(iii) if $p > 2$ then $\Lambda_{\max}(x) = \Lambda_{\min}(x)$,

(iv) $u(x)$ is a constant multiple of $v(x)$.

Practical usage of Picone identity

The operators l and L are half-linear operators with anisotropic p-Laplacian

Theorem 2. *Let u be a nontrivial solution of $l(u) = 0$ such that $u = 0$ on $\partial\Omega$ and let*

$$\int_\Omega \Big[(\lambda_{\min}(x) - K(x))\,\|\nabla u\|^p + (C(x) - c(x))|u|^p\Big]\,dx \geq 0.$$

Then every solution of $L(v) = 0$ has a zero in $\overline{\Omega}$.

Corollary 1. *Let u be a nontrivial solution of $l(u) = 0$ such that $u = 0$ on $\partial\Omega$.*

(i) If $\lambda_{\min}(x) \geq K(x)$ and $C(x) \geq c(x)$ in Ω, then every solution of $L(v) = 0$ has a zero in $\overline{\Omega}$.

(ii) If $\lambda_{\min}(x) = \lambda_{\max}(x)$, then every solution of $l(u) = 0$ has a zero in $\overline{\Omega}$.

Where to find details

Simona Fišnarová, Robert Mařík: *Generalized Picone and Riccati inequalities for half-linear differential operators with arbitrary elliptic matrices* Electronic J. Diferential Equations 2010 (2010), no. 111, pp. 1–13.