Oscillation of PDE with p-Laplacian

Robert Mařík, Mendel University, Brno, CZ

Mendel University of Agriculture and Forestry in Brno

Faculty of Forestry and Wood Technology

Elliptic half-linear PDE

$$\boxed{\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+\Big\langle \vec b(x),\|\nabla u\|^{p-2}\nabla u\Big\rangle + c(x)|u|^{p-2}u=0} \tag{E}$$

- $p>1$, $x=(x_1,\dots,x_n)\in\mathbb{R}^n$,
- $A(x)$ is elliptic $n\times n$ matrix with differentiable components,
- $c(x)$ is Hölder continuous function,
- $\vec b(x)=\big(b_1(x),\dots,b_n(x)\big)$ is continuous n-vector function,
- q is a conjugate number to the number p, i.e., $q=\dfrac{p}{p-1}$,
- $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^n$, $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$,
- $\Omega(a,b)=\{x\in\mathbb{R}^n: a\le\|x\|\le b\}$, $\Omega(a)=\{x\in\mathbb{R}^n: a\le\|x\|\}$, $S(a)=\{x\in\mathbb{R}^n:\|x\|=a\}$

Solution

Solution of (E) in $\Omega\subseteq\mathbb{R}^n$ is a differentiable function $u(x)$ such that $A(x)\|\nabla u(x)\|^{p-2}\nabla u(x)$ is also differentiable and u satisfies (E) in Ω.

Concept of oscillation

Equation (E) is said to be *oscillatory* if it possesses no solution $u(x)$ which is positive for large $\|x\|$.

Sketch of typical proof of oscillation criterion

(i) Suppose by contradiction that the PDE is nonoscillatory and possesses eventually positive solution.

(ii) Using transformation $\vec w(x)=A(x)\dfrac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}$ convert positive solutions of

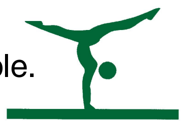
$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+c(x)|u|^{p-2}u=0$$

into

$$\operatorname{div}\vec w + c(x) + \Big\langle \vec w, b(x)A^{-1}(x)+(p-1)\frac{\nabla u(x)}{u(x)}\Big\rangle = 0. \tag{1}$$

(iii) Integrating over spheres and using standard tools and some algebraic gymnastics derive a Riccati type inequality in one variable.

(iv) Proceed as in the proof of oscillation criterion for ODE.



Observations and natural questions

The following natural questions suggest that the approach mentioned in previous paragraph needs to be revisited.

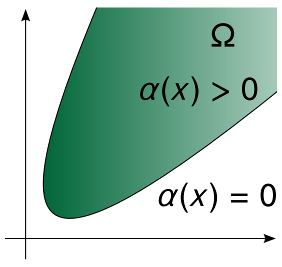
- Oscillation criteria depend in fact on the mean value of $c(x)$ over spheres centered in the origin. Is it possible to detect oscillation in such an extreme case as $\displaystyle\int_{S(r)} c(x)\,\mathrm{d}\sigma = 0$?
- Function $\lambda(r)\ge\max\limits_{x\in S(r)}\lambda_{\max}(x)$ plays a crucial role in the linear case ($p=2$) and $\rho(r)\ge\max\limits_{x\in S(r)}\dfrac{\|A(x)\|^p}{\lambda_{\min}^{p-1}(x)}$ plays similar role in general case $p>1$. Why such a discrepancy appears?

Oscillation deduced from $c(x)|_{x\in\Omega}$

Theorem 1. *Let Ω be unbounded simply connected domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$ and $\mathrm{meas}\big(\Omega\cap S(t)\big)>0$ for $t>1$. Let $k\in(1,\infty)$ real number and $\alpha\in C^1(\Omega\cap\Omega(1),\mathbb{R}^+)\cap C_0(\overline{\Omega},\mathbb{R})$ function satisfying*

(i) $\alpha(x)=0$ *iff* $x\notin\Omega\cap\Omega(1)$,

(ii) $\displaystyle\int_1^\infty\left(\int_{\Omega\cap S(t)}\alpha(x)\,\mathrm{d}\sigma\right)^{1-q}\mathrm{d}t=\infty.$



If

$$\lim_{t\to\infty}\int_{\Omega\cap\Omega(1,t)}\alpha(x)\left(c(x)-\frac{k}{(p\alpha(x))^p}\|\nabla\alpha(x)\|^p\right)\mathrm{d}x=\infty,$$

then the equation

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right)+c(x)|u|^{p-2}u=0$$

is oscillatory.

Application – oscillation in upper halfplane

Corollary 1. *If $n=2$ and*

$$\lim_{t\to\infty}\frac{1}{\ln t}\int_1^t r\int_0^\pi c(r,\phi)\sin^2(\phi)\,\mathrm{d}\phi\,\mathrm{d}r>\frac{\pi}{2}, \tag{2}$$

then the equation $\Delta u + c(x)u=0$ is oscillatory.

The function $c(r,\phi)=\dfrac{A}{r^2}\sin\phi$ satisfies $\displaystyle\int_{S(r)}c(x)\,\mathrm{d}\sigma=0$ and the oscillation cannot be deduced from "usual" oscillation criteria. However, condition (2) can be used for A sufficiently large.

Oscillation deduced from ODE

The simplest method how to detect oscillation is to employ oscillation criteria for ordinary differential equations.

Theorem 2. *For a real number $l>1$ define*

$$a(r)=(l^*)^{p-1}\int_{S(r)}\|A(x)\|^p\lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma,$$

$$b(r)=\int_{S(r)}\left[c(x)-\frac{l^{p-1}}{\lambda_{\min}^{p-1}(x)}\frac{\|\vec b(x)\|^p}{p^p}\right]\mathrm{d}\sigma,$$

$$l^*=1 \text{ if } \|\vec b(x)\|=0 \text{ and } l^*=\frac{l}{l-1} \text{ otherwise.}$$

If $\left(a(r)|u'|^{p-2}u'\right)'+b(r)|u|^{p-2}u=0$ is oscillatory, then (E) is also oscillatory.

Oscillation deduced from ODE ($p\le 2$)

Theorem 3. *Let $1<p\le 2$. For a real number $l>1$ define*

$$\overline{a}(r)=(l^*)^{p-1}\int_{S(r)}\lambda_{\max}(x)\,\mathrm{d}\sigma,$$

$$\overline{b}(r)=\int_{S(r)}\left[c(x)-\frac{l^{p-1}}{p^p}\lambda_{\max}(x)\left\|\vec b(x)A^{-1}(x)\right\|^p\right]\mathrm{d}\sigma,$$

$$l^*=1 \text{ if } \|\vec b(x)\|=0 \text{ and } l^*=\frac{l}{l-1} \text{ otherwise.}$$

If $\left(\overline{a}(r)|u'|^{p-2}u'\right)'+\overline{b}(r)|u|^{p-2}u=0$ is oscillatory, then (E) is also oscillatory.

Application

Corollary 2. *Suppose $\phi, k\in C^1([r_0,\infty),\mathbb{R}^+)$*

(i) *$H(r,r)=0$ and $H(r,s)>0$ for $r>s\ge r_0$,*

(ii) *$\partial H(r,s)\,/\,\partial s$ is continuous and nonpositive,*

(iii)

$$h(r,s):=-\frac{\partial}{\partial s}\Big[H(r,s)k(s)\Big]-H(r,s)k(s)\frac{\phi'(s)}{\phi(s)}$$

$$\int_{r_0}^r H^{1-p}(r,s)|h(r,s)|^p\,\mathrm{d}s<\infty$$

$$\limsup_{r\to\infty}\frac{1}{H(r,r_0)}\int_a^r\left\{H(r,s)k(s)\phi(s)\int_{S(s)}c(x)\,\mathrm{d}\sigma - \frac{1}{p^p}\Big[H(r,s)k(s)\Big]^{1-p}\Theta(s)\phi(s)|h(r,s)|^p\right\}\mathrm{d}s=\infty,$$

where

$$\Theta(s)=\begin{cases}\displaystyle\int_{S(s)}\lambda_{\min}^{1-p}(x)\|A(x)\|^p\,\mathrm{d}\sigma & \text{if } p>2,\\ \displaystyle\int_{S(s)}\lambda_{\max}(x)\,\mathrm{d}\sigma & \text{if } 1<p\le 2.\end{cases}$$

Then $\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+c(x)|u|^{p-2}u=0$ is oscillatory.

Remark. It holds

$$\lambda_{\min}^{1-p}(x)\|A(x)\|^p=\left(\frac{\lambda_{\max}(x)}{\lambda_{\min}(x)}\right)^{p-1}\lambda_{\max}(x)\ge\lambda_{\max}(x)$$

and hence the case $1<p\le 2$ is sharper than the general case $p>1$. Corollary 2 is sharper than corresponding result published by Xu,Xing (2005).

Possible extensions

The above suggested approach can be used whenever the study of an equation can be restricted to the partial differential equation (1) (or the corresponding inequality with $=$ replaced by $\le$). This covers for example the equation with *mixed nonlinearities* such as

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+\Big\langle\vec b(x),\|\nabla u\|^{p-2}\nabla u\Big\rangle+c(x)|u|^{p-2}u+\sum_{i=1}^m c_i(x)|u|^{p_i-2}u=0.$$