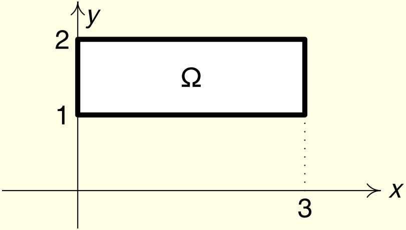
VÝPOČET DVOJNÉHO INTEGRÁLU

Robert Mařík

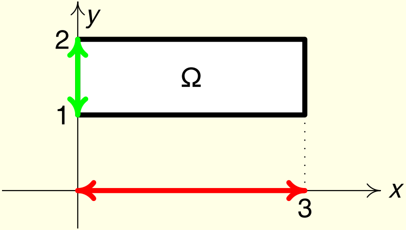
2. října 2009

Obsah



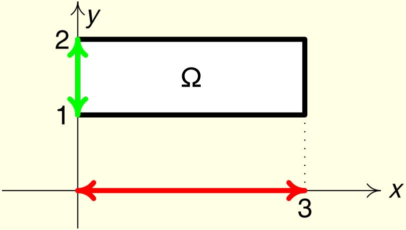
$$\iint_\Omega (x+y)\,\mathrm{d}x\,\mathrm{d}y$$

Budeme počítat integrál funkce $x + y$ na obdélníku Ω.



$$\iint_{\Omega} (x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 \left[\int_0^3 (x+y)\,\mathrm{d}x\right] \mathrm{d}y$$

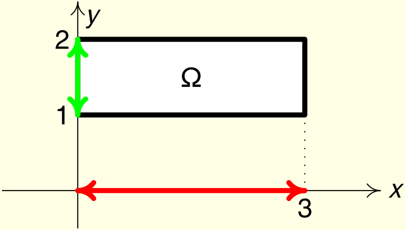
Z obrázku určíme meze pro jednotlivé proměnné. Budeme nejprve integrovat podle y a potom podle x (ale šlo by to i naopak).



$$\iint_\Omega (x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 \left[\int_0^3 (x+y)\,\mathrm{d}x\right]\mathrm{d}y = \int_1^2 \left[\frac{x^2}{2} + xy\right]_0^3 \mathrm{d}y$$

$$\int x\,\mathrm{d}x = \frac{x^2}{2}$$

$$\int y\,\mathrm{d}x = y\int 1\,\mathrm{d}x = yx$$



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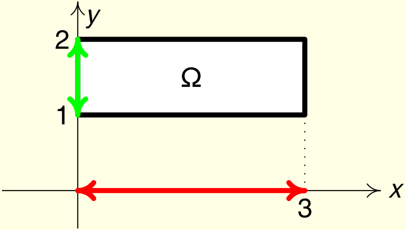
$$= \int_1^2 \left[\frac{9}{2}+3y-\left(\frac{0}{2}+0y\right)\right]\mathrm{d}y$$

Použijeme Newtonovu-Leibnizovu větu.



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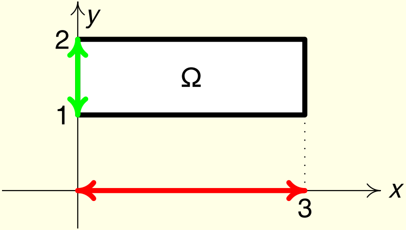
$$= \int_1^2 \left[\frac{9}{2} + 3y - \left(\frac{0}{2} + 0y\right)\right]\mathrm{d}y = \int_1^2 \left(\frac{9}{2} + 3y\right)\mathrm{d}y$$

Upravíme



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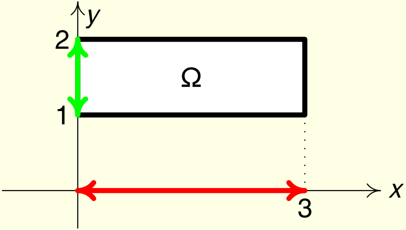
$$= \left[\frac{9}{2}y+3\frac{y^2}{2}\right]_1^2$$

Integrujeme podle y



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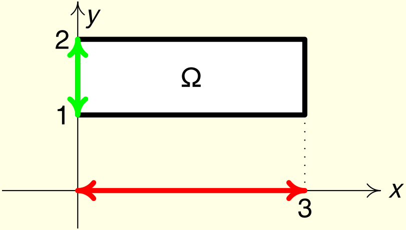
$$= \left[\frac{9}{2}y+3\frac{y^2}{2}\right]_1^2 = \frac{9}{2}\cdot 2+3\cdot\frac{4}{2}-\left(\frac{9}{2}+3\cdot\frac{1}{2}\right)$$

Použijeme Newtonovu-Leibnizovu větu.



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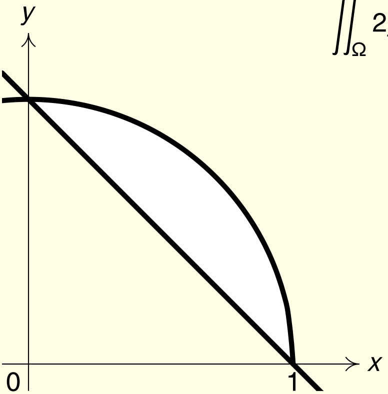
$$= \left[\frac{9}{2}y + 3\frac{y^2}{2}\right]_1^2 = \frac{9}{2}\cdot 2 + 3\cdot\frac{4}{2} - \left(\frac{9}{2} + 3\cdot\frac{1}{2}\right)$$

$$= 9 + 6 - 6 = 9$$

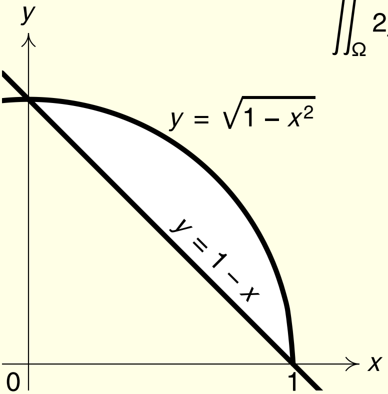
Hotovo.

$$\iint_\Omega 2y \,\mathrm{d}x \,\mathrm{d}y$$



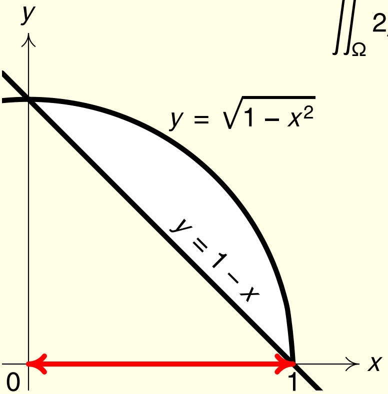
Máme vypočítat $\iint_\Omega 2y \,\mathrm{d}x \,\mathrm{d}y$ kde množina Ω leží v prvním kvadrantu, shora je ohraničena jednotkovou kružnicí $x^2 + y^2 = 1$ a zdola přímkou $x + y - 1 = 0$.

$$\iint_\Omega 2y\,\mathrm{d}x\,\mathrm{d}y = \int \left(\int \qquad 2y\,\mathrm{d}y\right)\mathrm{d}x$$



$x^2 + y^2 = 1 \Rightarrow y = \sqrt{1 - x^2}$
$x + y - 1 = 0 \Rightarrow y = 1 - x$
Budeme integrovat nejprve podle y (v proměnných mezích) a potom podle x (v pevných mezích).

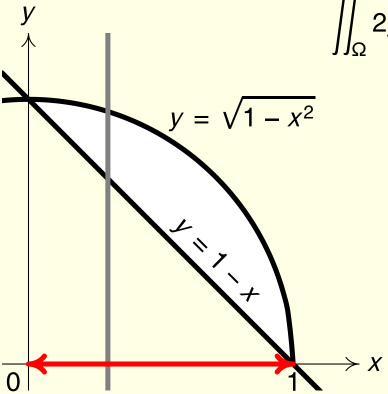
$$\iint_\Omega 2y\,dx\,dy = \int_0^1 \left(\int \qquad 2y\,dy\right) dx$$



$x_{min} = 0,$
$x_{max} = 1,$

Meze pro x jsou patrné z obrázku.

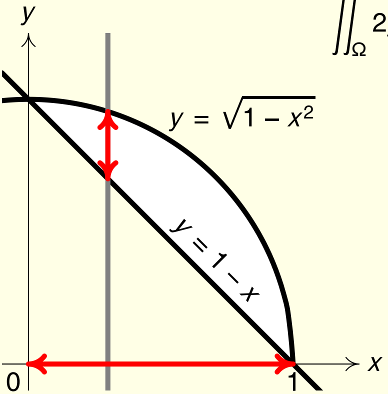
$$\iint_\Omega 2y\,dx\,dy = \int_0^1 \left(\int \qquad 2y\,dy\right) dx$$



$x_{min} = 0$,
$x_{max} = 1$,

Stanovíme meze pro y. Uvažujeme libovolnou hodnotu x mezi x_{min} a x_{max} a hledáme omezení pro y.

$$\iint_\Omega 2y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left(\int_{1-x}^{\sqrt{1-x^2}} 2y\,\mathrm{d}y\right)\mathrm{d}x$$



$x_{\min} = 0,$
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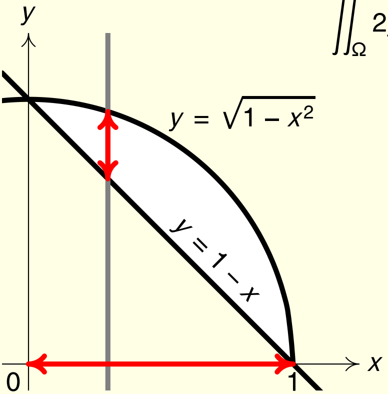
$y_{\min} = 1 - x,$
$y_{\max} = \sqrt{1-x^2}$

Hodnoty y začínají na přímce a končí na kružnici. Rovnice přímky tedy bude dolním a rovnice kružnice horním omezením na y.



$$\iint_\Omega 2y \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left(\int_{1-x}^{\sqrt{1-x^2}} 2y \,\mathrm{d}y \right) \mathrm{d}x$$

$$= \int_0^1 \left(\left[y^2 \right]_{1-x}^{\sqrt{1-x^2}} \right) \mathrm{d}x$$

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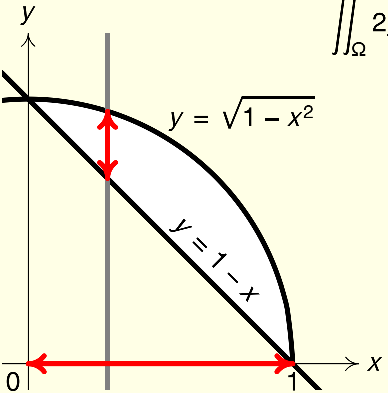

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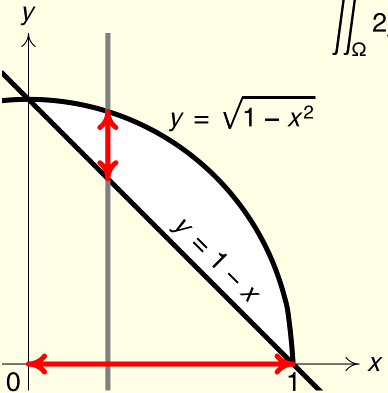
$$\iint_\Omega 2y\,dx\,dy = \int_0^1 \left(\int_{1-x}^{\sqrt{1-x^2}} 2y\,dy\right) dx$$

$$= \int_0^1 \left(\left[y^2\right]_{1-x}^{\sqrt{1-x^2}}\right) dx$$

$$= \int_0^1 \left(\left[1-x^2-(1-x)^2\right]\right) dx$$



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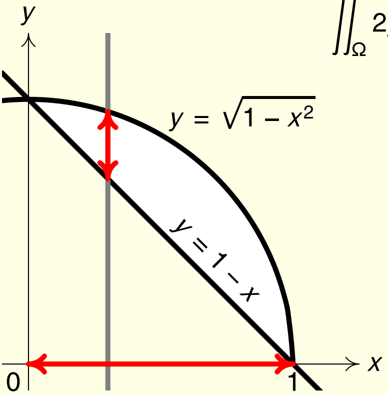
$$\iint_\Omega 2y \, dx \, dy = \int_0^1 \left(\int_{1-x}^{\sqrt{1-x^2}} 2y \, dy \right) dx$$
$$= \int_0^1 \left(\left[y^2 \right]_{1-x}^{\sqrt{1-x^2}} \right) dx$$
$$= \int_0^1 \left(\left[1 - x^2 - (1-x)^2 \right] \right) dx$$
$$= \int_0^1 \left(1 - x^2 - (1 - 2x + x^2) \right) dx$$



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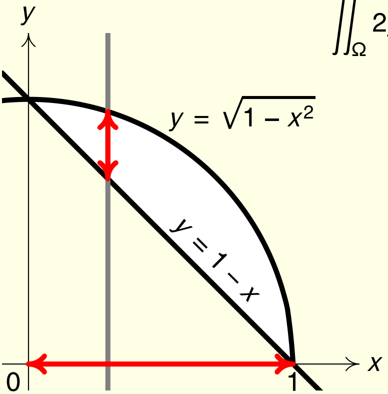
$$= \int_0^1 \left(1 - x^2 - (1 - 2x + x^2)\right)\mathrm{d}x$$
$$= \int_0^1 \left(2x - 2x^2\right)\mathrm{d}x$$



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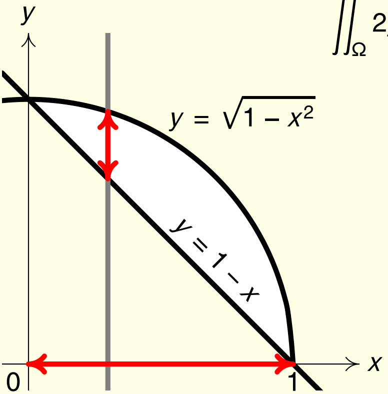
$$= \left[x^2 - \frac{2}{3} x^3 \right]_0^1$$



$x_{\min} = 0,$
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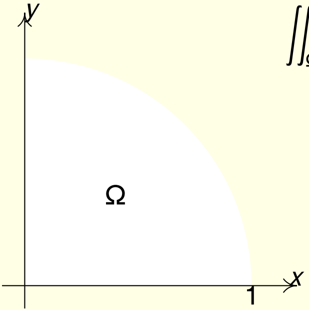
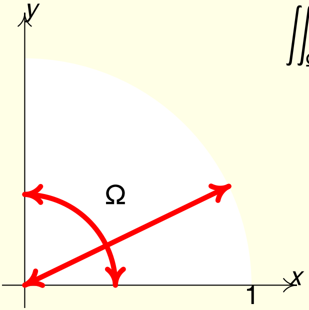
$$\begin{aligned}
\iint_\Omega 2y \, dx \, dy &= \int_0^1 \left(\int_{1-x}^{\sqrt{1-x^2}} 2y \, dy \right) dx \\
&= \int_0^1 \left(\left[y^2 \right]_{1-x}^{\sqrt{1-x^2}} \right) dx \\
&= \int_0^1 \left(\left[1 - x^2 - (1-x)^2 \right] \right) dx \\
&= \int_0^1 \left(1 - x^2 - (1 - 2x + x^2) \right) dx \\
&= \int_0^1 \left(2x - 2x^2 \right) dx \\
&= \left[x^2 - \frac{2}{3} x^3 \right]_0^1 \\
&= 1 - \frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}$$

Hotovo.



$$\iint_\Omega x \,\mathrm{d}x \,\mathrm{d}y$$

- Budeme počítat integrál $\iint x \,\mathrm{d}x \,\mathrm{d}y$ přes část jednotkového kruhu, která leží v prvním kvadrantu.
- Protože integrační obor je část kruhu, zdá se býti vhodné přejít do polárních souřadnic.

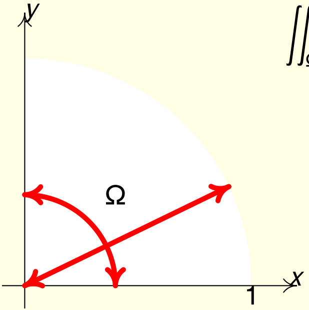


$r \in (0, 1]$

$\phi \in [0, \frac{\pi}{2}]$

$$\iint_\Omega x \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left(\int_0^{\frac{\pi}{2}} \underbrace{r\cos\phi}_{\text{funkce}} \underbrace{r}_{\text{Jakobián}} \,\mathrm{d}\phi \right) \mathrm{d}r$$

- Meze pro r a ϕ jsou konstanty.
- Za x dosazujeme podle transformačních vztahů $x = r\cos\phi$.
- Přidáme Jakobián, který je v polárních souřadnicích roven r.



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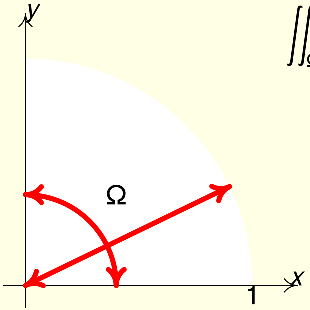
$$= \int_0^1 \Big[r^2 \sin\phi\Big]_0^{\frac{\pi}{2}}\,\mathrm{d}r$$

Integrujeme přes ϕ.

$$\int \cos\phi\,\mathrm{d}\phi = \sin\phi$$



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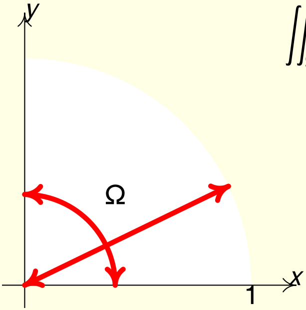
$$= \int_0^1 \Bigl[r^2 \sin\phi \Bigr]_0^{\frac{\pi}{2}} \mathrm{d}r$$

$$= \int_0^1 \Bigl[r^2 \sin\frac{\pi}{2} - r\sin 0 \Bigr] \mathrm{d}r$$

Použijeme Newtonovu-Leibnizovu větu.



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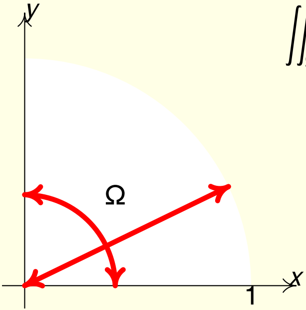
$$\iint_\Omega x \, dx \, dy = \int_0^1 \left(\int_0^{\frac{\pi}{2}} \underbrace{r\cos\phi}_{\text{funkce}} \underbrace{r}_{\text{Jakobián}} \, d\phi \right) dr$$

$$= \int_0^1 \left[r^2 \sin\phi \right]_0^{\frac{\pi}{2}} dr$$

$$= \int_0^1 \left[r^2 \sin\frac{\pi}{2} - r\sin 0 \right] dr$$

$$= \int_0^1 \left[r^2 \right] dr$$

Upravíme.



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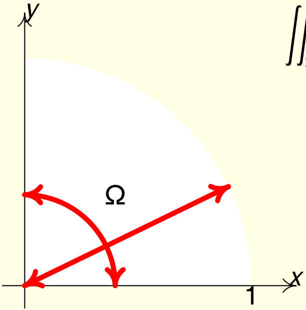
$$= \int_0^1 \Bigl[r^2 \sin\frac{\pi}{2} - r\sin 0\Bigr]\,\mathrm{d}r$$

$$= \int_0^1 \Bigl[r^2\Bigr]\,\mathrm{d}r$$

$$= \Bigl[\frac{r^3}{3}\Bigr]_0^1$$

Integrujeme přes r.



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$$= \int_0^1 \left[r^2 \sin\phi\right]_0^{\frac{\pi}{2}} \mathrm{d}r$$

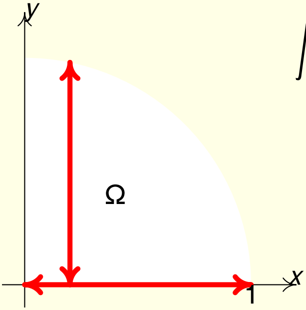
$$= \int_0^1 \left[r^2 \sin\frac{\pi}{2} - r\sin 0\right] \mathrm{d}r$$

$$= \int_0^1 \left[r^2\right] \mathrm{d}r$$

$$= \left[\frac{r^3}{3}\right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

- Použijeme Newtonovu-Leibnizovu větu a upravíme.
- Hotovo.

V kartézských souřadnicích:



$$\iint_\Omega x\,dx\,dy = \int_0^1 \left(\int_0^{\sqrt{1-x^2}} x\,dy\right) dx$$
$$= \int_0^1 \left[xy\right]_0^{\sqrt{1-x^2}} dx$$
$$= \int_0^1 x\sqrt{1-x^2}\,dx$$
$$= \text{substituční metodou} \dots$$
$$= \left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$$
$$= -\frac{1}{3}(0)^{\frac{3}{2}} - \left(-\frac{1}{3}(1)^{\frac{3}{2}}\right)$$
$$= \frac{1}{3}$$

Pro srovnání ukažme výpočet v kartézských souřadnicích

Konec