Conjugacy criteria for half-linear ODE in theory of PDE with generalized p-Laplacian and mixed powers

Robert Mařík

Dpt. of Mathematics
Mendel University
Brno, CZ

Faculty of Forestry and Wood Technology

$$
\begin{aligned}
\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2}\,\nabla y \right\rangle \\
+ c(x)|y|^{p-2}y + \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y = e(x),
\end{aligned} \tag{E}
$$

- $x = (x_1, \dots, x_n)_{i=1}^n \in \mathbb{R}^n$, $p > 1$, $p_i > 1$,
- $A(x)$ is elliptic $n \times n$ matrix with differentiable components, $c(x)$ and $c_i(x)$ are Hölder continuous functions, $\vec{b}(x) = \big(b_1(x), \dots, b_n(x)\big)$ is continuous n-vector function,
- $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)_{i=1}^n$ and $\operatorname{div} = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}$ is are the usual nabla and divergence operators,
- q is a conjugate number to the number p, i.e., $q = \frac{p}{p-1}$,
- $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$, $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$, $\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\} = \lambda_{\max}$ is the spectral norm
- **solution** of (E) in $\Omega \subseteq \mathbb{R}^n$ is a differentiable function $u(x)$ such that $A(x)\|\nabla u(x)\|^{p-2}\nabla u(x)$ is also differentiable and u satisfies (E) in Ω
- $S(a) = \{x \in \mathbb{R}^n : \|x\| = a\}$,
  $\Omega(a) = \{x \in \mathbb{R}^n : a \leq \|x\|\}$,
  $\Omega(a, b) = \{x \in \mathbb{R}^n : a \leq \|x\| \leq b\}$

$$u'' + c(x)u = 0 \tag{1}$$

- Equation is oscillatory if $c(x)$ is large enough. Many oscillation criteria are expressed in terms of the integral $\int^{\infty} c(x)\,\mathrm{d}x$ (Hille and Nehari type)
- There are oscillation criteria which can detect oscillation even if $\int^{\infty} c(x)\,\mathrm{d}x$ is extremly small. These criteria are often in fact series of conjugacy criteria.

$$(p(t)u')' + c(t)u + \sum_{i=1}^{m} c_i(t)|u|^{\alpha_i} \operatorname{sgn} u = e(t) \tag{2}$$

where $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$.

Theorem A (Sun,Wong (2007)). *If for any $T \geq 0$ there exists a_1, b_1, a_2, b_2 such that $T \leq a_1 < b_1 \leq a_2 < b_2$ and*

$$\begin{cases} c_i(t) \geq 0 & t \in [a_1, b_1] \cup [a_2, b_2], \ i = 1, 2, \dots, n \\ e(x) \leq 0 & t \in [a_1, b_1] \\ e(x) \geq 0 & t \in [a_2, b_2] \end{cases}$$

and there exists a continuously differentiable function $u(t)$ satisfying $u(a_i) = u(b_i) = 0$, $u(t) \neq 0$ on (a_i, b_i) and

$$\int_{a_i}^{b_i} \Big\{ p(t)u'^2(t) - Q(t)u^2(t) \Big\} \,\mathrm{d}t \leq 0 \tag{3}$$

for $i = 1, 2$, where

$$Q(t) = k_0 |e(t)|^{\eta_0} \prod_{i=1}^{m} \Big(c_i^{\eta_i}(t) \Big) + c(t),$$

$k_0 = \prod_{i=0}^{m} \eta_i^{-\eta_i}$ *and η_i, $i = 0, \dots, n$ are positive constants satisfying* $\sum_{i=1}^{m} \alpha_i \eta_i = 1$ *and* $\sum_{i=0}^{m} \eta_i = 1$, *then all solutions of (2) are oscillatory.*

$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2}\,\nabla y \right\rangle + c(x)|y|^{p-2}y + \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y = e(x), \tag{E}$$

Detection of oscillation from ODE

Theorem B (O. Došlý (2001)). *Equation*

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \tag{4}$$

is oscillatory, if the ordinary differential equation

$$\left(r^{n-1}|u'|^{p-2}u'\right)' + r^{n-1}\left(\frac{1}{\omega_n r^{n-1}}\int_{S(r)} c(x)\,\mathrm{d}x\right)|u|^{p-2}u = 0 \tag{5}$$

is oscillatory. The number ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for $A(x) = a(\|x\|)I$, $a(\cdot)$ differentiable).

Our aim

- Extend method used in Theorem A to (E). Derive a general result, like Theorem B.
- Derive a result which does depend on more general expression, than the mean value of $c(x)$ over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2009) excluded the possibility $p_i > p$ for every i).

Faculty of Forestry and Wood Technology

$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2}\,\nabla y \right\rangle + \boldsymbol{c(x)|y|^{p-2}y} + \boldsymbol{\sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y} = \boldsymbol{e(x)}, \tag{E}$$

Modus operandi

- Get rid of terms $\sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y$ and $e(x)$ (join with $c(x)|y|^{p-2}y$) and convert the problem into
$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2}\,\nabla y \right\rangle + \boldsymbol{C(x)|y|^{p-2}y} = 0.$$
- Derive Riccati type inequality in n variables.
- Derive Riccati type inequality in 1 variable.
- Use this inequality as a tool which transforms results from ODE to PDE.

Using generalized AG inequality $\sum \alpha_i \geq \prod \left(\frac{\alpha_i}{\eta_i}\right)^{\eta_i}$, if $\alpha_i \geq 0$, $\eta_i > 0$ and $\sum \eta_i = 1$ we eliminate the right-hand side and terms with mixed powers.

Lemma 1. *Let either $y > 0$ and $e(x) \leq 0$ or $y < 0$ and $e(x) \geq 0$. Let $\eta_i > 0$ be numbers satisfying $\sum_{i=0}^{m} \eta_i = 1$ and $\eta_0 + \sum_{i=1}^{m} p_i \eta_i = p$ and let $c_i(x) \geq 0$ for every i. Then*

$$\frac{1}{|y|^{p-2}y}\left(-e(x) + \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y\right) \geq C_1(x),$$

where

$$C_1(x) := \left|\frac{e(x)}{\eta_0}\right|^{\eta_0} \prod_{i=1}^{m} \left(\frac{c_i(x)}{\eta_i}\right)^{\eta_i}. \tag{6}$$

Remark: The numbers η_i from Lemma 1 exist, if $p_i > p$ for some i.

Lemma 2. *Suppose $c_i(x) \geq 0$. Let $\eta_i > 0$ be numbers satisfying $\sum_{i=1}^{m} \eta_i = 1$ and $\sum_{i=1}^{m} p_i \eta_i = p$. Then*

$$\frac{1}{|y|^{p-2}y} \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y \geq C_2(x),$$

where

$$C_2(x) := \prod_{i=1}^{m} \left(\frac{c_i(x)}{\eta_i}\right)^{\eta_i} \tag{7}$$

Remark: The numbers η_i from Lemma 2 exist iff $p_i > p$ for some i and $p_j < p$ for some j.

Lemma 3. *Let y be a solution of (E) which does not have zero on Ω. Suppose that there exists a function $C(x)$ such that*

$$C(x) \leq c(x) + \sum_{i=1}^{m} c_i(x)|y|^{p_i - p} - \frac{e(x)}{|y|^{p-2}y}$$

Denote $\vec{w}(x) = A(x)\dfrac{\|\nabla y\|^{p-2}\nabla y}{|y|^{p-2}y}$. The function $\vec{w}(x)$ is well defined on Ω and satisfies the inequality

$$\operatorname{div}\vec{w} + (p-1)\Lambda(x)\,\|\vec{w}\|^q + \left\langle \vec{w}, A^{-1}(x)\vec{b}(x) \right\rangle + C(x) \leq 0 \tag{8}$$

where

$$\Lambda(x) = \begin{cases} \lambda_{\max}^{1-q}(x) & 1 < p \leq 2, \\ \lambda_{\min}\lambda_{\max}^{-q}(x) & p > 2. \end{cases} \tag{9}$$

Lemma 4. *Let (8) hold. Let $l > 1$, $l^* = \dfrac{l}{l-1}$ be two mutually conjugate numbers and $\alpha \in C^1(\Omega, \mathbb{R}^+)$ be a smooth function positive on Ω. Then*

$$\operatorname{div}(\alpha(x)\vec{w}) + (p-1)\frac{\Lambda(x)\alpha^{1-q}(x)}{l^*}\,\|\alpha(x)\vec{w}\|^q - \frac{l^{p-1}\alpha(x)}{p^p\Lambda^{p-1}(x)}\left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p + \alpha(x)C(x) \leq 0$$

holds on Ω. If $\left\| A^{-1}\vec{b} - \dfrac{\nabla\alpha}{\alpha} \right\| \equiv 0$ on Ω, then this inequality holds with $l^* = 1$.

Lemma 5. *Let the n-vector function $\vec{w}_0$ satisfy inequality*

$$\operatorname{div}\vec{w}_0 + \overline{C}(x) + (p-1)\overline{\Lambda}(x)\,\|\vec{w}_0\|^q \leq 0$$

on $\Omega(a,b)$. Denote $\widetilde{C}(r) = \int_{S(r)} \overline{C}(x)\,\mathrm{d}\sigma$ and $\widetilde{R}(r) = \int_{S(r)} \overline{\Lambda}^{1-p}\,\mathrm{d}\sigma$. Then the half-linear ordinary differential equation

$$\left(\widetilde{R}(r)|u'|^{p-2}u\right)' + \widetilde{C}(r)|u|^{p-2}u = 0, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}r}$$

is disconjugate on $[a,b]$ and it possesses solution which has no zero on $[a,b]$.

Theorem 1 ($\alpha \equiv 1$ in Lemma 4.). *Let $l > 1$. Let $l^* = 1$ if $\left\|\vec{b}\right\| \equiv 0$ and $l^* = \dfrac{l}{l-1}$ otherwise. Further, let $c_i(x) \geq 0$ for every i. Denote*

$$\widetilde{R}(r) = (l^*)^{p-1}\int_{S(r)} \Lambda^{1-p}(x)\,\mathrm{d}\sigma$$

and

$$\widetilde{C}(r) = \int_{S(r)} c(x) + C_1(x) - \frac{l^{p-1}}{p^p\Lambda^{p-1}(x)}\left\|A^{-1}(x)\vec{b}(x)\right\|^p\,\mathrm{d}\sigma\,,$$

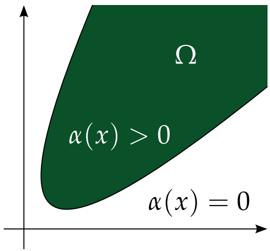
where $\Lambda(x)$ is defined by (9) and $C_1(x)$ is defined by (6).
Suppose that the equation

$$\left(\widetilde{R}(r)|u'|^{p-2}u'\right)' + \widetilde{C}(r)|u|^{p-2}u = 0$$

has conjugate points on $[a,b]$.
If $e(x) \leq 0$ on $\Omega(a,b)$, then equation (E) has no positive solution on $\Omega(a,b)$.
If $e(x) \geq 0$ on $\Omega(a,b)$, then equation (E) has no negative solution on $\Omega(a,b)$.

With general α we can get rid of "bad" parts.

If $\alpha \equiv 0$ outside Ω, then the integral $\int_{\Omega(a)} \alpha(x)c(x)\,\mathrm{d}x$ depends in fact on the restriction $c(x)\Big|_{x\in\Omega}$ only.



Theorem 2 (non-radial variant of Theorem 1). *Let $l > 1$ and let $\Omega \subset \Omega(a,b)$ be an open domain with piecewise smooth boundary such that $\operatorname{meas}(\Omega \cap S(r)) \neq 0$ for every $r \in [a,b]$. Let $c_i(x) \geq 0$ on Ω for every i and let $\alpha(x)$ be a function which is positive and continuously differentiable on Ω and vanishes on the boundary and outside Ω. Let $l^* = 1$ if $\left\| A^{-1}\vec{b} - \frac{\nabla\alpha}{\alpha} \right\| \equiv 0$ on Ω and $l^* = \frac{l}{l-1}$ otherwise. In the former case suppose also that the integral*

$$\int_{S(r)} \frac{\alpha(x)}{\Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p \mathrm{d}\sigma$$

which may have singularity on $\partial\Omega$ if $\Omega \neq \Omega(a,b)$ is convergent for every $r \in [a,b]$. Denote

$$\widetilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \alpha(x)\Lambda^{1-p}(x)\,\mathrm{d}\sigma$$

and

$$\widetilde{C}(r) = \int_{S(r)} \boldsymbol{\alpha(x)} \left(c(x) + C_1(x) - \frac{l^{p-1}}{p^p\Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p \right) \mathrm{d}\sigma\,,$$

where $\Lambda(x)$ is defined by (9) and $C_1(x)$ is defined by (6) and suppose that equation

$$\left(\widetilde{R}(r)|u'|^{p-2}u'\right)' + \widetilde{C}(r)|u|^{p-2}u = 0$$

has conjugate points on $[a,b]$.
If $e(x) \leq 0$ on $\Omega(a,b)$, then equation (E) has no positive solution on $\Omega(a,b)$.
If $e(x) \geq 0$ on $\Omega(a,b)$, then equation (E) has no negative solution on $\Omega(a,b)$.

Faculty of Forestry and Wood Technology

Theorem 3. *Let l, Ω, $\alpha(x)$, $\Lambda(x)$ and $\widetilde{R}(r)$ be defined as in Theorem 2 and let $c_i(x) \geq 0$ and* $\boldsymbol{e(x) \equiv 0}$ *on $\Omega(a,b)$. Denote*

$$\widetilde{C}(r) = \int_{S(r)} \alpha(x) \left(c(x) + C_2(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \right) \mathrm{d}\sigma \,,$$

where $C_2(x)$ is defined by (7). If the equation

$$\left(\widetilde{R}(r)|u'|^{p-2}u'\right)' + \widetilde{C}(r)|u|^{p-2}u = 0$$

has conjugate points on $[a,b]$, then every solution of equation (E) has zero on $\Omega(a,b)$.

Similar theorems can be derived also for estimates of terms with mixed powers based on different methods than AG inequality (see R. M., Nonlinear Analysis TMA 73 (2010)).

Conjugacy criteria for half-linear ODE in theory of PDE with generalized p-Laplacian and mixed powers

Robert Mařík

Dpt. of Mathematics
Mendel University
Brno, CZ

$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right)+\left\langle\vec b(x),\|\nabla y\|^{p-2}\,\nabla y\right\rangle + c(x)|y|^{p-2}y+\sum_{i=1}^m c_i(x)|y|^{p_i-2}y=e(x), \qquad \text{(E)}$$

- $x=(x_1,\dots,x_n)_{i=1}^n\in\mathbb{R}^n$, $p>1$, $p_i>1$,
- $A(x)$ is elliptic $n\times n$ matrix with differentiable components, $c(x)$ and $c_i(x)$ are Hölder continuous functions, $\vec b(x)=(b_1(x),\dots,b_n(x))$ is continuous n-vector function,
- $\nabla=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)_{i=1}^n$ and $\operatorname{div}=\frac{\partial}{\partial x_1}+\dots+\frac{\partial}{\partial x_n}$ is are the usual nabla and divergence operators,
- q is a conjugate number to the number p, i.e., $q=\frac{p}{p-1}$,
- $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^n$, $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$, $\|A\| = \sup\{\|Ax\|: x\in\mathbb{R}^n \text{ with } \|x\|=1\}=\lambda_{\max}$ is the spectral norm
- **solution** of (E) in $\Omega\subseteq\mathbb{R}^n$ is a differentiable function $u(x)$ such that $A(x)\|\nabla u(x)\|^{p-2}\nabla u(x)$ is also differentiable and u satisfies (E) in Ω
- $S(a)=\{x\in\mathbb{R}^n:\|x\|=a\}$,
  $\Omega(a)=\{x\in\mathbb{R}^n: a\le\|x\|\}$,
  $\Omega(a,b)=\{x\in\mathbb{R}^n: a\le\|x\|\le b\}$

Concept of oscillation for ODE

$$u''+c(x)u=0 \qquad (1)$$

- Equation is oscillatory if $c(x)$ is large enough. Many oscillation criteria are expressed in terms of the integral $\int^\infty c(x)\,\mathrm{d}x$ (Hille and Nehari type)
- There are oscillation criteria which can detect oscillation even if $\int^\infty c(x)\,\mathrm{d}x$ is extremly small. These criteria are often in fact series of conjugacy criteria.

Equation with mixed powers

$$(p(t)u')'+c(t)u+\sum_{i=1}^m c_i(t)|u|^{\alpha_i}\operatorname{sgn}u=e(t) \qquad (2)$$

where $\alpha_1>\dots>\alpha_m>1>\alpha_{m+1}>\dots>\alpha_n>0$.

Theorem A (Sun,Wong (2007))**.** *If for any $T\ge 0$ there exists a_1, b_1, a_2, b_2 such that $T\le a_1<b_1\le a_2<b_2$ and*

$$\begin{cases} c_i(t)\ge 0 & t\in[a_1,b_1]\cup[a_2,b_2],\ i=1,2,\dots,n\\ e(x)\le 0 & t\in[a_1,b_1]\\ e(x)\ge 0 & t\in[a_2,b_2]\end{cases}$$

and there exists a continuously differentiable function $u(t)$ satisfying $u(a_i)=u(b_i)=0$, $u(t)\ne 0$ on (a_i,b_i) and

$$\int_{a_i}^{b_i}\left\{p(t)u'^2(t)-Q(t)u^2(t)\right\}\,\mathrm{d}t\le 0 \qquad (3)$$

for $i=1,2$, where

$$Q(t)=k_0|e(t)|^{\eta_0}\prod_{i=1}^m\left(c_i^{\eta_i}(t)\right)+c(t),$$

$k_0=\prod_{i=0}^m\eta_i^{-\eta_i}$ and η_i, $i=0,\dots,n$ are positive constants satisfying $\sum_{i=1}^m\alpha_i\eta_i=1$ and $\sum_{i=0}^m\eta_i=1$, then all solutions of (2) are oscillatory.

$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right)+\left\langle\vec b(x),\|\nabla y\|^{p-2}\,\nabla y\right\rangle + c(x)|y|^{p-2}y+\sum_{i=1}^m c_i(x)|y|^{p_i-2}y=e(x), \qquad \text{(E)}$$

Detection of oscillation from ODE

Theorem B (O. Došlý (2001))**.** *Equation*

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u)+c(x)|u|^{p-2}u=0 \qquad (4)$$

is oscillatory, if the ordinary differential equation

$$\left(r^{n-1}|u'|^{p-2}u'\right)'+r^{n-1}\left(\frac{1}{\omega_n r^{n-1}}\int_{S(r)}c(x)\,\mathrm{d}x\right)|u|^{p-2}u=0 \qquad (5)$$

is oscillatory. The number ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for $A(x)=a(\|x\|)I$, $a(\cdot)$ differentiable).

Our aim

- Extend method used in Theorem A to (E). Derive a general result, like Theorem B.
- Derive a result which does depend on more general expression, than the mean value of $c(x)$ over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2009) excluded the possibility $p_i>p$ for every i).

$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right)+\left\langle\vec b(x),\|\nabla y\|^{p-2}\,\nabla y\right\rangle + \boldsymbol{c(x)|y|^{p-2}y}+\boldsymbol{\sum_{i=1}^m c_i(x)|y|^{p_i-2}y=e(x)}, \qquad \text{(E)}$$

Modus operandi

- Get rid of terms $\sum_{i=1}^m c_i(x)|y|^{p_i-2}y$ and $e(x)$ (join with $c(x)|y|^{p-2}y$) and convert the problem into
  $$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right)+\left\langle\vec b(x),\|\nabla y\|^{p-2}\,\nabla y\right\rangle+\boldsymbol{C(x)|y|^{p-2}y}=0.$$
- Derive Riccati type inequality in n variables.
- Derive Riccati type inequality in 1 variable.
- Use this inequality as a tool which transforms results from ODE to PDE.

Using generalized AG inequality $\sum\alpha_i\ge\prod\left(\frac{\alpha_i}{\eta_i}\right)^{\eta_i}$, if $\alpha_i\ge 0$, $\eta_i>0$ and $\sum\eta_i=1$ we eliminate the right-hand side and terms with mixed powers.

Lemma 1. *Let either $y>0$ and $e(x)\le 0$ or $y<0$ and $e(x)\ge 0$. Let $\eta_i>0$ be numbers satisfying $\sum_{i=0}^m\eta_i=1$ and $\eta_0+\sum_{i=1}^m p_i\eta_i=p$ and let $c_i(x)\ge 0$ for every i. Then*

$$\frac{1}{|y|^{p-2}y}\left(-e(x)+\sum_{i=1}^m c_i(x)|y|^{p_i-2}y\right)\ge C_1(x),$$

where

$$C_1(x):=\left|\frac{e(x)}{\eta_0}\right|^{\eta_0}\prod_{i=1}^m\left(\frac{c_i(x)}{\eta_i}\right)^{\eta_i}. \qquad (6)$$

Remark: The numbers η_i from Lemma 1 exist, if $p_i>p$ for some i.

Lemma 2. *Suppose $c_i(x)\ge 0$. Let $\eta_i>0$ be numbers satisfying $\sum_{i=1}^m\eta_i=1$ and $\sum_{i=1}^m p_i\eta_i=p$. Then*

$$\frac{1}{|y|^{p-2}y}\sum_{i=1}^m c_i(x)|y|^{p_i-2}y\ge C_2(x),$$

where

$$C_2(x):=\prod_{i=1}^m\left(\frac{c_i(x)}{\eta_i}\right)^{\eta_i} \qquad (7)$$

Remark: The numbers η_i from Lemma 2 exist iff $p_i>p$ for some i and $p_j<p$ for some j.

Lemma 3. *Let y be a solution of (E) which does not have zero on Ω. Suppose that there exists a function $C(x)$ such that*

$$C(x)\le c(x)+\sum_{i=1}^m c_i(x)|y|^{p_i-p}-\frac{e(x)}{|y|^{p-2}y}$$

Denote $\vec w(x)=A(x)\frac{\|\nabla y\|^{p-2}\nabla y}{|y|^{p-2}y}$. The function $\vec w(x)$ is well defined on Ω and satisfies the inequality

$$\operatorname{div}\vec w+(p-1)\Lambda(x)\,\|\vec w\|^q+\left\langle\vec w,A^{-1}(x)\vec b(x)\right\rangle+C(x)\le 0 \qquad (8)$$

where

$$\Lambda(x)=\begin{cases}\lambda_{\max}^{1-q}(x) & 1<p\le 2,\\ \lambda_{\min}\lambda_{\max}^{-q}(x) & p>2.\end{cases} \qquad (9)$$

Lemma 4. *Let (8) hold. Let $l>1$, $l^*=\frac{l}{l-1}$ be two mutually conjugate numbers and $\alpha\in C^1(\Omega,\mathbb{R}^+)$ be a smooth function positive on Ω. Then*

$$\operatorname{div}(\alpha(x)\vec w)+(p-1)\frac{\Lambda(x)\alpha^{1-q}(x)}{l^*}\|\alpha(x)\vec w\|^q-\frac{l^{p-1}\alpha(x)}{p^p\Lambda^{p-1}(x)}\left\|A^{-1}(x)\vec b(x)-\frac{\nabla\alpha(x)}{\alpha(x)}\right\|^p+\alpha(x)C(x)\le 0$$

holds on Ω. If $\left\|A^{-1}\vec b-\frac{\nabla\alpha}{\alpha}\right\|\equiv 0$ on Ω, then this inequality holds with $l^*=1$.

Lemma 5. *Let the n-vector function $\vec w_0$ satisfy inequality*

$$\operatorname{div}\vec w_0+\overline{C}(x)+(p-1)\overline{\Lambda}(x)\,\|\vec w_0\|^q\le 0$$

on $\Omega(a,b)$. Denote $\widetilde C(r)=\int_{S(r)}\overline{C}(x)\,\mathrm{d}\sigma$ and $\widetilde R(r)=\int_{S(r)}\overline{\Lambda}^{1-p}\,\mathrm{d}\sigma$. Then the half-linear ordinary differential equation

$$\left(\widetilde R(r)|u'|^{p-2}u'\right)'+\widetilde C(r)|u|^{p-2}u=0, \qquad {}'=\frac{\mathrm{d}}{\mathrm{d}r}$$

is disconjugate on $[a,b]$ and it possesses solution which has no zero on $[a,b]$.

Theorem 1 ($\alpha\equiv 1$ in Lemma 4.)**.** *Let $l>1$. Let $l^*=1$ if $\left\|\vec b\right\|\equiv 0$ and $l^*=\frac{l}{l-1}$ otherwise. Further, let $c_i(x)\ge 0$ for every i. Denote*

$$\widetilde R(r)=(l^*)^{p-1}\int_{S(r)}\Lambda^{1-p}(x)\,\mathrm{d}\sigma$$

and

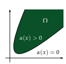
$$\widetilde C(r)=\int_{S(r)}c(x)+C_1(x)-\frac{l^{p-1}}{p^p\Lambda^{p-1}(x)}\left\|A^{-1}(x)\vec b(x)\right\|^p\,\mathrm{d}\sigma,$$

where $\Lambda(x)$ is defined by (9) and $C_1(x)$ is defined by (6).
Suppose that the equation

$$\left(\widetilde R(r)|u'|^{p-2}u'\right)'+\widetilde C(r)|u|^{p-2}u=0$$

has conjugate points on $[a,b]$.
If $e(x)\le 0$ on $\Omega(a,b)$, then equation (E) has no positive solution on $\Omega(a,b)$.
If $e(x)\ge 0$ on $\Omega(a,b)$, then equation (E) has no negative solution on $\Omega(a,b)$.

With general α we can get rid of "bad" parts.
If $\alpha\equiv 0$ outside Ω, then the integral $\int_{\Omega(a)}\alpha(x)c(x)\,\mathrm{d}x$ depends in fact on the restriction $c(x)\big|_{x\in\Omega}$ only.



Theorem 2 (non-radial variant of Theorem 1)**.** *Let $l>1$ and let $\Omega\subset\Omega(a,b)$ be an open domain with piecewise smooth boundary such that $\operatorname{meas}(\Omega\cap S(r))\ne 0$ for every $r\in[a,b]$. Let $c_i(x)\ge 0$ on Ω for every i and let $\alpha(x)$ be a function which is positive and continuously differentiable on Ω and vanishes on the boundary and outside Ω. Let $l^*=1$ if $\left\|A^{-1}\vec b-\frac{\nabla\alpha}{\alpha}\right\|\equiv 0$ on Ω and $l^*=\frac{l}{l-1}$ otherwise. In the former case suppose also that the integral*

$$\int_{S(r)}\frac{\alpha(x)}{\Lambda^{p-1}(x)}\left\|A^{-1}(x)\vec b(x)-\frac{\nabla\alpha(x)}{\alpha(x)}\right\|^p\,\mathrm{d}\sigma$$

which may have singularity on $\partial\Omega$ if $\Omega\ne\Omega(a,b)$ is convergent for every $r\in[a,b]$. Denote

$$\widetilde R(r)=(l^*)^{p-1}\int_{S(r)}\alpha(x)\Lambda^{1-p}(x)\,\mathrm{d}\sigma$$

and

$$\widetilde C(r)=\int_{S(r)}\boldsymbol{\alpha(x)}\left(c(x)+C_1(x)-\frac{l^{p-1}}{p^p\Lambda^{p-1}(x)}\left\|A^{-1}(x)\vec b(x)-\frac{\nabla\alpha(x)}{\alpha(x)}\right\|^p\right)\mathrm{d}\sigma,$$

where $\Lambda(x)$ is defined by (9) and $C_1(x)$ is defined by (6) and suppose that equation

$$\left(\widetilde R(r)|u'|^{p-2}u'\right)'+\widetilde C(r)|u|^{p-2}u=0$$

has conjugate points on $[a,b]$.
If $e(x)\le 0$ on $\Omega(a,b)$, then equation (E) has no positive solution on $\Omega(a,b)$.
If $e(x)\ge 0$ on $\Omega(a,b)$, then equation (E) has no negative solution on $\Omega(a,b)$.

Theorem 3. *Let l, Ω, $\alpha(x)$, $\Lambda(x)$ and $\widetilde R(r)$ be defined as in Theorem 2 and let $c_i(x)\ge 0$ and* $\boldsymbol{e(x)\equiv 0}$ *on $\Omega(a,b)$. Denote*

$$\widetilde C(r)=\int_{S(r)}\alpha(x)\left(c(x)+C_2(x)-\frac{l^{p-1}}{p^p\Lambda^{p-1}(x)}\left\|A^{-1}(x)\vec b(x)-\frac{\nabla\alpha(x)}{\alpha(x)}\right\|^p\right)\mathrm{d}\sigma,$$

where $C_2(x)$ is defined by (7). If the equation

$$\left(\widetilde R(r)|u'|^{p-2}u'\right)'+\widetilde C(r)|u|^{p-2}u=0$$

has conjugate points on $[a,b]$, then every solution of equation (E) has zero on $\Omega(a,b)$.

Similar theorems can be derived also for estimates of terms with mixed powers based on different methods than AG inequality (see R. M., Nonlinear Analysis TMA 73 (2010)).