Ordinary differential equations in the oscillation theory of partial half-linear differential equation

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$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \Big\rangle + c(x)|u|^{p-2}u = 0 \tag{E}$$

- $x = (x_1, \dots, x_n)_{i=1}^n \in \mathbb{R}^n$,
- $A(x)$ is elliptic $n \times n$ matrix with differentiable components,
- $c(x)$ is Hölder continuous function,
- $\vec{b}(x) = \big(b_1(x), \dots, b_n(x)\big)$ is continuous n-vector function,
- $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)_{i=1}^n$ is the usual nabla operator,
- $\operatorname{div} = \frac{\partial}{\partial x_1} + \dots + \frac{\partial}{\partial x_n}$ is the usual divergence operator,
- q is a conjugate number to the number p, i.e., $q = \frac{p}{p-1}$,
- $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$, $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$,
- **solution** of (E) in $\Omega \subseteq \mathbb{R}^n$ is a differentiable function $u(x)$ such that $A(x)\|\nabla u(x)\|^{p-2}\nabla u(x)$ is also differentiable and u satisfies (E) in Ω

$$\boxed{\mathrm{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u\Big\rangle + c(x)|u|^{p-2}u = 0} \tag{E}$$

Matrix norms

Spectral norm:

$$\|A\| = \sup\left\{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\right\} = \lambda_{\mathrm{max}}$$

Frobenius norm:

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2} \qquad\qquad \|A\| \le \|A\|_F \le \sqrt{n}\|A\|$$

Sets in $\mathbb{R}^n$

$$\Omega(a,b) = \{x \in \mathbb{R}^n : a \le \|x\| \le b\}$$
$$\Omega(a) = \{x \in \mathbb{R}^n : a \le \|x\|\}$$
$$S(a) = \{x \in \mathbb{R}^n : \|x\| = a\}$$

$$\boxed{\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \Big\rangle + c(x)|u|^{p-2}u = 0} \tag{E}$$

Concept of oscillation

Equation (E) is said to be *oscillatory* if it possesses no solution $u(x)$ which is positive for large $\|x\|$.

Detection of oscillation from ODE

Theorem A (O. Došlý (2001)). *Equation*

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \tag{1}$$

is oscillatory, if the ordinary differential equation

$$\left(r^{n-1}|u'|^{p-2}u'\right)' + r^{n-1}\left(\frac{1}{\omega_n r^{n-1}}\int_{S(r)} c(x)\,\mathrm{d}\sigma\right)|u|^{p-2}u = 0 \tag{2}$$

is oscillatory. The number ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for $A(x) = a(\|x\|)I$, $a(\cdot)$ differentiable).

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$$\boxed{\mathrm{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \Big\rangle + c(x)|u|^{p-2}u = 0} \tag{E}$$

Known results (specified for $\vec{b} = \vec{o}$)

Theorem B (Xu (2006)). $\theta \in C^1([r_0,\infty], \mathbb{R}^+)$, $m > 1$, $\lambda \in C([r_0,\infty), \mathbb{R}^+)$. *If*

$$\lim_{r\to\infty} \int_{r_0}^{r} \left[\theta(s)\int_{\boldsymbol{S(s)}} \boldsymbol{c(x)}\, \mathrm{d}\boldsymbol{\sigma} - \lambda(s)\frac{m}{4}\frac{\theta'^2(s)}{\theta(s)}\right] \mathrm{d}s = \infty$$

and

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} \frac{\mathbf{1}}{\boldsymbol{\theta(\|x\|)\lambda(\|x\|)}}\, \mathrm{d}x = \infty, \qquad \textit{where } \lambda(r) \geq \max_{\boldsymbol{x\in S(r)}} \boldsymbol{\lambda}_{\max}\boldsymbol{(x)}$$

then $\boxed{\mathrm{div}\Big(\boldsymbol{A(x)}\nabla u\Big) + \boldsymbol{c(x)}u = 0}$ *is oscillatory.*

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Theorem C (Xu, Xing (2005)). *Suppose* φ, $k \in C^1([r_0,\infty),\mathbb{R}^+)$

(i) $H(r,r)=0$ *and* $H(r,s)>0$ *for* $r>s\geq r_0$,
$\partial H(r,s)\,/\,\partial s$ *is continuous and nonpositive,*

(ii)

$$h(r,s) := -\frac{\partial}{\partial s}\Big[H(r,s)k(s)\Big] - H(r,s)k(s)\frac{\varphi'(s)}{\varphi(s)}$$

$$\int_{r_0}^{r} H^{1-p}(r,s)|h(r,s)|^p\,\mathrm{d}s < \infty$$

$$\limsup_{r\to\infty}\frac{1}{H(r,r_0)}\int_a^r\Big\{H(r,s)k(s)\varphi(s)\int_{S(s)} c(x)\,\mathrm{d}\sigma - \frac{1}{p^p}\big[H(r,s)k(s)\big]^{1-p}\Theta_{Xu}(s)\varphi(s)|h(r,s)|^p\Big\}\,\mathrm{d}s = \infty,$$

where $\quad\Theta_{Xu}(s)=\rho(s)\omega_n s^{n-1}\quad$ *and* $\quad\rho(s)\geq\max\limits_{x\in S(s)}\dfrac{\|A(x)\|_F^p}{\lambda_{\min}^{p-1}(x)}$.

Then $\boxed{\mathrm{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+c(x)|u|^{p-2}u=0}$ *is oscillatory.*

$$\boxed{\operatorname{div}\Bigl(A(x)\|\nabla u\|^{p-2}\nabla u\Bigr) + \Bigl\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \Bigr\rangle + c(x)|u|^{p-2}u = 0} \tag{E}$$

Sketch of proofs of most oscillation criteria

(i) Find in the literature oscillation criterion for ODE which has not been extended to PDE yet :-).

(ii) Using transformation $\vec{w}(x) = A(x)\dfrac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}$ convert positive solutions of

$$\operatorname{div}\Bigl(A(x)\|\nabla u\|^{p-2}\nabla u\Bigr) + c(x)|u|^{p-2}u = 0$$

into

$$\operatorname{div}\vec{w} + c(x) + (p-1)\left\langle \vec{w}, \frac{\nabla u(x)}{u(x)} \right\rangle = 0. \tag{3}$$

(iii) Integrating (3) over spheres and using standard tools (such as eigenvalues, Schwarz and Hölder inequalities) derive a Riccati type inequality which is similar to the inequality from the proof of onedimensional criterion.

(iv) Get contradiction by repeating steps from the proof of the corresponding oscillation criterion for ODE (with necessary modifications).

$$\boxed{\operatorname{div}\Big(\boldsymbol{A(x)}\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u\Big\rangle + \boldsymbol{c(x)}|u|^{p-2}u = 0} \tag{E}$$

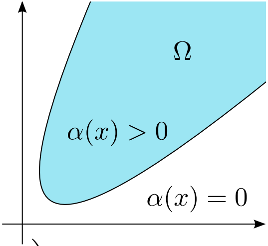
Questions

- Oscillation criteria depend in fact on the mean value of $c(x)$ over spheres centered in the origin. Is it possible to detect oscillation in such an extreme case as $\displaystyle\int_{\boldsymbol{S(||x||)}} \boldsymbol{c(x)}\,\mathrm{d}\boldsymbol{\sigma} = 0$?
- Is it possible to replace all these steps (i)–(v) by method suggested in Theorem A? Is it possible to deduce oscillation of (E) from oscillation of certain ODE?
- Function $\boxed{\lambda(r) \geq \max_{x\in S(r)} \boldsymbol{\lambda}_{\max}\boldsymbol{(x)}}$ plays a crucial role in the linear case and $\boxed{\rho(r) \geq \max_{x\in S(r)} \frac{\|\boldsymbol{A(x)}\|_F^{\boldsymbol{p}}}{\boldsymbol{\lambda}_{\min}^{\boldsymbol{p-1}}\boldsymbol{(x)}}}$ plays similar role if $p > 1$. This phenomenon can be observed also in other oscillation criteria than Theorems B and C. We know that $\rho(r) \geq \lambda(r)$. Why such a disharmony appears?

Theorem 1. *Let Ω be unbounded simply connected domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$ and* meas$\big(\Omega\cap S(t)\big)>0$ *pro* $t>1$. *Let* $k\in(1,\infty)$ *real number and* $\alpha\in C^1(\Omega\cap\Omega(1),\mathbb{R}^+)\cap C_0(\overline{\Omega},\mathbb{R})$ *function satisfying*

(i) $\alpha(x)=0$ *iff* $x\notin\Omega\cap\Omega(1)$,

(ii) $$\int_1^\infty\left(\int_{\Omega\,\cap\,S(t)}\alpha(x)\ \mathrm{d}\sigma\right)^{1-q}\mathrm{d}t=\infty.$$



If

$$\lim_{t\to\infty}\int_{\Omega\,\cap\,\Omega(1,t)}\alpha(x)\left(\boldsymbol{c(x)}-\frac{k}{(p\alpha(x))^p}\,\|\nabla\alpha(x)\|^p\right)\mathrm{d}x=\infty,$$

then the equation

$$\mathrm{div}\left(\|\nabla u\|^{p-2}\nabla u\right)+\boldsymbol{c(x)}|u|^{p-2}u=0$$

is oscillatory.

Oscillation in upper halfplane

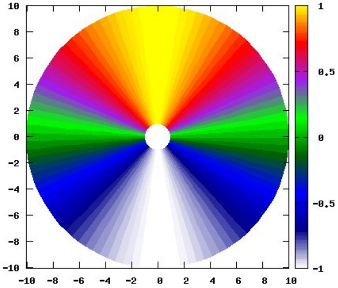
If $n = 2$ and

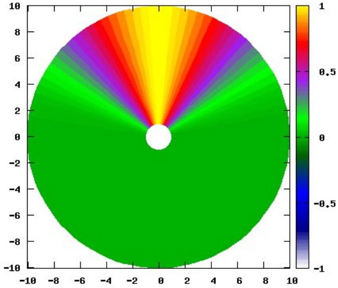
$$\lim_{t\to\infty} \frac{1}{\ln t} \int_1^t r \int_0^\pi c(r,\varphi)\,\sin^2(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}r > \frac{\pi}{2},$$

then the equation $\boxed{\Delta u + c(x)u = 0}$ is oscillatory.

$$c(r,\varphi) = \frac{A}{r^2}\sin\varphi$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\varphi\,\mathrm{d}\varphi = 0 \qquad \frac{2}{\pi}\int_0^{\pi} \sin^3\varphi\,\mathrm{d}\varphi = \frac{8}{3\pi} \approx 0.85$$





$$\boxed{\mathrm{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u\Big\rangle + c(x)|u|^{p-2}u = 0} \qquad \text{(E)}$$

From oscillation of ODE to oscillation of PDE

Theorem 2. *For a real number $l > 1$ define*

$$a(r) = (l^*)^{p-1}\int_{S(r)} \|A(x)\|^p \lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma\,,$$

$$b(r) = \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{\lambda_{\min}^{p-1}(x)}\frac{\|\vec{b}(x)\|^p}{p^p}\right]\mathrm{d}\sigma\,,$$

$$l^* = 1 \textit{ if } \|\vec{b}(x)\| = 0 \textit{ and } l^* = \frac{l}{l-1} \textit{ otherwise.}$$

If $\boxed{\Big(a(r)|u'|^{p-2}u'\Big)' + b(r)|u|^{p-2}u = 0}$ *is oscillatory, then* (E) *is also oscillatory.*

$$\boxed{\mathrm{div}\Big(\boldsymbol{A(x)}\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u\Big\rangle + \boldsymbol{c(x)}|u|^{p-2}u = 0} \tag{E}$$

From oscillation of ODE to oscillation of PDE ($p \le 2$)

Theorem 3. *Let $1 < p \le 2$. For a real number $l > 1$ define*

$$\overline{a}(r) = (l^*)^{p-1} \int_{S(r)} \boldsymbol{\lambda}_{\max}\boldsymbol{(x)}\, \mathrm{d}\sigma\,,$$

$$\overline{b}(r) = \int_{S(r)} \left[\boldsymbol{c(x)} - \frac{l^{p-1}}{p^p}\boldsymbol{\lambda}_{\max}\boldsymbol{(x)} \left\|\vec{b}(x)\boldsymbol{A}^{-1}\boldsymbol{(x)}\right\|^p\right] \mathrm{d}\sigma\,,$$

$$l^* = 1 \textit{ if } \|\vec{b}(x)\| = 0 \textit{ and } l^* = \frac{l}{l-1} \textit{ otherwise.}$$

If $\boxed{\Big(a(r)|u'|^{p-2}u'\Big)' + b(r)|u|^{p-2}u = 0}$ *is oscillatory, then* (E) *is also oscillatory.*

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Corollary 1. *The function*

$$\Theta_{Xu}(s) = \int_{S(s)} \lambda_{\mathsf{min}}^{1-p}(x) \max_{x \in S(s)} \|A(x)\|_F^p \, \mathsf{d}\sigma$$

from Theorem C can be replaced by smaller function

$$\Theta(s) = \begin{cases} \displaystyle\int_{S(s)} \lambda_{\mathsf{min}}^{1-p}(x) \|A(x)\|^p \, \mathsf{d}\sigma \\ \displaystyle\int_{S(s)} \lambda_{\mathsf{max}}(x) \, \mathsf{d}\sigma & \textit{if } 1 < p \leq 2. \end{cases}$$

Remark. It holds

$$\begin{aligned} \lambda_{\mathsf{min}}^{1-p}(x) \|A(x)\|^p &= \lambda_{\mathsf{max}}(x) \left(\frac{\lambda_{\mathsf{max}}(x)}{\lambda_{\mathsf{min}}(x)} \right)^{p-1} \\ &\geq \lambda_{\mathsf{max}}(x) \end{aligned}$$

The difference between $p > 2$ and $1 < p \leq 2$

$$\vec{w}(x) = A(x)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}$$

$\boxed{p > 1 \text{ arbitrary}}$

$$\boxed{\|\vec{w}\| \leq \|A\|\frac{\|\nabla u\|^{p-1}}{|u|^{p-1}} \implies \frac{\|\nabla u\|^{p}}{|u|^{p}} \geq \frac{\|\vec{w}\|^{q}}{\|A\|^{q}}}$$

$\boxed{1 < p \leq 2}$

$$\boxed{\|\vec{w}\| \leq \|A\|\frac{\|\nabla u\|^{p-1}}{|u|^{p-1}} \implies \frac{\|\nabla u\|^{2-p}}{|u|^{2-p}} \geq \frac{\|\vec{w}\|^{(2-p)/(p-1)}}{\|A\|^{(2-p)/(p-1)}}}$$