Oscillation criteria for equations with p–Laplacian

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$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + c(x)\Phi(u) = 0 \qquad \text{(E)}$$

$\Phi(u) = |u|^{p-2}u$, $p > 1$, $c(x) : \mathbb{R}^n \to \mathbb{R}$, $c(\cdot) \not\equiv 0$

Particular cases

$n = 1,\ p = 2 \ \to\ u'' + c(x)u = 0$	*extensive literature*
$n = 1 \to \big(\Phi(u')\big)' + c(x)\Phi(u) = 0$	Došlý, Elbert, Lomtatidze, Mirzov
$p = 2 \quad \to \quad \Delta u + c(x)u = 0$	Alegretto, Fiedler, Müller-Pfeiffer, Toraev

Other "similar" equations

$(\lvert u'\rvert^{\alpha-2}u')' + c(t)\lvert u\rvert^{\beta-2}u = 0$	Cecchi, Došlá, Drábek, Marini
$\Delta u + B(x, u) = 0$	Swanson, Noussair, Kusano

CONCEPT OF OSCILLATION

Function u is oscillatory if the set of zeros of u is unbounded
Equation (E) is *oscillatory* if every its solution is oscillatory

METHODS OF STUDY

$u'' + c(x)u = 0$; $\quad \int_a^b [y'^2 - c(x)y^2]\,\mathrm{d}x > 0$ for $y \in W_0^{1,2}([a, b])$, $y \not\equiv 0$;
$w(x) = u'(x) \,/\, u(x)$, $\qquad w' + c(x) + w^2 = 0$

Došlý – Mařík, Jaroš – Kusano – Yoshida

$u(x)$ be positive solution of (E) on $\Omega \subseteq \mathbb{R}^n$

• **Variational technique**: $y \in W_0^{1,p}(\Omega)$, $y \not\equiv 0$

$$\mathcal{J}(y; \Omega) := \int_\Omega \Big(||\nabla y||^p - c(x)|y|^p\Big)\,\mathrm{d}x > 0$$

• **Riccati technique**: $\quad \vec{w}(x) = \Phi\left(\frac{||\nabla u||}{u}\right)\frac{\nabla u}{||\nabla u||}, \quad \frac{1}{p} + \frac{1}{q} = 1$

$$\operatorname{div}\vec{w} + c(x) + (p-1)||\vec{w}||^q = 0$$

HARTMAN–WINTNER TYPE CRITERIA

Hartman, Wintner (1952): $u'' + c(x)u = 0, \quad \overline{C}(t) = \frac{1}{t}\int_1^t \int_1^s c(x)\,\mathrm{d}x\ \mathrm{d}s$

$\lim_{t\to\infty} \overline{C}(x) = \infty$

$-\infty < \liminf_{t\to\infty} \overline{C}(x) < \limsup_{t\to\infty} \overline{C}(x)$

$$C(t) = \frac{p-1}{t^{p-1}} \int_1^t s^{p-2} \int_{1\le ||x|| \le s} ||x||^{1-n} c(x)\,\mathrm{d}x\ \mathrm{d}s$$

$$\lim_{t\to\infty} C(t) = \infty$$

$$-\infty < \liminf_{t\to\infty} C(t) < \limsup_{t\to\infty} C(t)$$

$$\lim_{t\to\infty} \int_{||x||\le t} ||x||^{1-n} c(x)\,\mathrm{d}x = \infty$$

$p \ge n$ and $\lim_{t\to\infty} \int_{||x||\le t} c(x)\,\mathrm{d}x = \infty$

$p - 1 \ge n$ and there exists $\alpha \in (-\frac{n}{p}, p-n-1]$ such that

$$\lim_{t\to\infty} \frac{1}{t^{\alpha+1}} \int_0^t s^{\alpha} \int_{||x||\le s} c(x)\,\mathrm{d}x\ \mathrm{d}s = \infty$$

HILLE AND NEHARI TYPE CRITERIA

$$u'' + c(x)u = 0, \qquad c(x) \geq 0, \quad \int^{\infty} c(x)\,\mathrm{d}x < +\infty$$

Hille (1948): $\overline{Q}(t) = t \displaystyle\int_t^{\infty} c(x)\,\mathrm{d}x$

$$\liminf_{t\to\infty} \overline{Q}(t) > \frac{1}{4}$$

$$\limsup_{t\to\infty} \overline{Q}(t) > 1$$

Nehari (1957): $\overline{H}(t) = \dfrac{1}{t} \displaystyle\int_1^t x^2 c(x)\,\mathrm{d}x$

$$\liminf_{t\to\infty} \overline{H}(t) > \frac{1}{4}$$

$$C_0 = \lim_{t\to\infty} C(t) \qquad Q(t) = t^{p-1}\left(C_0 - \int_{1\leq||x||\leq t} ||x||^{1-n} c(x)\,\mathrm{d}x\right)$$

$$H(t) = \frac{1}{t} \int_{1\leq||x||\leq t} ||x||^{p-n+1} c(x)\,\mathrm{d}x$$

$$Q_* = \liminf_{t\to\infty} Q(t) \qquad Q^* = \limsup_{t\to\infty} Q(t)$$

$$H_* = \liminf_{t\to\infty} H(t) \qquad H^* = \limsup_{t\to\infty} H(t)$$

ω_n — measure of the n-dimensional unit sphere in $\mathbb{R}^n$

$$\limsup_{t\to\infty} \frac{t^{p-1}}{\ln t}[C_0 - C(t)] > \left|\frac{p-n}{p}\right|^p \omega_n$$

$$Q_* > -\infty, \ \limsup_{t\to\infty} \frac{1}{\ln t} \int_{1\leq||x||\leq t} ||x||^{p-n} c(x)\,\mathrm{d}x > \left|\frac{n-p}{p}\right|^p \omega_n$$

$$Q_* > \frac{1}{p-1}\left|\frac{n-p}{p}\right|^p \omega_n$$

$$H_* > \left|\frac{n-p}{p}\right|^p \omega_n$$

$$p = n \text{ and } \limsup_{t\to\infty} t^{p-1}[C_0 - C(t)] = \infty$$

$$p = n, \ Q_* > -\infty \text{ and } \limsup_{t\to\infty} \int_{1\leq||x||\leq t} c(x)\,\mathrm{d}x = \infty$$

Assumptions:

$$\frac{(n-1)-p(p-1)}{p(p-1)}\Phi\Big(\frac{n-1}{p}\Big)\omega_n \le Q_* \le \Big|\frac{n-p}{p}\Big|^p \frac{\omega_n}{p-1}, \tag{1}$$

$$\frac{1-n}{p}\Phi\Big(\frac{p-n+1}{p}\Big)\omega_n \le H_* \le \Big|\frac{n-p}{p}\Big|^p \omega_n. \tag{2}$$

Notation: If (1), then A denotes the smaller of roots of

$$(p-1)\omega_n^{-q/p}|x|^q + (n-p)x + (p-1)Q_* = 0$$

If (2), then B denotes the larger of roots of

$$(p-1)\omega_n^{-q/p}|x|^q + (n-p)x + H_* = 0$$

$u'' + c(x)u = 0$, $0 \le Q_*, H_* \le \frac{1}{4}$, $A = \frac{1}{2} - \sqrt{\frac{1}{4} - Q_*}$, $B = \frac{1}{2} + \sqrt{\frac{1}{4} - H_*}$

(1) and $H^* > \Big|\frac{p-n+1}{p}\Big|^p \omega_n - A$

(1), (2) and $H^* > H_* - A + B$

(2) and $Q^* > \frac{1}{p-1}\Big|\frac{1-n}{p}\Big|^p \omega_n + B$

(1), (2) and $Q^* > Q_* - A + B$

$\liminf_{t\to\infty}[Q(t) + H(t)] > \frac{p}{p-1}\Big|\frac{n-p}{p}\Big|^p \omega_n$

$\limsup_{t\to\infty}[Q(t) + H(t)] > \Big|\frac{1-n}{p}\Big|^p \frac{\omega_n}{p-1} + \Big|\frac{p-n+1}{p}\Big|^p \omega_n$

(1), (2) and $\limsup_{t\to\infty}[Q(t) + H(t)] > Q_* + H_* - A + B$

NONEXISTENCE OF POSITIVE SOLUTIONS

$$\text{If}\quad p \geq n \quad\text{and}\quad \liminf_{r\to\infty} \int_{||x||\leq r} c(x)\,\mathrm{d}x \geq 0, \qquad \text{or}$$

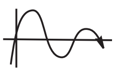
$$p < n \quad\text{and}\quad \sup_{r>0} r^{p-n} \int_0^r \Big(\int_{||x||<t} c(x)\,\mathrm{d}x \Big)^q t^{\frac{1-n}{p-1}}\,\mathrm{d}t > \frac{(n-p)^{p-1}}{(p-1)^p}\omega_n^q$$

then (E) has no solution positive on $\mathbb{R}^n$.

ODE METHODS

$$\boxed{\Big(r^{n-1}\Phi(u') \Big)' + r^{n-1} \frac{\int_{||x||=r} c(x)\,\mathrm{d}\sigma}{r^{n-1}\omega_n} \Phi(u) = 0} \tag{ODE}$$

If (ODE) is conjugate on $[a, b]$, then (E) has no positive solution on the set $\{x \in \mathbb{R}^n : a \leq ||x|| \leq b\}$.

 (ODE) is oscillatory

OPEN QUESTIONS

1. Is it possible to formulate integral nonsocillation criteria (even in the linear case)?
2. Is the nodal oscillation equivalent to weak oscillation?

   $D \subseteq \mathbb{R}^n$ is a *nodal domain of* (E) if there exists solution of (E) such that $u|_{\partial D} = 0$, $u \not\equiv 0$. Equation (E) is *nodally oscillatory* if it has nodal domain outside of every ball in $\mathbb{R}^n$.
3. How can be these criteria modified for various types of unbounded domains?