Integral averages and oscillation criteria for half–linear PDE

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$$\boxed{\operatorname{div}(||\nabla u||^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0} \qquad p > 1 \qquad \text{(E)}$$

$$\Delta u + c(x)u = 0 \qquad p = 2 \qquad \text{(L)}$$

Concept of oscillation

Function u is oscillatory in $\Omega \subseteq \mathbb{R}^n$ if the set of zeros of u in Ω is unbounded with respect to the norm.

Equation (E) is *oscillatory in* Ω if every its solution defined in Ω is oscillatory in Ω.

Methods of study

- Comparison with radial equation and application of 1-dimensional oscillation criteria
- Riccati technique
- Variational technique

Known results

☞ **Theorem A** [Jaroš-Kusano-Yoshida (2000), Došlý-M.(2001)].
Let $\widehat{c}(r) = \frac{1}{\omega_n r^{n-1}} \int_{||x||=r} c(x)\,\mathrm{dS}$.

If $(r^{n-1}|y'|^{p-2}y')' + r^{n-1}\widehat{c}(r)|y|^{p-2}y = 0$ is oscillatory, then also (E) is oscillatory.

☞ **Theorem B** [M.(2000)].
Denote $C(t) = \frac{p-1}{t^{p-1}} \int_1^t s^{p-2} \int_{1\leq||x||\leq s} ||x||^{1-n} c(x)\,\mathrm{d}x\ \mathrm{d}s$.

If $-\infty < \liminf_{t\to\infty} C(t) < \limsup_{t\to\infty} C(t) \leq \infty$ or if $\lim_{t\to\infty} C(t) = \infty$, then equation (E) is oscillatory.

Philos, Grace, Kong, Li, Manojlovic, Yeh

$$(|u'|^{p-2}u')' + c(x)|u|^{p-2}u = 0 \qquad t \geq t_0 \tag{1}$$

$$D = \{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_0\} \qquad H(t,s) \in C(D, \mathbb{R}_0^+) \qquad H(t,t) = 0$$

Wang(2001): $k \in C([t_0,\infty), \mathbb{R}^+) \quad \frac{\partial}{\partial s}\Big(k(s)H(t,s)\Big) \leq 0$

$$\rho \in C([t_0,\infty), \mathbb{R}^+) \qquad h(t,s) = \frac{\partial H(t,s)}{\partial s} + H(t,s)\frac{\mathrm{d}\,\rho(s)}{\mathrm{d}\,s}$$

$$\int_{t_0}^{t} H^{1-p}(t,s)|h(t,s|^p \rho(s)\,\mathrm{d}s < \infty$$

✍ **Example:** $H(t,s) = (t-s)^\lambda$, $\lambda > 1$, $\rho(s) \equiv k(s) \equiv 1$

☞ **Theorem D.** (1) is oscillatory if:

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)\rho(s)c(s) - \frac{|h(t,s)|^p \rho(s)}{p^p H^{p-1}(t,s)}\Big]\,\mathrm{d}s = \infty$$

☞ **Theorem E.** (1) is oscillatory if:

$$0 < \inf_{s\geq t_0}\Big\{\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\Big\} \tag{2}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{|h(t,s)|^p \rho(s)}{p^p H^{p-1}(t,s)}\,\mathrm{d}s < \infty \tag{3}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big[H(t,s)\rho(s)c(s) - \frac{|h(t,s)|^p \rho(s)}{p^p H^{p-1}(t,s)}\Big]\,\mathrm{d}s \geq A(T) \tag{4}$$

$$\int_{t_0}^{\infty} (A_+(T))^q k^{-1}(T)\rho^{1-q}(T)\,\mathrm{d}T = \infty \tag{5}$$

☞ **Theorem F.** (1) is oscillatory if: (2), (5)

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)c(s)\,\mathrm{d}x < \infty \tag{6}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big[H(t,s)\rho(s)c(s) - \frac{|h(t,s)|^p \rho(s)}{p^p H^{p-1}(t,s)}\Big]\,\mathrm{d}s \geq A(T) \tag{7}$$

$D = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n) : t \geq ||x|| \geq t_0\}$, $H(t,x) \in C(D, \mathbb{R}_0^+)$

$||x|| = t \Rightarrow H(t,x) = 0,$

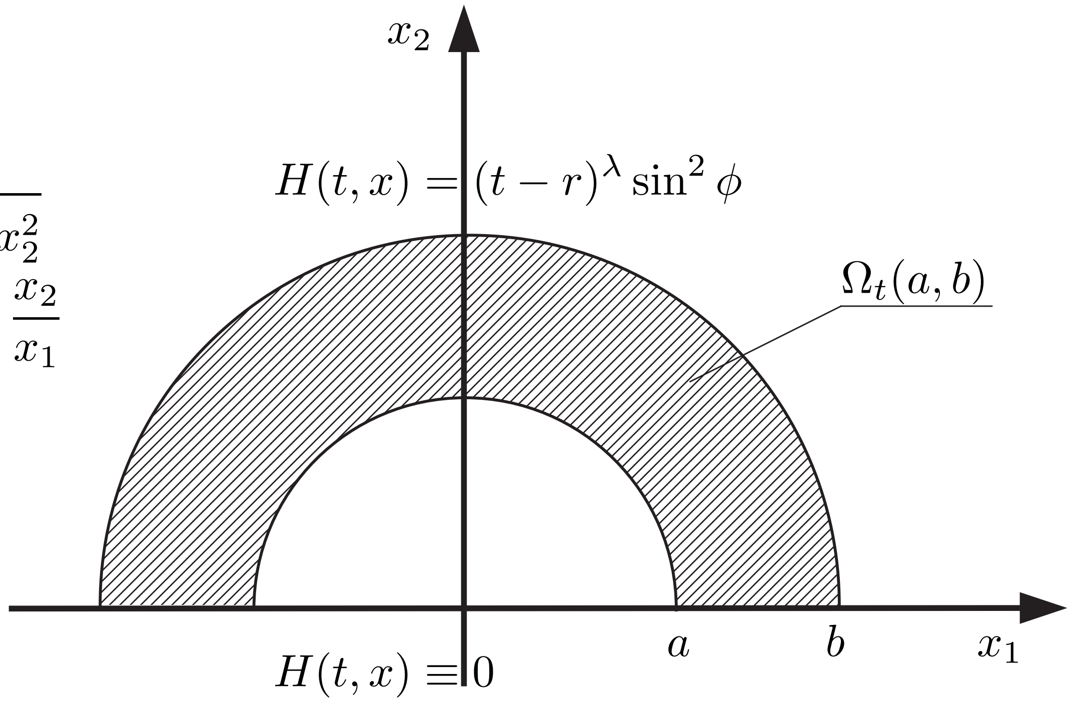
$H(t,x) = 0$ for $||x|| < t$, then $||\nabla H(t,x)|| = 0$ $\qquad \nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$

$$\Omega(a) = \{x \in \mathbb{R}^n : ||x|| \geq a\}$$
$$S(a) = \{x \in \mathbb{R}^n : ||x|| = a\}$$
$$\Omega_t(a,b) = \{x \in \mathbb{R}^n : b \geq ||x|| \geq a,\ H(t,x) \neq 0\}$$
$$\mathcal{H}(t,s) = \int_{S(s)} H(t,x)\,\mathrm{dS} > 0 \qquad s < t$$
$$k \in C(\mathbb{R}, \mathbb{R}^+) \quad \frac{\partial}{\partial s}\Big(k(s)\mathcal{H}(t,s)\Big) \leq 0 \qquad s < t$$
$$\rho \in C(\mathbb{R}, \mathbb{R}^+) \quad \vec{h}(t,x) = \nabla H(t,x) + H(t,x)\frac{\nabla \rho(x)}{\rho(x)}$$
$$\int_{\Omega_t(t_0,t)} H^{1-p}(t,x)||\vec{h}(t,x)||^p \rho(x)\,\mathrm{d}x < \infty$$

✍ **Example:** $\qquad \Delta u + c(x)u = 0 \qquad n = 2$

$r = ||x|| = \sqrt{x_1^2 + x_2^2}$

$\phi = \arg x = \arctan \frac{x_2}{x_1}$



a) $k \equiv \frac{1}{r}, \rho \equiv 1$

b) $k \equiv \rho \equiv \frac{1}{r}$

☞ **Theorem 1.** If

$$\limsup_{t\to\infty}\frac{1}{\mathcal{H}(t,t_0)}\int_{\Omega_t(t_0,t)}\left[H(t,x)\rho(x)c(x)-\frac{||\vec{h}(t,x)||^p\rho(x)}{p^pH^{p-1}(t,x)}\right]\mathrm{d}x=\infty,$$

then (E) is oscillatory.

☞ **Theorem 2.** If

$$\limsup_{t\to\infty}\frac{1}{\mathcal{H}(t,t_0)}\int_{\Omega_t(t_0,t)}\frac{||\vec{h}(t,x)||^p\rho(x)}{H^{p-1}(t,x)}\,\mathrm{d}x<\infty \tag{1}$$

$$0<\inf_{s\ge t_0}\left\{\liminf_{t\to\infty}\frac{k(s)\mathcal{H}(t,s)}{k(t_0)\mathcal{H}(t,t_0)}\right\} \tag{2}$$

there exists a function $A\in C(\Omega(t_0),\mathbb{R})$ such that

$$\inf_{t\in(T,\infty)}\left\{\frac{1}{\mathcal{H}(t,T)}\int_{\Omega_t(T,t)}\left[H(t,x)\rho(x)c(x)-\frac{||\vec{h}(t,x)||^p\rho(x)}{p^pH^{p-1}(t,x)}\right]\mathrm{d}x\right\}\ge A(T) \tag{3}$$

$$\int_{t_0}^{\infty}\Big(A_+(T)\Big)^q\widehat{\rho}^{1-q}(T)k^{-1}(T)\,\mathrm{d}T=\infty \tag{4}$$

$$A_+(T)=\max\{A(T),0\}$$

$$\widehat{\rho}(T)=\sup_{t>T}\left\{\frac{1}{\mathcal{H}(t,T)}\int_{S(T)}\rho(x)H(t,x)\,\mathrm{dS}\right\},$$

then (E) is oscillatory.

☞ **Theorem 3.** If (2), (3), (4) and

$$\liminf_{t\to\infty}\frac{1}{\mathcal{H}(t,t_0)}\int_{\Omega(t_0,t)}H(t,x)\rho(x)c(x)\,\mathrm{d}x<\infty. \tag{5}$$

then (E) is oscillatory.

✍ **Example:** $\Delta u + c(x)u = 0$ $n = 2$

$r = ||x|| = \sqrt{x_1^2 + x_2^2}$

$\phi = \arg x = \arctan \dfrac{x_2}{x_1}$

Every of the following conditions is sufficient for oscillation of the equation on the half–plane $x_2 \geq 0$:

$$\limsup_{t\to\infty} t^{-\lambda} \int_{M(t)} \left[c(x(r,\phi))(t-r)^{\lambda} \sin^2\phi - \frac{(t-r)^{\lambda}}{r^2} \cos^2\phi \right] \mathrm{d}x\,(r,\phi) = \infty$$

$$\limsup_{t\to\infty} t^{-\lambda} \int_{M(t)} c(x(r,\phi))(t-r)^{\lambda} r^{-1} \sin^2\phi \, \mathrm{d}x\,(r,\phi) = \infty$$

where $M(t) = \{(x_1, x_2) \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq t^2, x_2 > 0\}$.