Ordinary differential equations in the oscillation theory of partial half-linear differential equation

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$$\boxed{\operatorname{div}\Bigl(A(x)\|\nabla u\|^{p-2}\nabla u\Bigr) + \Bigl\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \Bigr\rangle + c(x)|u|^{p-2}u = 0} \tag{E}$$

- $x = (x_1, \dots, x_n)_{i=1}^n \in \mathbb{R}^n$,
- $A(x)$ is elliptic $n \times n$ matrix with differentiable components,
- $c(x)$ is Hölder continuous function,
- $\vec{b}(x) = \bigl(b_1(x), \dots, b_n(x)\bigr)$ is continuous n-vector function,
- $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)_{i=1}^n$ is the usual nabla operator,
- $\operatorname{div} = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}$ is the usual divergence operator,
- q is a conjugate number to the number p, i.e., $q = \frac{p}{p-1}$,
- $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$,
- $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$,
- **solution** of (E) in $\Omega \subseteq \mathbb{R}^n$ is a differentiable function $u(x)$ such that $A(x)\|\nabla u(x)\|^{p-2}\nabla u(x)$ is also differentiable and u satisfies (E) in Ω

$$\boxed{\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+\Big\langle \vec{b}(x),\|\nabla u\|^{p-2}\nabla u\Big\rangle + c(x)|u|^{p-2}u=0} \tag{E}$$

Concept of oscillation

Equation (E) is said to be *oscillatory* if for every nontrivial solution $u(x)$ and every number t^* there exists x^* with properties $u(x^*)=0$ and $\|x^*\|>t^*$.

Radial case, $A=I$, $\vec{b}=\vec{o}$

If the function $c(x)$ is radial, i.e. $c(x)=\widetilde{c}(\|x\|)$, then the equation for radial solution $u(x)=\widetilde{u}(\|x\|)$ of

$$\operatorname{div}\Big(\|\nabla u\|^{p-2}\nabla u\Big)+c(x)|u|^{p-2}u=0 \tag{1}$$

is

$$\Big(r^{n-1}|\widetilde{u}'|^{p-2}\widetilde{u}'\Big)'+r^{n-1}\,\widetilde{c}(r)\,|\widetilde{u}|^{p-2}\widetilde{u}=0. \qquad ' = \frac{\mathrm{d}}{\mathrm{d}r} \tag{2}$$

If (2) is oscillatory, then (1) is also oscillatory.

Detection of oscillation from ODE, $A=I$, $\vec{b}=\vec{o}$

Oscillation of partial differential equation can be detected from oscillation of ordinary differential equation.

Theorem A (O. Došlý (2001)). *Equation*

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u)+c(x)|u|^{p-2}u=0 \tag{3}$$

is oscillatory, if the ordinary differential equation

$$\Big(r^{n-1}|u'|^{p-2}u'\Big)'+r^{n-1}\left(\frac{1}{\omega_n r^{n-1}}\int_{S(r)} c(x)\,\mathrm{d}x\right)|u|^{p-2}u=0 \tag{4}$$

is oscillatory. The number ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for $A(x)=a(\|x\|)I$, $a(\cdot)$ differentiable).

Matrix norms

Spectral norm:

$$\|A\| = \sup\left\{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\right\} = \lambda_{\mathsf{max}}$$

Frobenius norm:

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}$$

$$\|A\| \le \|A\|_F \le \sqrt{n}\|A\|$$

Sets in $\mathbb{R}^n$

$$\Omega(a,b) = \{x \in \mathbb{R}^n : a \le \|x\| \le b\}$$
$$\Omega(a) = \{x \in \mathbb{R}^n : a \le \|x\|\}$$
$$S(a) = \{x \in \mathbb{R}^n : \|x\| = a\}$$

$\vec{v}(x)$ is the normal unit vector to the sphere $S(\|x\|)$ oriented outwards

Theorem B (Xu (2006)). *$\theta \in C^1([r_0,\infty],\mathbb{R}^+)$, $m>1$, $\lambda \in C([r_0,\infty),\mathbb{R}^+)$. If*

$$\lim_{r\to\infty}\int_{\Omega(r_0,r)}\left[\theta(\|x\|)c(x)-\lambda(\|x\|)\frac{m}{4}\frac{\theta'^2(\|x\|)}{\theta(\|x\|)}\right]\mathrm{d}x=\infty$$

and

$$\lim_{r\to\infty}\int_{\Omega(r_0,r)}\frac{1}{\theta(\|x\|)\lambda(\|x\|)}\mathrm{d}x=\infty, \qquad \text{where } \lambda(r)\ge \max_{x\in S(r)}\lambda_{\max}(x)$$

then $\boxed{\operatorname{div}\Big(A(x)\nabla u\Big)+c(x)u=0}$ *is oscillatory.*

Theorem C (Xu, Xing (2005)). *Suppose ϕ, $k\in C^1([r_0,\infty),\mathbb{R}^+)$*

(i) *$H(r,r)=0$ and $H(r,s)>0$ for $r>s\ge r_0$,*
$\partial H(r,s)/\partial s$ is continuous and nonpositive,

(ii)

$$h(r,s):=-\frac{\partial}{\partial s}\Big[H(r,s)k(s)\Big]-H(r,s)k(s)\frac{\phi'(s)}{\phi(s)}$$

$$\int_{r_0}^{r}H^{1-p}(r,s)|h(r,s)|^p\,\mathrm{d}s<\infty$$

(iii)

$$\limsup_{r\to\infty}\frac{1}{H(r,r_0)}\int_a^r\Big\{H(r,s)k(s)\phi(s)\int_{S(s)}c(x)\,\mathrm{d}\sigma - \frac{1}{p^p}\left[H(r,s)k(s)\right]^{1-p}\Theta_{Xu}(s)\phi(s)|h(r,s)|^p\Big\}\mathrm{d}s=\infty,$$

where

$$\Theta_{Xu}(s)=\rho(s)\omega_n s^{n-1} \qquad \text{and} \qquad \rho(s)\ge\max_{x\in S(s)}\frac{\|A(x)\|_F^p}{\lambda_{\min}^{p-1}(x)}$$

Then $\boxed{\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+c(x)|u|^{p-2}u=0}$ *is oscillatory.*

Sketch of proof

The method used to prove most of oscillation criteria for half-linear PDE

(i) Start with a proof of oscillation criterion for ODE.
(ii) Suppose by contradiction that the PDE is nonoscillatory and possesses eventually positive solution.
(iii) Using transformation $\vec{w}(x) = A(x)\dfrac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}$ convert positive solutions of

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + c(x)|u|^{p-2}u = 0$$

into

$$\operatorname{div}\vec{w} + c(x) + (p-1)\left\langle \vec{w}, \frac{\nabla u(x)}{u(x)} \right\rangle = 0. \tag{5}$$

(iv) Integrating (5) over spheres and using standard tools (such as eigenvalues, Schwarz and Hölder inequalities) derive a Riccati type inequality which is similar to the inequality from the proof of onedimensional criterion.
(v) Repeat steps from the proof of oscillation criterion for ODE which yield a contradiction.

Questions

- Is it possible to replace all these steps (i)–(v) by method suggested in Theorem A? Is it possible to deduce oscillation of

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \left\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \right\rangle + c(x)|u|^{p-2}u = 0$$

from oscillation of certain ODE?

- Function $\boxed{\lambda(r) \geq \max_{x\in S(r)} \lambda_{\mathsf{max}}(x)}$ plays a crucial role in the linear case and $\boxed{\rho(r) \geq \max_{x\in S(r)} \dfrac{\|A(x)\|_F^p}{\lambda_{\mathsf{min}}^{p-1}(x)}}$ plays similar role if $p > 1$. This phenomenon can be oserved also in other oscillation criteria than Theorems B and C. We know that $\rho(r) \geq \lambda(r)$. Why such a discrepancy appears?

Main result

Theorem 1. *For a real number $l > 1$ define*

$$a(r) = (l^*)^{p-1} \int_{S(r)} \|A(x)\|^p \lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma,$$

$$b(r) = \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{\lambda_{\min}^{p-1}(x)} \frac{\|\vec{b}(x)\|^p}{p^p} \right] \mathrm{d}\sigma,$$

$$l^* = \begin{cases} 1 & \text{if } \|\vec{b}(x)\| = 0, \\ \frac{l}{l-1} & \text{otherwise.} \end{cases}$$

If $\boxed{\left(a(r)|u'|^{p-2}u'\right)' + b(r)|u|^{p-2}u = 0}$ *is oscillatory, then* (E) *is also oscillatory.*

Proof.

$$\vec{w}(x) = A(x) \frac{\|\nabla u(x)\|^{p-2} \nabla u(x)}{|u(x)|^{p-2} u(x)}$$

$$\operatorname{div} \vec{w} + c(x) - \left(\frac{l}{\lambda_{\min}}\right)^{p-1} \frac{1}{p^p} \|\vec{b}\|^p + (p-1)\lambda_{\min} \frac{1}{l^*} \frac{\|\nabla u\|^p}{|u|^p} \leq 0$$

$$W(r) = \int_{S(r)} \langle \vec{w}, \vec{\nu} \rangle\,\mathrm{d}\sigma \qquad \Longrightarrow \qquad W' + b(r) + (p-1)a^{1-q}(r)|W|^q \leq 0$$

□

Modified version

Theorem 1A. *Let $\rho \in C^1(\Omega(1), \mathbb{R}^+)$. Theorem 1 remains valid, if $a(r)$, $b(r)$ and l^* are replaced by*

$$a(r) = (l^*)^{p-1} \int_{S(r)} \rho(x) \|A(x)\|^p \lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma,$$

$$b(r) = \int_{S(r)} \rho(x) \left[c(x) - \frac{l^{p-1}}{p^p \lambda_{\min}^{p-1}(x)} \left\| \vec{b}(x) - \frac{\nabla \rho(x)}{\rho(x)} A(x) \right\|^p \right] \mathrm{d}\sigma,$$

$$l^* = \begin{cases} 1 & \text{if } \|\rho(x)\vec{b}(x) - \nabla\rho(x) A(x)\| = 0, \\ \frac{l}{l-1} & \text{otherwise.} \end{cases}$$

Proof. Consider $\vec{w}_\rho(x) = \rho(x)\vec{w}(x)$ instead of $\vec{w}(x)$. □

Main result improved for $1 < p \leq 2$

Theorem 2. *Let $1 < p \leq 2$. For a real number $l > 1$ define*

$$\overline{a}(r) = (l^*)^{p-1} \int_{S(r)} \lambda_{\max}(x)\, \mathrm{d}\sigma\,,$$

$$\overline{b}(r) = \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{p^p} \lambda_{\max}(x) \left\| \vec{b}(x) A^{-1}(x) \right\|^p \right] \mathrm{d}\sigma\,,$$

$$l^* = \begin{cases} 1 & \text{if } \|\vec{b}(x)\| = 0, \\ \frac{l}{l-1} & \text{otherwise.} \end{cases}$$

If $\boxed{\left(a(r)|u'|^{p-2}u'\right)' + b(r)|u|^{p-2}u = 0}$ *is oscillatory, then* (E) *is also oscillatory.*

Proof.

$$\vec{w}(x) = A(x) \frac{\|\nabla u(x)\|^{p-2} \nabla u(x)}{|u(x)|^{p-2} u(x)}$$

$$\operatorname{div} w + c(x) - \frac{l^{p-1}}{p^p} \lambda_{\max} \|\vec{b} A^{-1}\|^p + (p-1) \frac{1}{l^*} \lambda_{\max}^{1-q} \|\vec{w}\|^q \leq 0.$$

$$W(r) = \int_{S(r)} \langle \vec{w}, \vec{\nu} \rangle\, \mathrm{d}\sigma \qquad \Longrightarrow \qquad W' + \overline{b}(r) + (p-1)\overline{a}^{1-q}(r)|W|^q \leq 0$$

□

Example

Corollary 1. *The function*

$$\Theta_{X_u}(s) = \rho(s)\omega_n s^{n-1}, \qquad \rho(s) \geq \max_{x \in S(s)} \|A(x)\|_F^p \lambda_{\min}^{1-p}(x)$$

from Theorem C can be repaced by smaller function

$$\Theta(s) = \begin{cases} \displaystyle\int_{S(s)} \|A(x)\|^p \lambda_{\min}^{1-p}(x)\, \mathrm{d}\sigma & \text{if } p > 2, \\ \displaystyle\int_{S(s)} \lambda_{\max}(x)\, \mathrm{d}\sigma & \text{if } 1 < p \leq 2. \end{cases}$$

$$\vec{w}(x) = A(x)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}$$

$p > 1$ arbitrary

$$\operatorname{div}\vec{w} + c + \left\langle \vec{b}, \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u} \right\rangle + (p-1)\frac{\langle A\|\nabla u\|^{p-2}\nabla u, \nabla u\rangle}{|u|^p} = 0$$

$$\operatorname{div}\vec{w} + c + \left\langle \vec{b}, \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u} \right\rangle + (p-1)\lambda_{\mathsf{min}}\frac{\|\nabla u\|^p}{|u|^p} \le 0$$

$$\operatorname{div}\vec{w} + c + \left\langle \vec{b}, \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u} \right\rangle + (p-1)\left(\frac{1}{l} + \frac{1}{l^*}\right)\lambda_{\mathsf{min}}\frac{\|\nabla u\|^p}{|u|^p} \le 0$$

$$\operatorname{div}\vec{w} + c - \left(\frac{l}{\lambda_{\mathsf{min}}}\right)^{p-1}\frac{1}{p^p}\|\vec{b}\|^p + (p-1)\lambda_{\mathsf{min}}\frac{1}{l^*}\frac{\|\nabla u\|^p}{|u|^p} \le 0$$

$$\boxed{\|\vec{w}\| \le \|A\|\frac{\|\nabla u\|^{p-1}}{|u|^{p-1}} \implies \frac{\|\nabla u\|^p}{|u|^p} \ge \frac{\|\vec{w}\|^q}{\|A\|^q}}$$

$$\operatorname{div}\vec{w} + c - \left(\frac{l}{\lambda_{\mathsf{min}}}\right)^{p-1}\frac{1}{p^p}\|\vec{b}\|^p + (p-1)\lambda_{\mathsf{min}}\frac{1}{l^*\|A\|^q}\|\vec{w}\|^q \le 0$$

$1 < p \le 2$

$$\operatorname{div}\vec{w} + c + \left\langle \vec{b}, A^{-1}\vec{w}\right\rangle + (p-1)\left\langle \vec{w}, A^{-1}\vec{w}\right\rangle\frac{\|\nabla u\|^{2-p}}{|u|^{2-p}} = 0$$

$$\left\langle \vec{w}, A^{-1}\vec{w}\right\rangle \ge \|\vec{w}\|^2\frac{1}{\lambda_{\mathsf{max}}}$$

$$\boxed{\|\vec{w}\| \le \|A\|\frac{\|\nabla u\|^{p-1}}{|u|^{p-1}} \implies \frac{\|\nabla u\|^{2-p}}{|u|^{2-p}} \ge \frac{\|\vec{w}\|^{(2-p)/(p-1)}}{\|A\|^{(2-p)/(p-1)}}}$$

$$\operatorname{div}\vec{w} + c + \left\langle \vec{b}A^{-1}, \vec{w}\right\rangle + (p-1)\lambda_{\mathsf{max}}^{1-q}\|\vec{w}\|^q \le 0$$

$$\operatorname{div}\vec{w} + c + \left\langle \vec{b}A^{-1}, \vec{w}\right\rangle + (p-1)\left(\frac{1}{l} + \frac{1}{l^*}\right)\lambda_{\mathsf{max}}^{1-q}\|\vec{w}\|^q \le 0$$

$$\operatorname{div} w + c - \frac{l^{p-1}}{p^p}\lambda_{\mathsf{max}}\|\vec{b}A^{-1}\|^p + (p-1)\frac{1}{l^*}\lambda_{\mathsf{max}}^{1-q}\|\vec{w}\|^q \le 0$$