RICCATI TECHNIQUE
FOR HALF–LINEAR PDE

Robert Mařík
marik@mendelu.cz

Mendel University
Brno, Czech Republic

$$\boxed{\operatorname{div}(||\nabla u||^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0} \qquad p > 1 \qquad \text{(E)}$$

$$\Delta u + c(x)u = 0 \qquad p = 2 \qquad \text{(L)}$$

$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + \left\langle \vec{b}(x), ||\nabla u||^{p-2}\nabla u \right\rangle + c(x)|u|^{p-2}u = 0$$

$$\operatorname{div}\left(||\nabla u||^{p-2}\nabla u\right) + c(x)f(u) = 0$$

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Phi\left(\frac{\partial u}{\partial x_i}\right) + c(x)\Phi(u) = 0 \qquad \Phi(x) = |x|^{p-2}x$$

Concept of oscillation

Function u is oscillatory in $\Omega \subseteq \mathbb{R}^n$ if the set of zeros of u in Ω is unbounded with respect to the norm.

Equation (E) is *oscillatory in* Ω if every its solution defined in Ω is oscillatory in Ω.

Methods of study

☛ Comparison with radial equation and application of oscillation criteria for ODE

☞ **Theorem A** [Jaroš–Kusano–Yoshida (2000), Došlý–M. (2001)].

Let $\widehat{c}(r) = \dfrac{1}{\omega_n r^{n-1}} \displaystyle\int_{||x||=r} c(x)\,\mathrm{dS}$.

If $(r^{n-1}|y'|^{p-2}y')' + r^{n-1}\widehat{c}(r)|y|^{p-2}y = 0$ is oscillatory, then also (E) is oscillatory.

☛ Riccati technique

$$\vec{w}(x) = \frac{||\nabla u(x)||^{p-2}\nabla u}{|u(x)|^{p-2}u(x)} \implies \operatorname{div}\vec{w} + c(x) + (p-1)||\vec{w}||^{q} = 0$$

☛ Variational technique

$$\mathcal{F}_p(u;\Omega) := \int_{\Omega} \left[||\nabla u(x)||^{p} - c(x)|u(x)|^{p}\right] \mathrm{d}x$$

Philos, Grace, Kong, Li, Manojlovic, Yeh

$$(|u'|^{p-2}u')' + c(x)|u|^{p-2}u = 0 \qquad t \geq t_0 \tag{1}$$

$$D = \{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_0\} \qquad H(t,s) \in C(D, \mathbb{R}_0^+) \qquad H(t,t) = 0$$

Wang(2001): $k \in C([t_0,\infty), \mathbb{R}^+)$ $\qquad \dfrac{\partial}{\partial s}\Big(k(s)H(t,s)\Big) \leq 0$

$$\rho \in C([t_0,\infty), \mathbb{R}^+) \qquad h(t,s) = \frac{\partial H(t,s)}{\partial s} + H(t,s)\frac{\rho'(s)}{\rho(s)}$$

$$\int_{t_0}^{t} H^{1-p}(t,s)|h(t,s)|^p \rho(s)\,\mathrm{d}s < \infty$$

✍ **Example:** $H(t,s) = (t-s)^\lambda$, $\lambda > 1$, $\rho(s) \equiv k(s) \equiv 1$

☞ **Theorem B.** (1) is oscillatory if:

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)\rho(s)c(s) - \frac{|h(t,s)|^p \rho(s)}{p^p H^{p-1}(t,s)}\Big]\,\mathrm{d}s = \infty$$

INTEGRAL AVERAGING TECHNIQUE FOR HALF–LINEAR PDE

$D_n = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n) : t \geq ||x|| \geq t_0\}$, $H(t,x) \in C(D_n, \mathbb{R}_0^+)$

$||x|| = t \Rightarrow H(t,x) = 0$,

$H(t,x) = 0$ for $||x|| < t$, then $||\nabla H(t,x)|| = 0$ $\qquad \nabla = (\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n})$

$$\Omega(a) = \{x \in \mathbb{R}^n : ||x|| \geq a\} \qquad S(a) = \{x \in \mathbb{R}^n : ||x|| = a\}$$

$$\Omega_t(a,b) = \{x \in \mathbb{R}^n : b \geq ||x|| \geq a,\ H(t,x) \neq 0\}$$

$$\mathcal{H}(t,s) = \int_{S(s)} H(t,x)\,\mathrm{dS} > 0 \qquad s < t$$

$$k \in C(\mathbb{R}, \mathbb{R}^+) \quad \frac{\partial}{\partial s}\Big(k(s)\mathcal{H}(t,s)\Big) \leq 0 \qquad s < t$$

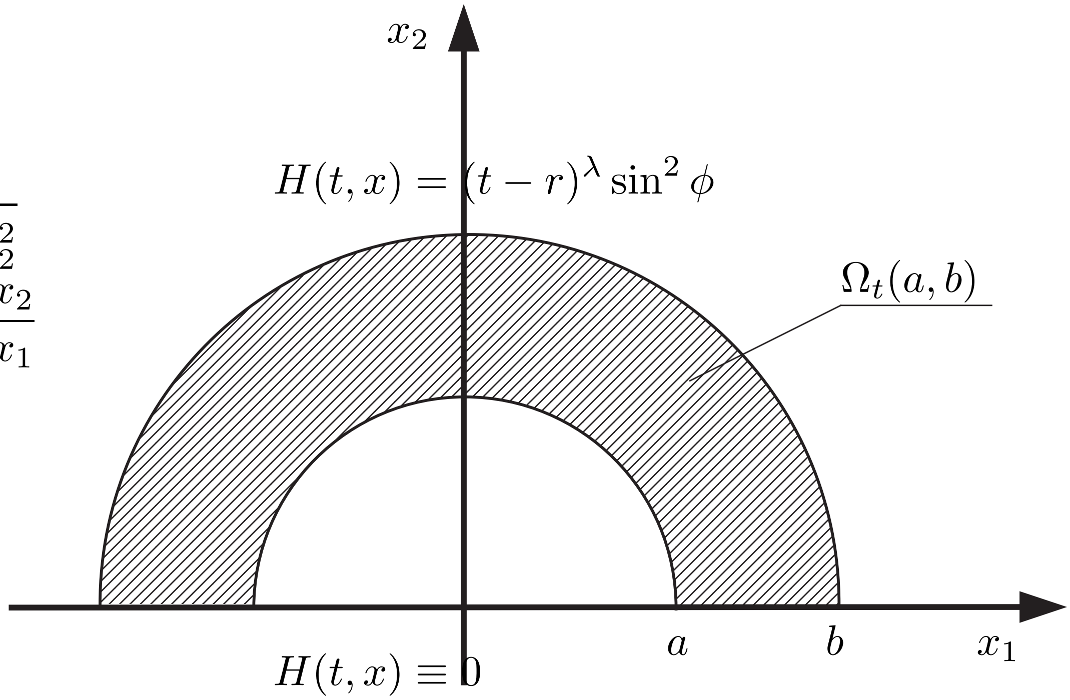
$$\rho \in C(\mathbb{R}, \mathbb{R}^+) \quad \vec{h}(t,x) = \nabla H(t,x) + H(t,x)\frac{\nabla\rho(x)}{\rho(x)}$$

$$\int_{\Omega_t(t_0,t)} H^{1-p}(t,x)||\vec{h}(t,x)||^p \rho(x)\,\mathrm{d}x < \infty$$

✍ **Example:** $\Delta u + c(x)u = 0$ $\qquad n = 2$

$r = ||x|| = \sqrt{x_1^2 + x_2^2}$

$\phi = \arg x = \arctan \dfrac{x_2}{x_1}$



a) $k \equiv \dfrac{1}{r}$, $\rho \equiv 1$

b) $k \equiv \rho \equiv \dfrac{1}{r}$

☞ **Theorem 1.** If

$$\limsup_{t\to\infty} \frac{1}{\mathcal{H}(t,t_0)} \int_{\Omega_t(t_0,t)} \left[H(t,x)\rho(x)c(x) - \frac{||\vec{h}(t,x)||^p \rho(x)}{p^p H^{p-1}(t,x)} \right] \mathrm{d}x = \infty,$$

then (E) is oscillatory.

Theorem 1 may detect oscillation even if the mean value of the potential function $c(x)$ over the sphere is small.

Q. Kong (1999)

$$y'' + q(t)y = 0 \tag{2}$$

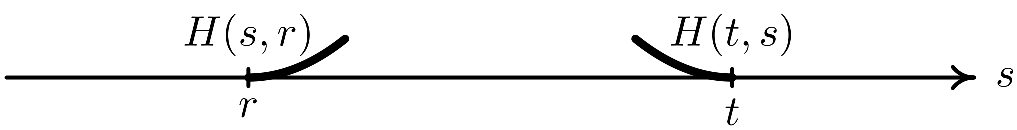
$$H(t,t) = 0, \qquad H(t,s) > 0 \text{ for } t > s$$

$$\frac{\partial H}{\partial t}(t,s) = h_1(t,s)\sqrt{H(t,s)}, \qquad \frac{\partial H}{\partial s}(t,s) = -h_2(t,s)\sqrt{H(t,s)}$$

☞ **Theorem C** (Kong). The equation is oscillatory if for some $H(t,s)$ and each $r \geq t_0$ the following inequalities hold:

$$\limsup_{t\to\infty} \int_r^t \left[H(s,r)q(s) - \frac{1}{4}h_1^2(s,r)\right] \mathrm{d}s > 0$$

$$\limsup_{t\to\infty} \int_r^t \left[H(t,s)q(s) - \frac{1}{4}h_2^2(t,s)\right] \mathrm{d}s > 0.$$



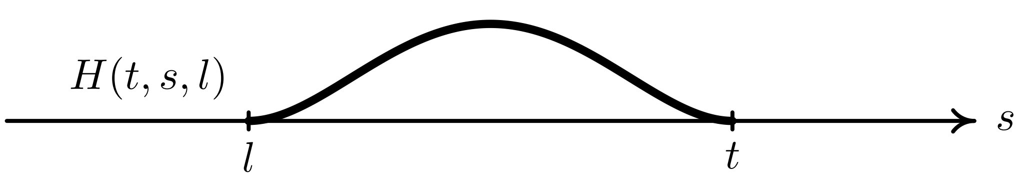
Sun (2004) (for more general equation than (2))

$H(t,s,l) > 0$ for $l < s < t$, $\qquad H(t,t,l) = 0 = H(t,l,l)$

☞ **Theorem D** (Sun). Equation (2) is oscillatory if for every $l \geq t_0$ there exists a function $H(t,s,l)$ such that

$$\limsup_{t\to\infty} \int_l^t H^2(t,s,l)\Big(q(s) - h^2(t,s,l)\Big) \mathrm{d}s > 0,$$

where $\dfrac{\partial H(t,s,l)}{\partial s} = h(t,s,l)H(t,s,l)$.



Extensions: Xu (2005) for elliptic PDE.

☛ The function $H(t,s) \in C(D,[0,\infty))$ is said to belong to the class $\mathcal{H}$ if

(i). $H(t,s) = 0$ if and only if $t = s$.

(ii). The partial derivative $\dfrac{\partial H}{\partial s}(t,s)$ exists.

(iii). $h_2(t,s) = -\dfrac{\partial H}{\partial s}(t,s)H^{-1}(t,s)$ for $(t,s) \in D$, $t \neq s$,
$h_2^p(t,s)H(t,s)$ is locally integrable on each compact subset in D.

☛ The function $H^*(t,s) \in C(D,[0,\infty))$ is said to belong to the class $\mathcal{H}^*$ if

(i). $H^*(t,s) = 0$ if and only if $t = s$.

(ii). The partial derivative $\dfrac{\partial H^*}{\partial t}(t,s)$ exists

(iii). $h_1^*(t,s) = \dfrac{\partial H^*}{\partial t}(t,s)\Big[H^*(t,s)\Big]^{-1}$ for $(t,s) \in D$, $t \neq s$
$\Big[h_1^*(t,s)\Big]^p H^*(t,s)$ is locally integrable on each compact subset in D.

☞ **Theorem 2.** Suppose that there exists real number $c \in (a,b)$, positive smooth function $\rho(x)$ and functions $H(t,s) \in \mathcal{H}$, $H^*(t,s) \in \mathcal{H}^*$, such that

$$\begin{aligned}
&\frac{1}{H^*(c,a)}\int_{\Omega(a,c)} H^*(\|x\|,a)\rho(x)c(x)\,\mathrm{d}x + \frac{1}{H(b,c)}\int_{\Omega(c,b)} H(b,\|x\|)\rho(x)c(x)\,\mathrm{d}x \\
&\quad > \frac{1}{H^*(c,a)}\int_{\Omega(a,c)} \left\|\frac{\nabla\rho(x)}{\rho(x)} + h_1^*(\|x\|,a)\vec{\nu}\right\|^p \rho(x)H^*(\|x\|,a)p^{-p}\,\mathrm{d}x \\
&\quad + \frac{1}{H(b,c)}\int_{\Omega(c,b)} \left\|\frac{\nabla\rho(x)}{\rho(x)} - h_2(b,\|x\|)\vec{\nu}\right\|^p \rho(x)H(b,\|x\|)p^{-p}\,\mathrm{d}x\,.
\end{aligned} \tag{3}$$

Then every solution of (E) has at least one zero inside $\Omega(a,b)$.

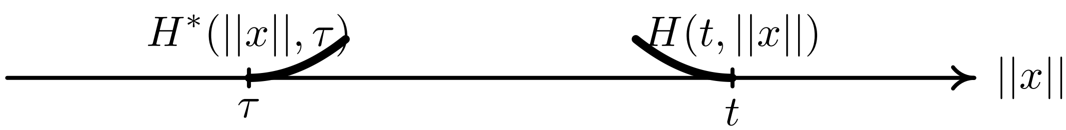
☞ **Theorem 3.** If there exist $t_0 > 0$, $H \in \mathcal{H}$, $H^* \in \mathcal{H}^*$, $\rho \in C^1(\Omega(t_0), \mathbb{R}^+)$ such that for every $\tau > t_0$ the inequalities

$$\limsup_{t\to\infty} \int_{\Omega(\tau,t)} H(t, \|x\|)\rho(x) \times \left[c(x) - p^{-p} \left\| \frac{\nabla\rho(x)}{\rho(x)} - h_2(t, \|x\|)\vec{\nu} \right\|^p \right] \mathrm{d}x > 0 \tag{4}$$

and

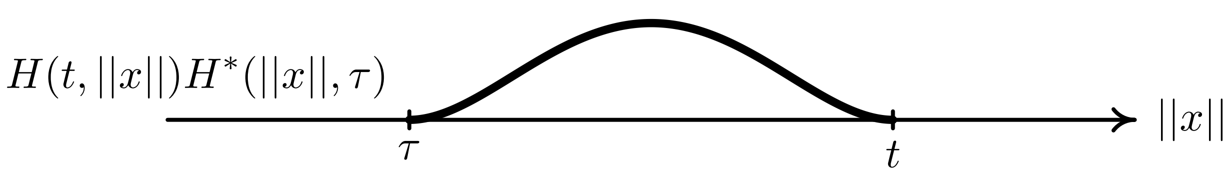
$$\limsup_{t\to\infty} \int_{\Omega(\tau,t)} H^*(\|x\|, \tau)\rho(x) \times \left[c(x) - p^{-p} \left\| \frac{\nabla\rho(x)}{\rho(x)} + h_1^*(\|x\|, \tau)\vec{\nu} \right\|^p \right] \mathrm{d}x > 0. \tag{5}$$

hold, then equation (E) is oscillatory.



☞ **Theorem 4.** Suppose that for every $T > t_0$ there exist $\tau > T$, $H \in \mathcal{H}$ and $H^* \in \mathcal{H}^*$ such that

$$\limsup_{t\to\infty} \int_{\Omega(\tau,t)} H(t, \|x\|)H^*(\|x\|, \tau)\rho(x) \times \left[c(x) - p^{-p} \left\| \frac{\nabla\rho(x)}{\rho(x)} - \left[h_2(t, \|x\|) - h_1^*(\|x\|, \tau) \right] \vec{\nu} \right\|^p \right] \mathrm{d}x > 0 \tag{6}$$



$$\operatorname{div}(\|\nabla u\|^{p-2}\,\nabla u) + c(x)|u|^{q-2}u = 0, \tag{7}$$

$p, q > 1$, no sign restricitons on $c(x)$

☞ **Theorem 5.** Suppose that there exist positive functions $a(x) \in C^1(\Omega(1), \mathbb{R}^+)$, $\phi \in C([1,\infty), \mathbb{R}^+)$ and a constant $l > 1$ such that

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} \left[a(x)c(x) - \frac{1}{p}\,\|\nabla a(x)\|^p \left[\frac{q-1}{l} p^* \phi(\|x\|) a(x)\right]^{-p/p^*} \right] \mathrm{d}x \;= \infty \tag{8}$$

and

$$\lim_{r\to\infty} \int^r \phi(r) \Big(r^{n-1} \tilde{a}(r) \Big)^{-p^*/p} \,\mathrm{d}r \;= \infty, \tag{9}$$

where $\tilde{a}(r)$ is the mean value of the function $a(x)$ on the sphere $S(r)$, i.e.

$$\tilde{a}(r) = \frac{1}{\omega_n r^{n-1}} \int_{S(r)} a(x)\,\mathrm{dS}\,.$$

Then equation (7) has no positive solution which satisfies

$$|u(x)|^{\frac{q-p}{p-1}} \geq \phi(\|x\|) \tag{10}$$

for large $\|x\|$.

☞ **Theorem 6.** Let

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} c(x)\,\mathrm{d}x \;= \infty. \tag{11}$$

Then equation (7) possesses no positive solution satisfying inequality

$$|u(x)|^{q-p} \geq \|x\|^{n-p} \tag{12}$$

on $\Omega(r)$ for arbitrary large r.