Oscillation theory of partial differential equations with p-Laplacian

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Oscilační teorie pro parciální diferenciální rovnice s p-Laplaciánem

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Abstract

This book is devoted to the study of a partial differential equation with p-Laplacian and nonlinearity of Emden-Fowler type. The equations with p-Laplacian arise in several problem in mathematical physics, such as glaciology, the study of non–Newtonian fluids and slow diffusion problems.

This book collects author's results in the oscillation theory of partial differentaial equations and gives an unified approach to these results. While most papers devoted to the oscillain theory of PDE's are based on an idea to replace a partial differetial equation by its radially symmetric majorant and solve the problem in the scope of theory of ordinary differential equations, the results in this book are somewhat different and more general. In the presented results we try to take into consideration some effects which may appear for partial equations, like non-radial criteria and oscillation on general domains.

The book consists of five chapters. In the first chapter we introduce the basic form of the half-linear PDE with p-Laplacian, explain some basic facts and introduce the notation common for all chapters. Three following chapters are devoted to three different equations (starting from the simplest equation to more general) and the last chapter contains shorter investigations on other related differential equations and inequalities.

Each chapter is a self-contained part of text. For this reason the numbering of equations and theorems is also independent in each chapter and any number of theorem or equation refers to the theorem or equation in actual chapter (unless stated explicitly otherwise).

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Keywords: partial differential equation, second order differential equation, p-Laplacian, oscillation theory

Contents

1 Introduction
1 Half-linear PDE with p-Laplacian
2 Basic facts from oscillation theory
3 Riccati transformation

2 Two terms PDE with p-Laplacian
1 Introduction
2 Hartman–Wintner type oscillation criteria
3 Hille and Nehari type oscillation criteria
4 Oscillation and weighted integral averages
5 Oscillation in general domains
6 Oscillation criteria on half-plane

3 Three terms PDE with p-Laplacian
1 Introduction
2 Riccati inequality
3 Oscillation of three terms half-linear equation
4 Interval type oscillation criteria

4 Three terms elliptic half-linear PDE
1 Introduction
2 Reduction into ODE
3 Applications

5 Related equations and inequalities
1 Inequality with p-Laplacian
2 Equation with degenerated p-Laplacian
3 Variational technique and nonradial criteria

Summary

Souhrn

Bibliography

Chapter 1

Introduction

1 Half-linear PDE with p-Laplacian

In this book we study the partial differential equation with $p-$Laplacian and the nonlinearity of Emden-Fowler type

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) + c(x)\Phi(u) = 0 \tag{1.1}$$

and several its generalizations introduced in subsequent chapters. Here $p > 1$, Φ is signed power function $\Phi(u) = |u|^{p-2}u = |u|^{p-1}\operatorname{sgn} u$, $x = (x_1, x_2, \dots, x_n)$, the vector norm $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^n$, $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ is the usual nabla operator and $\operatorname{div}(\cdot) = \sum_{i=1}^{n} \frac{\partial(\cdot)_i}{\partial x_i}$ is the usual divergence operator. The sets $\Omega(a)$, $\Omega(a, b)$ and $S(a)$ are sets in $\mathbb{R}^n$ defined as follows:

$$\begin{aligned} \Omega(a) &= \left\{x \in \mathbb{R}^n : a \le \|x\|\right\}, \\ \Omega(a,b) &= \left\{x \in \mathbb{R}^n : a \le \|x\| \le b\right\}, \\ S(a) &= \left\{x \in \mathbb{R}^n : \|x\| = a\right\}. \end{aligned}$$

The function $c(x)$ is assumed to be integrable on every compact subset of $\Omega(1)$. It is worth to mention that we do not assume anything concerning either the fixed sign or the radial symmetry of the potential $c(x)$. The solution of Eq. (1.1) we understand every differentiable function $u : \Omega(1) \to \mathbb{R}$ such that $\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_i}$ is differentiable with respect to x_i and u satisfies Eq. (1.1) on $\Omega(1)$.

The number q is the conjugate number to p, i.e., $q = \frac{p}{p-1}$. Among others, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $(p-1)(q-1) = 1$ hold. The number ω_n is the surface area of the unit sphere in $\mathbb{R}^n$ and the vector $\vec{\nu}(x)$ is the normal unit vector to the sphere $S(\|x\|)$ oriented outwards, i.e. $\vec{\nu}(x) = (x_1, \dots, x_n)\|x\|^{-1}$. Integration over the domain $\Omega(a, b)$ is performed introducing hyperspherical coordinates (r, θ), i.e.

$$\int_{\Omega(a,b)} f(x)\,\mathrm{d}x = \int_a^b \int_{S(r)} f\big(x(r,\theta)\big)\,\mathrm{d}\sigma\;\mathrm{d}r\,,$$

where $\mathrm{d}\sigma$ is the integral element of the surface of the sphere $S(r)$.

If $p = 2$, then Eq. (1.1) reduces to the linear Schrödinger equation

$$\Delta u + c(x)u = 0, \tag{1.2}$$

where $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{(\partial x_i)^2}$, and if $n = 1$ then (1.1) reduces to the half–linear ordinary differential equation

$$\Big(\Phi(u')\Big)' + c(x)\Phi(u) = 0. \qquad ' = \frac{\mathrm{d}}{\mathrm{d}x} \tag{1.3}$$

For the basic references about the half–linear equation (1.3) see the monograph by Došlý and Řehák (2005) and papers [Elbert, 1979; Kusano, Naito, Ogata, 1994; Kusano, Naito, 1997; Lomtatidze, 1996; Došlý, 1998; Kandelaki, Lomtatidze, Ugulava, 2000; Došlý, 2000; Došlý, Lomtatidze, 2006].

If the function $c(x)$ is radial, i.e. $c(x) = \widetilde{c}\big(\|x\|\big)$, then the equation for radial solution $u(x) = \widetilde{u}\big(\|x\|\big)$ of Eq. (1.1) becomes

$$\Big(r^{n-1}\Phi(\widetilde{u}')\Big)' + r^{n-1}\widetilde{c}(r)\Phi(\widetilde{u}) = 0, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}r} \tag{1.4}$$

where $r = \|x\|$ and this equation can be transformed into (1.3) by introducing new independent variable $s = r^{(p-n)/(p-1)}$.

If we put both $n = 1$ and $p = 2$, then (1.1) reduces to the ordinary differential equation

$$u'' + c(x)u = 0 \tag{1.5}$$

which has been studied extensively by many authors.

The p-Laplacian is known to be a convenient tool to describe several physical and biological phenomena, see [Díaz, 1985] for more details.

2 Basic facts from oscillation theory

One of the pioneering works in the comparison theory of elliptic partial differential equations is the work of Hartman and Wintner (1955). Probably the first general oscillation criterion for partial differential equations was obtained by Glazman (1958). Oscillation and comparison theory of second order elliptic linear partial differential equation and related equations has been further elaborated in the literature in 60's and 70's in works of Clark, Headley, Kreith, Noussair, Swanson, Travis and others, see [Clark, Swanson, 1965; Headley, Swanson, 1968; Swanson, 1968; Headley, 1970; Kreith, Travis, 1972; Kreith, 1974; Noussair, Swanson, 1979; Swanson, 1979; Noussair, Swanson, 1980; Swanson, 1983; Kreith, 1984] and the references therein.

Many of the oscillation criteria in the literature are based on radialization techniques which convert the problem in n variables into a problem in one variable and thus convert, in some sense, the partial differential equation into ordinary differential equation. Hence many of the methods and results for ordinary differential equations can be applied also for partial differential equations, see for example papers [Fiedler, 1988; Kusano, Naito, Ogata, 1994; Kusano, Naito, 1997] and also Chapter 4 of this thesis. Atakaryev and Toraev [Toraev, 1985; Atakarryev, Toraev, 1986] used direct variational technique rather than radialization methods and obtained oscillation criteria on various types of unbounded domain. Besides the variational technique, Riccati equation and Picone identity are useful tools in comparison and oscillation theory. For references about Riccati type substitution in partial differential equations see e.g. [Noussair, Swanson, 1980; Schminke, 1989; Došlý, Mařík, 2001], for

Picone identity see e.g. [Kreith, 1984; Allegretto, Huang, 1998; Jaroš, Kusano, Yoshida, 2000; Došlý, Mařík, 2001].

For further references concerning oscillatory properties of linear elliptic PDE and several its generalizations see e.g. [Müller–Pfeiffer, 1980; Schminke, 1989; Naito, Naito, Usami, 1997; Mařík, 2000^1; Mařík, 2000^2; Mařík, 2000^3] and the references therein.[1] The reader can see also the papers of Z. Xu and his coauthors [Xing, Xu, 2003; Xing, Xu, 2005; Xu, 2005; Xu, 2006^1; Xu, 2006^2; Xu, 2007; Xia, Xu, 2007] as an up-to-date reference and last progress in this field.

Remark 2.1 (two types of oscillation)**.** A well-known linear oscillation theory is established for Eq. (1.2) . According to this theory, there are two different concepts of oscillation – *weak oscillation* and *strong (nodal) oscillation*. Equation (1.2) is said to be *weakly oscillatory* if every its solution has a zero outside of every ball in $\mathbb{R}^n$ and *strongly oscillatory* if every solution has a nodal domain[2] outside of every ball in $\mathbb{R}^n$. Allegretto (1974) proved that both definitions are equivalent if the function $c(x)$ is sufficiently smooth. Moss and Piepenbrick (1978) improved Allegretto's result and relaxed the conditions on the function $c(x)$ – weak and strong oscillations are equivalent if the function $c(x)$ is locally Hölder continuous. As far as the author knows, the possible equivalence between both types of oscillation remains an open question for (1.1). In this book the weak oscillation is examined.

Definition 2.1 (oscillation)**.** The function u defined on $\Omega(1)$ is said to be *oscillatory*, if the set of the zeros of the function u is unbounded with respect to the norm, i.e. the function u has a zero in $\Omega(t)$ for every $t \geq 1$. Eq. (1.1) is said to be *oscillatory* if every its solution defined on $\Omega(1)$ is oscillatory. Conversely, the equation is said to be *nonoscillatory*, if it is not oscillatory.

Definition 2.2 (oscillation in Ω)**.** Let Ω be an unbounded domain in $\mathbb{R}^n$. The function u defined on $\Omega(1)$ is said to be *oscillatory in the domain* Ω, if the set of zeros of the function u which belong to the closure $\overline{\Omega}$ is unbounded with respect to the norm. Equation (1.1) is said to be *oscillatory in the domain* Ω if every its solution defined on $\Omega(1)$ is oscillatory in Ω. The equation is said to be *nonoscillatory in* Ω if it is not oscillatory in Ω.

Due to the homogeneity of the set of solutions, it follows from the definition that the equation which possesses a solution on $\Omega(1)$ is nonoscillatory, if it has a solution u which is positive on $\Omega(T)$ for some $T > 1$ and oscillatory otherwise. Further, the equation is nonoscillatory in Ω if it has a solution u such that u is positive on $\overline{\Omega} \cap \Omega(T)$ for some $T > 1$ and oscillatory otherwise.

Remark 2.2. The classical oscillation and comparison theory states that Eq. (1.5) is oscillatory if the function $c(x)$ is sufficiently large. This covers the following cases:

(i) The $c(x)$ is sufficiently large for large x – Kneser and Kneser-type oscillation criteria)

(ii) The interval $(1, \infty)$ is allowed to contain parts with small values of the function $c(x)$ in every neighborhood of ∞, but the integral of the function $c(x)$[3] is sufficiently large – Hartman-Wintner, Nehari, Hille, Kamenev and similar oscillation criteria.

[1]The results in [Schminke, 1989] are expressed in spectral terms, concerning the lower spectrum of Schrödinger operator.

[2]A bounded domain $\Omega \subseteq \mathbb{R}^n$ is said to be the nodal domain of a nontrivial solution u of (1.2), if $u|_{\partial\Omega} = 0$.

[3]or, more generally, some integral involving this function

(iii) The integral of the function $c(x)$ is allowed to be small, but there is a sequence of intervals (a_i, b_i) with property $\lim_{i\to\infty} a_i = \infty$ such that the function $c(x)$ is sufficiently large on these intervals – interval-type criteria.

It is known that these classical results can be extended to Eq. (1.1) and thus Eq. (1.1) is oscillatory if the function $c(x)$ is sufficiently large. Most authors keep the terminology explained in this remark[4] also for Eq. (1.1), the only difference is that $c(x)$ is replaced by integral mean value of the function $c(x)$ over spheres centered in the origin.

Remark 2.3 (radial and nonradial criteria)**.** The function $c(x)$ is usually included in the integrals over spheres in absolute majority of oscillation criteria. Let us introduce the following classification of these criteria.

(i) In many cases the oscillation criteria in fact depend on the mean value over spheres centered in the origin only, i.e. on the function $g(r) = \int_{S(r)} c(x)\,\mathrm{d}\sigma$. The criteria of this group[5] will be called *radial oscillation criteria*. As a consequence of the fact that the radial criteria depend on the integral of the potential function over the sphere only it follows that though these criteria are proved to be sharp in the cases when the function $c(x)$ is radially symmetric, these criteria may fail to detect the contingent oscillation of the equation in the cases when the mean value of the function $c(x)$ over the balls centered in the origin is small.

(ii) To remove the disadvantage of radial criteria we derive also several oscillation results in which the distribution of the potential $c(x)$ over spheres is also allowed to play a role. These criteria will be called *nonradial oscillation criteria*.

Let us emphasize that following the nonradial approach we obtain oscillation criteria which are applicable also to the cases when the equation is strongly asymmetric with respect to origin and the mean value of the potential $c(x)$ is small. The possible applications include for example criteria which depend on the function $g(r) = \int_{S(r)} \rho(x)c(x)\,\mathrm{d}\sigma$, where $\rho(x)$ is n-variable function (which does not depend on $\|x\|$ only). The oscillation criteria of this type are applicable also in such extreme cases when $\int_{S(r)} c(x)\,\mathrm{d}S = 0$ and these criteria can be used also to detect oscillation over more general exterior domains, than the exterior of a ball. The author believes that nonradial criteria are more natural for partial differential equations and provide deeper insight into the oscillation properties specific for partial differential equations. Moreover, the oscillation of radially symmetric PDE's can be studied in the scope of ODE's (see Eq. (1.4)) and oscillation of PDE's with "sufficiently large" mean value of the potential function can be detected via oscillation of certain ordinary differential equation, as has been proved independently in [Jaroš, Kusano, Yoshida, 2000] and [Došlý, Mařík, 2001].[6]

Remark that there are only few results in the literature concerning the oscillation on other types of unbounded domains, than an exterior of a ball. Let us mention the paper [Atakarryev, Toraev, 1986], where Kneser–type oscillation criteria for various types of unbounded domains were derived for the linear equation

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + p(x)u = 0.$$

[4] like Kneser, Nehari, Kamenev type criteria

[5] This group covers absolute majority of known results.

[6] The results from [Jaroš, Kusano, Yoshida, 2000] and [Došlý, Mařík, 2001] are cited and extended in Chapter 4 of this thesis.

The forced superlinear equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)|u|^{\beta-1}u = f(x), \quad \beta > 1$$

is studied in the paper [Jaroš, Kusano, Yoshida, 2001] via the Picone identity and the results concerning oscillation on the domains with piecewise smooth boundary are established.

The methods presented in this thesis are often applicable also to several similar equations, like (for $p = 2$) the nonlinear equation

$$\Delta u + c(x)f(u) = 0, \tag{2.1}$$

where the continuous function f satisfies sign condition $uf(u) > 0$ for $u \neq 0$ and $f'(u) > \mu > 0$ for some $\mu \in \mathbb{R}$ and every $u > 0$. However, in order to keep our ideas transparent, we use the term $c(x)\Phi(u)$ rather than replacing this term by a term of the type $c(x)f(u)$ and hence consider the simpler Eq. (1.1) only. Several authors consider even more general quasilinear equation

$$\text{div}\left(\|\nabla u\|^{p-2}\nabla u\right) + B(x,u) = 0, \tag{2.2}$$

where $B(x,u)$ satisfies, roughly speaking, some conditions which imply that (2.2) is a majorant (in the sense of Sturmian theory) for (1.1). Some generalizations of this type are introduced in Chapter 5, see also Remark 2.2 on page 61.

3 Riccati transformation

First we introduce main ideas of Riccati technique, which is the main tool in most our results. It is well known that the Riccati differential equation

$$w' + w^2 + c(x) = 0 \tag{3.1}$$

plays an important role in the study of the second order linear differential equation

$$u'' + c(x)u = 0. \tag{3.2}$$

In fact, if (3.2) has a positive solution u on an interval I, then the function $w = u'/u$ is a solution of (3.1), defined on I. Conversely, if the Riccati *inequality*

$$w' + w^2 + c(x) \leq 0$$

has a solution w, defined on I, then (3.2) has a positive solution on I. Another important aspect which concerns the substitution $w = u'/u$ and terms from Riccati equation is that these terms are embedded into the Picone identity which forms the link between the so-called Riccati technique and variational technique in the oscillation theory of Eq. (3.2) (and its generalizations).

It is also well known that the Riccati type substitution can be extended to several other types of second order differential equations and inequalities, which include the selfadjoint second order differential equation, the half-linear differential equation , the Schrödinger equation and also Eq. (1.1). See for example [Swanson, 1968; Swanson, 1979; Noussair, Swanson, 1980; Schminke, 1989; Kandelaki, Lomtatidze, Ugulava, 2000].[7]

The main idea of the Riccati technique is contained in the following Lemma.

Lemma 3.1. *Let u be solution of* (1.1) *positive on the domain Ω. The vector function $\vec{w}(x)$ defined by*

$$\vec{w}(x) = \frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)} \tag{3.3}$$

is well defined on Ω and satisfies the Riccati equation

$$\operatorname{div}\vec{w} + c(x) + (p-1)\|\vec{w}\|^q = 0 \tag{3.4}$$

for every $x \in \Omega$.

Proof. From (3.3) it follows (the dependence on the variable x is suppressed in the notation)

$$\operatorname{div}\vec{w} = \frac{\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right)}{|u|^{p-2}u} - (p-1)\frac{\|\nabla u\|^p}{|u|^p}$$

on the domain Ω. Since u is a positive solution of (1.1) on Ω it follows

$$\operatorname{div}\vec{w} = -c - (p-1)\frac{\|\nabla u\|^p}{|u|^p}.$$

Application of (3.3) gives $\operatorname{div}\vec{w} = -c - (p-1)\|\vec{w}\|^q$ on Ω. Hence (3.4) follows. □

Sometimes it is convenient to use a modified Riccati substitution multiplied by a smooth function in one variable (see e.g. (1.5) on page 74)

$$\vec{w}(x) = -\alpha(\|x\|)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{\Phi(u(x))}, \quad \alpha \in C^1([a_0,\infty),\mathbb{R}^+),$$

or n variables (see e.g. (4.4) on page 24)

$$\vec{w}(x) = \rho(x)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{\Phi(u(x))}, \quad \rho \in C^1(\Omega(1),\mathbb{R}^+),$$

or by a $n \times n$ matrix (see e.g. (2.3) on page 59)

$$\vec{w}(x) = A(x)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{\Phi(u(x))}, \quad A(x) = (a_{ij}(x)), a_{ij} \in C^1(\Omega(1),\mathbb{R}^+).$$

[7] Concerning the Riccati-equation methods in the oscillation theory of PDE's, [Noussair, Swanson, 1980] used the transformation

$$\vec{w}(x) = -\frac{\alpha(\|x\|)}{\varphi(u)}(A\nabla u)(x)$$

to detect nonexistence of eventually positive solution of the semilinear inequality

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + p(x)\varphi(u) \le 0,$$

which seems to be one of the first papers dealing with the transformation of PDE into the Riccati type equation.

Chapter 2

Two terms PDE with p-Laplacian

1 Introduction

In this chapter we consider the two terms half-linear partial differential equation with p-Laplacian in the form

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) + c(x)\Phi(u) = 0. \tag{1.1}$$

As we pointed out in the previous chapter, if the function $u(x)$ is a function which has no zero on the domain Ω, then the substitution $\vec{w} = \frac{\|\nabla u\|^{p-2}\nabla u}{\Phi(u)}$ converts (1.1) into vector equation

$$\operatorname{div}\vec{w} + c(x) + (p-1)\|\vec{w}\|^{q} = 0, \tag{1.2}$$

where q is a conjugate number to the number p.

The results from the first part of this chapter are motivated by the papers [Chantladze, Kandelaki, Lomtatidze, 1999; Kandelaki, Lomtatidze, Ugulava, 2000] and [Schminke, 1989], where the Riccati technique is used to establish new oscillation criteria for the half–linear ordinary differential equation

$$\Big(\Phi(u')\Big)' + c(x)\Phi(u) = 0, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}x}$$

and the linear Schrödinger equation

$$\Delta u + c(x)u = 0, \tag{1.3}$$

respectively. In the first parts of this chapter we present an extension of Hartman–Wintner, Hille and Nehari type oscillatin ocriteria, proved in [Mařík, 2000^2; Mařík, 2000^3]. The results in the last three parts of this chapter are based on an idea to use Philos's type averaging function $H(t,x)$ in oscillation criteria. It is worth to mention that the idea to use this approach to detect oscillation of partial differential equations in more general domains than exterior of a ball is new even in the linear case (1.3)

The behavior of the following function $C_p(t)$ in a neighborhood of infinity plays a crucial role in the oscillation theory of (1.1)

$$C_p(t) = \frac{p-1}{t^{p-1}}\int_1^t s^{p-2}\int_{\Omega(1,s)} \|x\|^{1-n}c(x)\,\mathrm{d}x\ \mathrm{d}s\,.$$

It turns out to be useful to distinguish two complementary cases (see also [Chantladze, Kandelaki, Lomtatidze, 1999; Kandelaki, Lomtatidze, Ugulava, 2000]). In the first case the finite limit

$$\lim_{t\to\infty} C_p(t)$$

fails to exist and in the second case this limit exists as a finite number C_0, i.e.

$$\lim_{t\to\infty} C_p(t) =: C_0. \tag{1.4}$$

It can be proved that Eq. (1.1) is oscillatory in the first case (see Theorem 2.1 below).

In the second case, following [Chantladze, Kandelaki, Lomtatidze, 1999; Kandelaki, Lomtatidze, Ugulava, 2000], we formulate oscillation criteria for Eq. (1.1) in terms of the functions $C_p(t)$, $Q(t)$, $H(t)$ and numbers Q_*, Q^*, H_*, H^*, A and B defined as follows:

$$\begin{aligned}
&Q(t) = t^{p-1}\left(C_0 - \int_{\Omega(1,t)} \|x\|^{1-n} c(x)\,\mathrm{d}x\right);\\
&H(t) = \frac{1}{t}\int_{\Omega(1,t)} \|x\|^{p-n+1} c(x)\,\mathrm{d}x\ ;\\
&Q_* = \liminf_{t\to\infty} Q(t), \quad Q^* = \limsup_{t\to\infty} Q(t);\\
&H_* = \liminf_{t\to\infty} H(t), \quad H^* = \limsup_{t\to\infty} H(t).
\end{aligned}$$

If $Q_* \leq \left|\frac{p-n}{p}\right|^p \frac{\omega_n}{p-1}$, then the equation

$$(p-1)\omega_n^{-q/p}|x|^q + (n-p)x + (p-1)Q_* = 0 \tag{1.5}$$

has two zeros (including multiplicity, see Lemma 3.1 below). We denote by A the smaller of them. Similarly, if $H_* \leq \left|\frac{p-n}{p}\right|^p \omega_n$, then B denotes the larger of the zeros of the equation

$$(p-1)\omega_n^{-q/p}|x|^q + (n-p)x + H_* = 0. \tag{1.6}$$

First let us explain the role which the above defined functions play in the oscillation theory of Sturm–Liouville ODE

$$u'' + c(x)u = 0. \tag{1.7}$$

The function $C_p(t)$ reads for $p = 2$, $n = 1$ as follows

$$C_2(t) = \frac{1}{t}\int_1^t \int_1^s c(\xi)\,\mathrm{d}\xi\ \mathrm{d}s$$

and it is well known from the Hartman–Wintner oscillation criterion.

Theorem A (Hartman–Wintner)**.** *If*

$$-\infty < \liminf_{t\to\infty} C_2(t) < \limsup_{t\to\infty} C_2(t) \leq \infty, \quad \textit{or} \quad \lim_{t\to\infty} C_2(t) = \infty,$$

then Eq. (1.7) *is oscillatory.*

In the following theorems it is assumed that $c(x)$ is a positive function and the integral

$$\int_1^\infty c(x)\,\mathrm{d}x$$

exists. If it does not, (1.7) is known to be oscillatory [Leighton 1950].

If $c(t) \geq 0$, $n = 1$ and $p = 2$, then

$$Q(t) = t\int_t^\infty c(\xi)\,\mathrm{d}\xi$$

which is the function from the following Hille oscillation criterion (see [Swanson, 1968, Theorem 2.1]).

Theorem B (Hille)**.** *Let $c(x)$ be positive. The conditions*

$$\liminf_{t\to\infty} t\int_t^\infty c(x)\,\mathrm{d}x \;\leq \frac{1}{4}$$
$$\limsup_{t\to\infty} t\int_t^\infty c(x)\,\mathrm{d}x \;\leq 1$$

are necessary conditions for (1.7) *to be nonoscillatory.*

In our notation, Hille proved that Eq. (1.7) is oscillatory if $c(x)$ is positive and either $Q_* > \frac{1}{4}$, or $Q^* > 1$.

Nehari (1957) proved the following oscillation criterion.

Theorem C (Nehari)**.** *Let $c(x)$ be positive. The condition*

$$\limsup_{t\to\infty} \frac{1}{t}\int_1^t x^2 c(x)\,\mathrm{d}x \;> 1$$

is sufficient for Eq. (1.7) *to be oscillatory.*

For $n = 1$, $p = 2$, $c(x) \geq 0$ the Nehari's sufficient condition for oscillation of ordinary linear differential equation (1.7) can be written as $H^* > 1$.

In this sense the criteria including the functions $H(t)$, $Q(t)$ and limes inferior (superior) of them will be referred as Hille and Nehari type.

2 Hartman–Wintner type oscillation criteria

We start with investigations of nonoscillatory equation and provide an integral characterization of the fact that finite limit (1.4) exists.

Lemma 2.1. *Let $\vec{w}$ be the solution of Riccati equation* (1.2) *defined on $\Omega(a)$ for some $a > 1$. The following statements are equivalent*:

(i)

$$\int_{\Omega(a)} \|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}x \;< \infty; \tag{2.1}$$

(ii) *there exists a finite limit* (1.4)

(iii)

$$\liminf_{t\to\infty} C_p(t) > -\infty. \tag{2.2}$$

Proof. We multiply the Riccati equation (1.2) by $\|x\|^{1-n}$ and integrate on $\Omega(a,t)$. Application of the identity

$$\|x\|^{1-n}\operatorname{div}\vec{w} = \operatorname{div}(\|x\|^{1-n}\vec{w}) - (1-n)\|x\|^{-n}\langle\vec{w},\vec{\nu}\rangle,$$

and Gauss divergence theorem yield

$$\begin{aligned}\int_{S(t)}\|x\|^{1-n}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma - \int_{S(a)}\|x\|^{1-n}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma \\ -(1-n)\int_{\Omega(a,t)}\|x\|^{-n}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}x + (p-1)\int_{\Omega(a,t)}\|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}x \\ +\int_{\Omega(a,t)}\|x\|^{1-n}c(x)\,\mathrm{d}x = 0.\end{aligned} \tag{2.3}$$

We prove three implications: “(i)=>(ii)”, “(ii)=>(iii)” and “(iii)=>(i)”.

“(i)=>(ii)” Suppose that (2.1) holds. The Hölder inequality implies

$$\begin{aligned}\int_{\Omega(a,t)}\|x\|^{1-n}|\langle\vec{w},\vec{\nu}\rangle|\,\mathrm{d}x &\le \left(\int_{\Omega(a,t)}\|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}x\right)^{1/q}\left(\int_{\Omega(a,t)}\|x\|^{1-n-p}\,\mathrm{d}x\right)^{1/p} \\ &\le \left(\int_{\Omega(a)}\|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}x\right)^{1/q}\left(\omega_n\int_a^t s^{-p}\,\mathrm{d}s\right)^{1/p}.\end{aligned}$$

Hence

$$\int_{\Omega(a)}\|x\|^{-n}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}x \le \infty. \tag{2.4}$$

Denote

$$\begin{aligned}\widehat{C} = -(p-1)\int_{\Omega(a)}\|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}x + \int_{S(a)}\|x\|^{1-n}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma \\ +(1-n)\int_{\Omega(a)}\|x\|^{-n}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma + \int_{\Omega(1,a)}\|x\|^{1-n}c(x)\,\mathrm{d}x.\end{aligned}$$

We will show that $\widehat{C} = C_0$. Equation (2.3) can be written in the form

$$\begin{aligned}\widehat{C} - \int_{\Omega(1,t)}\|x\|^{1-n}c(x)\,\mathrm{d}x = \int_{S(t)}\|x\|^{1-n}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma \\ -(p-1)\int_{\Omega(t)}\|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}x + (1-n)\int_{\Omega(t)}\|x\|^{-n}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

Multiplying (2.5) by t^{p-2}, integrating over $[a,t]$ and multiplying by $\frac{p-1}{t^{p-1}}$ we obtain

$$\widehat{C} - \left(\frac{a}{t}\right)^{p-1} \left[\widehat{C} - C_p(a)\right] - C_p(t) = \frac{p-1}{t^{p-1}} \int_a^t s^{p-2} \int_{S(s)} \|x\|^{1-n} \langle \vec{w}, \vec{\nu} \rangle \,\mathrm{d}\sigma \,\mathrm{d}s$$
$$- \frac{(p-1)^2}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(s)} \|x\|^{1-n} \|\vec{w}\|^q \,\mathrm{d}x \,\mathrm{d}s$$
$$+ \frac{(1-n)(p-1)}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(s)} \|x\|^{-n} \langle \vec{w}, \vec{\nu} \rangle \,\mathrm{d}x \,\mathrm{d}s\,. \quad (2.6)$$

The second and the third integrals on the right hand side tend to zero as t tends to infinity in view of (2.1) and (2.4). The Hölder inequality implies

$$\left| \frac{1}{t^{p-1}} \int_a^t s^{p-2} \int_{S(s)} \|x\|^{1-n} \langle \vec{w}, \vec{\nu} \rangle \,\mathrm{d}\sigma \,\mathrm{d}s \right|$$
$$\le \frac{1}{t^{p-1}} \int_a^t s^{p-2} \left(\int_{S(s)} \|x\|^{1-n} \|\vec{w}\|^q \,\mathrm{d}\sigma \right)^{1/q} \omega_n^{1/p} \,\mathrm{d}s$$
$$\le \frac{\omega_n^{1/p}}{t^{p-1}} \left(\int_{\Omega(a,t)} \|x\|^{1-n} \|\vec{w}\|^q \,\mathrm{d}x \right)^{1/q} \left(\int_0^t s^{p^2-2p} \,\mathrm{d}s \right)^{1/p} \quad (2.7)$$
$$\le \frac{\omega_n^{1/p}}{(p-1)^{2/p}} \left(\int_{\Omega(a)} \|x\|^{1-n} \|\vec{w}\|^q \,\mathrm{d}x \right)^{1/q} t^{\frac{(p-1)^2}{p} - (p-1)}$$

and the first integral in (2.6) tends to zero too. Hence

$$\lim_{t\to\infty} C_p(t) = \widehat{C} = C_0.$$

The implication "(ii)=>(iii)" is trivial.

"(iii)=>(i)" Suppose, by contradiction, that (2.2) holds and

$$\int_{\Omega(a)} \|x\|^{1-n} \|\vec{w}\|^q \,\mathrm{d}x = +\infty.$$

From (2.3) we get

$$\frac{1}{t^{p-1}} \int_a^t s^{p-2} \int_{S(s)} \|x\|^{1-n} \langle \vec{w}, \vec{\nu} \rangle \,\mathrm{d}\sigma \,\mathrm{d}s$$
$$+ \frac{p-1}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{1-n} \|\vec{w}\|^q \,\mathrm{d}x \,\mathrm{d}s$$
$$- \frac{1-n}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{-n} \langle \vec{w}, \vec{\nu} \rangle \,\mathrm{d}x \,\mathrm{d}s \quad (2.8)$$
$$= \frac{1}{t^{p-1}} \int_a^t s^{p-2} \,\mathrm{d}s \int_{S(a)} \|x\|^{1-n} \langle \vec{w}, \vec{\nu} \rangle \,\mathrm{d}\sigma$$
$$- \frac{1}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{1-n} c(x) \,\mathrm{d}x \,\mathrm{d}s\,.$$

Define the function

$$v(t) := (p-1) \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{1-n} \|\vec{w}\|^q \,\mathrm{d}x \,\mathrm{d}s\,.$$

The function v satisfies

$$\frac{v(t)}{t^{p-1}} \to \infty \text{ for } t \to \infty. \tag{2.9}$$

Because of the right hand side of the equality (2.8) is bounded from above, there exists t_a such that the right hand side of (2.8) is less than $\frac{v(t)}{3t^{p-1}}$ for $t \geq t_a$. Now we have from (2.8)

$$\frac{2}{3}v(t) < \left| \int_a^t s^{p-2} \int_{S(s)} \|x\|^{1-n} \langle \vec{w}, \vec{\nu} \rangle \, \mathrm{d}\sigma \, \mathrm{d}s \right| + \left| (1-n) \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{-n} \langle \vec{w}, \vec{\nu} \rangle \, \mathrm{d}x \, \mathrm{d}s \right| \tag{2.10}$$

for $t \geq t_a$. Similarly to (2.7) we have

$$\left| \int_a^t s^{p-2} \int_{S(s)} \|x\|^{1-n} \langle \vec{w}, \vec{\nu} \rangle \, \mathrm{d}\sigma \, \mathrm{d}s \right| \leq \left(\int_{\Omega(a,t)} \|x\|^{1-n} \|\vec{w}\|^q \, \mathrm{d}x \right)^{1/q} \frac{\omega_n^{1/p} t^{\frac{(p-1)^2}{p}}}{(p-1)^{2/p}} = K \left(t v'(t) \right)^{1/q}, \tag{2.11}$$

where $K = \omega_n^{1/p}(p-1)^{-\frac{2}{p}-\frac{1}{q}}$. The Hölder inequality gives

$$\begin{aligned}
&\left| (1-n) \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{-n} \langle \vec{w}, \vec{\nu} \rangle \, \mathrm{d}x \, \mathrm{d}s \right| \\
&\leq (n-1) \int_a^t s^{p-2} \left(\int_{\Omega(a,t)} \|x\|^{1-n} \|\vec{w}\|^q \, \mathrm{d}x \right)^{1/q} \left(\int_1^\infty \omega_n \xi^{-p} \, \mathrm{d}\xi \right)^{1/p} \mathrm{d}s \\
&\leq (n-1) \left(\int_a^t s^{p-2} \int_{\Omega(a,t)} \|x\|^{1-n} \|\vec{w}\|^q \, \mathrm{d}x \, \mathrm{d}s \right)^{1/q} \\
&\qquad \times \left(\int_1^\infty \omega_n s^{-p} \, \mathrm{d}s \right)^{1/p} \left(\int_0^t s^{p-2} \, \mathrm{d}s \right)^{1/p} \mathrm{d}s \\
&= (n-1) \Big(\frac{v(t)}{p-1} \Big)^{1/q} \frac{t^{\frac{p-1}{p}} \omega_n^{1/p}}{(p-1)^{2/p}} \\
&= \frac{(n-1)\omega_n^{1/p}}{(p-1)^{\frac{1}{q}+\frac{2}{p}}} v^{1/q}(t) t^{\frac{p-1}{p}}.
\end{aligned} \tag{2.12}$$

In view of the fact (2.9) there exists a number $t_b \geq t_a$ such that

$$\frac{(n-1)\omega_n^{1/p}}{(p-1)^{\frac{1}{q}+\frac{2}{p}}} t^{\frac{p-1}{p}} \leq \frac{1}{3} v^{1/p}(t) \tag{2.13}$$

for $t \geq t_b$. Combining (2.10), (2.11), (2.12) and (2.13) we get

$$\frac{1}{3}v(t) \leq K \big(t v'(t) \big)^{1/q}$$

for $t \geq t_b$. From here

$$\frac{v'(t)}{v^q(t)} \geq \frac{1}{t} \left(\frac{1}{3\,K} \right)^q$$

for $t \geq t_b$. Integration of this inequality from t_b to ∞ gives a convergent integral on the left hand side and divergent integral on the right hand side. This contradiction ends the proof. □

The following oscillation criterion now follows immediately from Lemma 2.1.

Theorem 2.1 (Hartman–Wintner type oscillation criterion)**.** *If*

$$-\infty < \liminf_{t\to\infty} C_p(t) < \limsup_{t\to\infty} C_p(t) \leq \infty$$

or if

$$\lim_{t\to\infty} C_p(t) = \infty,$$

then Eq. (1.1) *is oscillatory.*

Proof. From the assumptions of the theorem it follows that $\liminf_{t\to\infty} C_p(t) > -\infty$. Suppose, by contradiction, that (1.1) is nonoscillatory. If there is a number $a > 1$ such that (1.1) has a solution positive on $\Omega(a)$, then Theorem 2.1 implies that there exists a finite limit $\lim_{t\to\infty} C_p(t)$. This contradicts the assumptions and theorem is proved. □

Remark 2.1. Theorem 2.1 extends Theorem A proved using Riccati technique in monograph [Hartman, 1964, Chap. XI] for linear ordinary differential equation. Remark that there are several possibilities how to formulate the extension of this classical criterion. Another generalization of this result was published by Došlý and Mařík (2001, Theorem 3.4) under an additional condition $p \geq n+1$. Here, in Theorem 2.1, we proved an oscillation criterion without any restriction on p.

Corollary 2.1 (Leighton–Wintner type criterion)**.** *If*

$$\lim_{t\to\infty} \int_{\Omega(1,t)} \|x\|^{1-n} c(x)\,\mathrm{d}x \ = \infty, \tag{2.14}$$

then Eq. (1.1) *is oscillatory.*

Proof. If (2.14) holds, then $\lim_{t\to\infty} C_p(t) = \infty$ and the statement follows from Theorem 2.1. □

3 Hille and Nehari type oscillation criteria

Recall that we will suppose that the finite limit

$$\lim_{t\to\infty} C_p(t) =: C_0$$

exists and we formulate oscillation criteria in terms of functions $C_p(t)$, $Q(t)$, $H(t)$ and numbers Q_*, Q^*, H_*, H^*, A and B defined on page 8.

The following lemma simply ensures that the numbers A and B are well defined. More precisely, we prove that Eqs. (1.5) and (1.6) are solvable and introduce some useful properties of the function defined by the left hand side of these equations.

Lemma 3.1. *Let $\alpha \in \mathbb{R}$ be arbitrary number. The function*

$$y(x) = (p-1)\omega_n^{-q/p}|x|^q + \alpha x$$

has the following properties:

(i) $y(x)$ *has its global minimum at the point* $\widehat{x} = -\omega_n \Phi\left(\dfrac{\alpha}{p}\right)$ *and* $y(\widehat{x}) = -\omega_n \left|\dfrac{\alpha}{p}\right|^p$*;*

(ii) $y(x)$ *is decreasing on* $(-\infty, \widehat{x}]$ *and increasing on* $[\widehat{x}, \infty)$*;*

(iii) $\lim_{t\to\pm\infty} y(x) = \infty$.

Proof. Follows immediately from computation $y' = (p-1)q\omega_n^{-q/p}|x|^{q-2}x + \alpha$. □

If (1.1) is nonoscillatory and $\vec{w}$ is the solution of (1.2) defined on $\Omega(a)$, then denote

$$\rho(t) = t^{p-1}\int_{S(t)} \|x\|^{1-n}\langle \vec{w}, \vec{\nu}\rangle \,\mathrm{d}\sigma \qquad \text{for } t \geq a; \tag{3.1}$$

$$r = \liminf_{t\to\infty} \rho(t), \qquad R = \limsup_{t\to\infty} \rho(t);$$

$$\widehat{A} = -\Phi\left(\frac{n-1}{p}\right)\omega_n, \qquad \widehat{B} = -\Phi\left(\frac{n-p-1}{p}\right)\omega_n.$$

Below we prove appriori bounds for the numbers r, R. We start with one technical lemma.

Lemma 3.2. *Let* (1.4) *holds,* (1.1) *be nonoscillatory,* $\vec{w}$ *be the solution of* (1.2) *defined on* $\Omega(a)$ *for some* $a > 1$ *and* $\rho(t)$ *the function defined by* (3.1)*. The following estimations are true for every* $t \geq \tau \geq a$:

$$|\rho(t)|^q \leq \omega_n^{q/p} t^p \int_{S(t)} \|x\|^{1-n}\|\vec{w}\|^q \,\mathrm{d}\sigma\,; \tag{3.2}$$

$$Q(t) \leq \rho(t) - t^{p-1}\int_t^\infty \Big[(p-1){\omega_n}^{-q/p}|\rho(s)|^q - (1-n)\rho(s)\Big] s^{-p}\,\mathrm{d}s\,; \tag{3.3}$$

$$\begin{aligned} H(t) \leq& -\rho(t) - \frac{1}{t}\int_\tau^t \Big[(p-1)\omega_n^{-q/p}|\rho(s)|^q - (p-n+1)\rho(s)\Big]\,\mathrm{d}s \\ &+ \frac{\tau}{t}\Big[\rho(\tau) + H(\tau)\Big]; \end{aligned} \tag{3.4}$$

$$Q(t) \leq \rho(t) + \left|\frac{1-n}{p}\right|^p \frac{\omega_n}{p-1}; \tag{3.5}$$

$$H(t) \leq -\rho(t) + \left|\frac{p-n+1}{p}\right|^p \omega_n + \frac{\tau}{t}\Big[\rho(\tau) + H(\tau)\Big]. \tag{3.6}$$

Proof. The inequality (3.2) follows from the definition of the function $\rho(t)$ and from the Schwarz and Hölder inequalities. Multiplying Eq. (2.5) by t^{p-1} we obtain

$$Q(t) = \rho(t) - t^{p-1}\int_t^\infty \left[(p-1)\int_{S(s)} \|x\|^{1-n}\|\vec{w}\|^q \,\mathrm{d}s\, - (1-n)\rho(s)s^{-p}\right]\mathrm{d}s\,.$$

Now (3.2) implies (3.3). Differentiating (2.5) we obtain

$$
\begin{aligned}
-\int_{S(t)} \|x\|^{1-n} c(x)\,\mathrm{d}\sigma \;&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho(t)}{t^{p-1}} \\
&\quad + (p-1)\int_{S(t)} \|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}\sigma \;-(1-n)\int_{S(t)} \|x\|^{-n}\left\langle \vec{w},\vec{\nu}\right\rangle\,\mathrm{d}\sigma\,.
\end{aligned}
$$

Multiplying this equality by t^p and integrating from $\tau \geq a$ to t we obtain

$$
\begin{aligned}
-\int_{\Omega(\tau,t)} \|x\|^{p-n+1} c(x)\,\mathrm{d}x \;&= t\rho(t) - \tau\rho(\tau) - p\int_\tau^t \rho(s)\,\mathrm{d}s \\
&\quad + \int_\tau^t \left[(p-1)s^p\int_{S(s)} \|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}\sigma \;-(1-n)\rho(s)\right]\mathrm{d}s\,.
\end{aligned}
$$

Now the fact that

$$
tH(t) - \tau H(\tau) = \int_{\Omega(\tau,t)} \|x\|^{p-n+1} c(x)\,\mathrm{d}x
$$

combined with (3.2) implies (3.4). Terms in brackets in the inequalities (3.3) and (3.4) can be estimated using Lemma 3.1 and after integration we get (3.5) and (3.6). Note that $\widehat{A}$ and $\widehat{B}$ are the points, which, substituted to $\rho(s)$ in (3.3) and (3.4), realize the minimum value of the function in brackets. □

Theorem 3.1. *Let* (1.4) *holds and*

$$
\limsup_{t\to\infty} \frac{t^{p-1}}{\ln t}\left[C_0 - C_p(t)\right] > \left|\frac{p-n}{p}\right|^p \omega_n. \tag{3.7}
$$

Then Eq. (1.1) *is oscillatory.*

Proof of Theorem 3.1. Suppose, by contradiction, that (1.4) holds, there exists a solution $w(x)$ of the Riccati equation defined on $\Omega(a)$ for some $a > 1$ and (3.7) does not hold. We start with the equality (2.5). Multiplying it by $(p-1)t^{p-2}$, integrating over $[a,t]$ and using (3.1) we obtain

$$
\begin{aligned}
t^{p-1}\Big(C_0 - C_p(t)\Big) - a^{p-1}C_0 &= (p-1)\int_a^t \frac{\rho(s)}{s}\,\mathrm{d}s \;-(p-1)t^{p-1}\int_{\Omega(t)} \|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}x \\
&\quad + (p-1)a^{p-1}\int_{\Omega(a)} \|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}x \\
&\quad - (p-1)\int_a^t s^{p-1}\int_{S(s)} \|x\|^{1-n}\|\vec{w}\|^q\,\mathrm{d}\sigma\;\mathrm{d}s \\
&\quad + (1-n)t^{p-1}\int_{\Omega(t)} \|x\|^{-n}\left\langle \vec{w},\vec{\nu}\right\rangle\,\mathrm{d}x \;-(1-n)a^{p-1}\int_{\Omega(a)} \|x\|^{-n}\left\langle \vec{w},\vec{\nu}\right\rangle\,\mathrm{d}x \\
&\quad + (1-n)\int_a^t \frac{\rho(s)}{s}\,\mathrm{d}s\,,
\end{aligned}
$$

where on the right hand side the last six terms appeared from integration by parts from the last two terms in (2.5). From here and from the inequality (3.2) we conclude

$$t^{p-1}\Big(C_0 - C_p(t)\Big) \le K - t^{p-1}\int_t^\infty \Big[(p-1)\omega_n^{-q/p}|\rho(s)|^q - (1-n)\rho(s)\Big] s^{-p}\,\mathrm{d}s$$
$$- \int_a^t \Big[(p-1)\omega_n^{-q/p}|\rho(s)|^q - (p-n)\rho(s)\Big]\frac{1}{s}\,\mathrm{d}s\,,$$

where the constant terms are joint in one constant K. The terms in the integrals can be estimated by Lemma 3.1 and after integration we obtain

$$t^{p-1}\left(C_0 - C_p(t)\right) \le K + \left|\frac{1-n}{p}\right|^p \frac{\omega_n}{p-1} + \left|\frac{p-n}{p}\right|^p \omega_n \ln\frac{t}{a} \tag{3.8}$$

for $t \ge a$, which contradicts (3.7). Theorem is proved. □

Suitable modifications of the left hand side of (3.7) lead to the following corollary.

Corollary 3.1. *Let* (1.4) *holds. Every of the following conditions is sufficient for Eq.* (1.1) *to be oscillatory*:

(i) $Q_* > -\infty$ *and*

$$\limsup_{t\to\infty} \frac{1}{\ln t}\int_{1\le\|x\|\le t} \|x\|^{p-n} c(x)\,\mathrm{d}x > \left|\frac{n-p}{p}\right|^p \omega_n; \tag{3.9}$$

(ii)

$$\liminf_{t\to\infty}[Q(t) + H(t)] > \frac{p}{p-1}\left|\frac{n-p}{p}\right|^p \omega_n; \tag{3.10}$$

(iii)

$$Q_* > \frac{1}{p-1}\left|\frac{n-p}{p}\right|^p \omega_n; \tag{3.11}$$

(iv)

$$H_* > \left|\frac{n-p}{p}\right|^p \omega_n. \tag{3.12}$$

Proof of Corollary 3.1. Suppose that (1.4) holds. From the definition of the function $Q(t)$ it follows

$$t^{p-1}\Big(C_0 - C_p(t)\Big) = Q(t) + t^{p-1}\int_{\Omega(1,t)} \|x\|^{1-n} c(x)\,\mathrm{d}x$$
$$- (p-1)\int_1^t s^{p-2}\int_{\Omega(1,s)} \|x\|^{1-n} c(x)\,\mathrm{d}x\;\mathrm{d}s\,.$$

Integration by parts in the second integral gives

$$t^{p-1}\Big(C_0 - C_p(t)\Big) = Q(t) + \int_{\Omega(1,t)} \|x\|^{p-n} c(x)\,\mathrm{d}x\,. \tag{3.13}$$

Now the statement (i) follows from Theorem 3.1. Similarly from the definition of the function $Q(t)$ it follows

$$t^{p-1}\Big(C_0 - C_p(t)\Big) = (p-1)\int_1^t \frac{Q(s)}{s}\,\mathrm{d}s + C_0.$$

From here and from Theorem 3.1 it follows that (1.1) is oscillatory if (iii) holds. Integrating by parts in the last equality we have

$$t^{p-1}\Big(C_0 - C_p(t)\Big) = (p-1)\left[\frac{1}{t}\int_1^t Q(s)\,\mathrm{d}s + \int_1^t \frac{1}{s^2}\int_1^s Q(\xi)\,\mathrm{d}\xi\,\mathrm{d}s\right] + C_0. \quad (3.14)$$

Further

$$\begin{aligned}\int_1^t Q(s)\,\mathrm{d}s =& tQ(t) - Q(1) - \int_1^t sQ'(s)\,\mathrm{d}s\\ =& tQ(t) - C_0 - C_0\frac{p-1}{p}(t^p - 1) + (p-1)\int_1^t s^{p-1}\int_{\Omega(1,s)} \|x\|^{1-n}c(x)\,\mathrm{d}x\,\mathrm{d}s\\ &+ \int_1^t s^p \int_{S(s)} \|x\|^{1-n}c(x)\,\mathrm{d}\sigma\,\mathrm{d}s\,.\end{aligned}$$

Integration by parts in the first integral on the right hand side and the definitions of the functions $Q(t)$ and $H(t)$ gives

$$\int_1^t Q(s)\,\mathrm{d}s = \frac{1}{p}\Big(tQ(t) + tH(t) - C_0\Big).$$

Hence

$$Q(t) + H(t) = \frac{p}{t}\int_1^t Q(s)\,\mathrm{d}s + \frac{C_0}{t}$$

holds for every $t \geq 1$. If (ii) holds, then

$$\liminf_{t\to\infty} \frac{1}{t}\int_1^t Q(s)\,\mathrm{d}s > \frac{1}{p-1}\left|\frac{n-p}{p}\right|^p \omega_n$$

and the equality (3.14) with the Theorem 3.1 implies oscillation of Eq. (1.1). The last statement follows from the equalities

$$C_p(t) - C_p(\tau) = (p-1)\int_\tau^t \frac{\ln s}{s^p}\frac{1}{\ln s}\int_{\Omega(1,s)} \|x\|^{p-n}c(x)\,\mathrm{d}x\,\mathrm{d}s$$

$$\frac{1}{\ln t}\int_{\Omega(1,t)} \|x\|^{p-n}c(x)\,\mathrm{d}x = \frac{H(t)}{\ln t} + \frac{1}{\ln t}\int_1^t \frac{H(s)}{s}\,\mathrm{d}s\,,$$

which can be derived in a similar way. □

For a related result proved by a different technique and for slightly different equation see Corollary 1.2 of Chapter 5 (page 76).

Remark 3.1. If the limit

$$\lim_{t\to\infty}\int_{\Omega(1,t)}\|x\|^{1-n}c(x)\,\mathrm{d}x \tag{3.15}$$

exists, then the limit (1.4) exists too and both limits are equal. If the limit (3.15) is finite, then $Q(t)$ takes the form

$$Q(t)=t^{p-1}\int_{\Omega(t)}\|x\|^{1-n}c(x)\,\mathrm{d}x\,.$$

On the other hand, the existence of (3.15) is not necessary for existence of the limit (1.4). For $p=2$, $n=1$, $c(x)\geq 0$ is the criterion (3.11) the well–known Hille's criterion.

If $p=n$, then the oscillation constant in Theorem 3.1 and Corollary 3.1 equals zero. In this case the criteria including lim sup, i.e. criteria (3.7) and (3.9), can be restated in the following sharper form.

Theorem 3.2. *Let* $p=n$ *and* (1.4) *holds. Each of the following conditions guarantees oscillation of Eq.* (1.1):

(i)

$$\limsup_{t\to\infty} t^{n-1}[C_0-C_p(t)]=\infty;$$

(ii)

$$Q_*>-\infty \quad \text{and} \quad \limsup_{t\to\infty}\int_{\Omega(1,t)}c(x)\,\mathrm{d}x=\infty.$$

The following theorem completes the criterion (3.10) in some sense.

Theorem 3.3. *Let* (1.4) *and*

$$\limsup_{t\to\infty}[Q(t)+H(t)]>\left|\frac{1-n}{p}\right|^p\frac{\omega_n}{p-1}+\left|\frac{p-n+1}{p}\right|^p\omega_n. \tag{3.16}$$

Then Eq. (1.1) *is oscillatory.*

Proof of Theorem 3.3. Suppose, by contradiction, that Eq. (1.1) is nonoscillatory and (1.4) holds. Then the inequalities (3.5) and (3.6) holds. The sum of this two inequalities contradicts (3.16). This contradiction ends the proof. □

Remark 3.2. Putting $x=\frac{1}{p-n+1}$ into the inequality

$$|1-px|^p+p-1\geq p|1-x|^p$$

and multiplying it by $\left|\frac{p-n+1}{p}\right|^p\frac{\omega_n}{p-1}$ we obtain

$$\left|\frac{1-n}{p}\right|^p\frac{\omega_n}{p-1}+\left|\frac{p-n+1}{p}\right|^p\omega_n\geq\frac{p}{p-1}\left|\frac{n-p}{p}\right|^p\omega_n.$$

For $p=n-1$ is this inequality trivial. From here it follows that the oscillation constant in the criterion (3.10), $\liminf_{t\to\infty}[Q(t)+H(t)]$, is smaller than that one in the criterion (3.16), $\limsup_{t\to\infty}[Q(t)+H(t)]$.

In view of the previous results, it is natural to focus our attention to the cases, when (3.11) and (or) (3.12) does not hold. Suppose

$$\frac{(n-1)-p(p-1)}{p(p-1)}\Phi\left(\frac{n-1}{p}\right)\omega_n \leq Q_* \leq \left|\frac{n-p}{p}\right|^p \frac{\omega_n}{p-1}, \tag{3.17}$$

and (or)

$$\frac{1-n}{p}\Phi\left(\frac{p-n+1}{p}\right)\omega_n \leq H_* \leq \left|\frac{n-p}{p}\right|^p \omega_n. \tag{3.18}$$

Remark 3.3. Putting $x = \frac{p-1}{n-1}$ into the inequality

$$1 - px \leq |1-x|^p \tag{3.19}$$

and multiplying this inequality by $\left|\frac{n-1}{p-1}\right|^p \frac{\omega_n}{p-1}$ we obtain

$$\frac{(n-1)-p(p-1)}{p(p-1)}\Phi\left(\frac{n-1}{p}\right)\omega_n \leq \left|\frac{n-p}{p}\right|^p \frac{\omega_n}{p-1}.$$

In the case $n = 1$ is this inequality trivial. Similarly we can obtain

$$\frac{1-n}{p}\Phi\left(\frac{p-n+1}{p}\right)\omega_n \leq \left|\frac{n-p}{p}\right|^p \omega_n$$

from (3.19) choosing $x = \frac{1}{p-n+1}$ and multiplying by $\left|\frac{p-n+1}{p}\right|^p \omega_n$. Hence both (3.17) and (3.18) are meaningful.

In the folowing two technical lemmas we present an estimate for numbers A and B. Recall that these numbers are defined on page 8.

Lemma 3.3. *Let* (1.1) *be nonoscillatory and let* (1.4) *and* (3.17) *hold. Then*

$$r \geq A \geq \widehat{A}, \tag{3.20}$$

where A denotes the smaller of zeros of Eq. (1.5)*.*

Proof. Let w be the solution of the Riccati equation (1.2) defined on $\Omega(a)$, $a > 1$. If $r = \infty$, there is nothing to prove. Suppose that $r < \infty$. If $Q_* = \frac{(n-1)-p(p-1)}{p(p-1)}\Phi\left(\frac{n-1}{p}\right)\omega_n$ then $\widehat{A}$ solves Eq. (1.5) and lies on the left hand side from the global minimum of the function defined by the left hand side of this equation. Then Lemma 3.1 implies that $A = \widehat{A}$. Now inequality (3.5) implies

$$r \geq Q_* - \left|\frac{1-n}{p}\right|^p \frac{\omega_n}{p-1} = \widehat{A} = A$$

and (3.20) holds.

Suppose that $\varepsilon_0 := Q_* - \frac{(n-1)-p(p-1)}{p(p-1)}\Phi\left(\frac{n-1}{p}\right)\omega_n > 0$. Then clearly $A > \widehat{A}$. For every $\varepsilon \in (0, \varepsilon_0)$ there exists $t_\varepsilon > a$ such that for every $t \geq t_\varepsilon$ the following inequality holds.

$$\rho(t) \geq r - \varepsilon > r - Q_* + \frac{(n-1)-p(p-1)}{p(p-1)}\Phi\left(\frac{n-1}{p}\right)\omega_n$$

From the inequality (3.5) we have

$$Q_* \le r + \left|\frac{1-n}{p}\right|^p \frac{\omega_n}{p-1}$$

and combining the last two inequalities we get

$$\rho(t) > r - \varepsilon > -\Phi\left(\frac{n-1}{p}\right)\omega_n$$

for $t \ge t_\varepsilon$. From here and from Lemma 3.1 it follows that the right hand side of (3.3) can be increased substituting $\rho(s)$ by $r-\varepsilon$, if $t \ge t_\varepsilon$. Hence, after integration, we have

$$(p-1)Q(t) < (p-1)\rho(t) - (p-1)\omega_n^{-q/p}|r-\varepsilon|^q + (1-n)(r-\varepsilon)$$

for every $t \le t_\varepsilon$. The limit process $\lim_{\varepsilon\to 0^+} \liminf_{t\to\infty}$ gives

$$(p-1)\omega_n^{-q/p}|r|^q + (n-p)r + (p-1)Q_* \le 0$$

which implies $r \ge A$. The lemma is proved. □

Lemma 3.4. *Let* (1.4) *and* (3.18) *hold and let* (1.1) *be nonoscillatory. Then*

$$R \le B \le \widehat{B}, \tag{3.21}$$

where the number B denotes the larger zero of Eq. (1.6).

Proof. The proof is almost the same as the proof of Lemma 3.3. Let w be the solution of the Riccati equation (1.2) defined on $\Omega(a)$, $a > 1$. If $R = -\infty$, there is nothing to prove. Suppose that $R > -\infty$. If $H_* = \frac{1-n}{p}\Phi\left(\frac{p-n+1}{p}\right)\omega_n$ then $\widehat{B}$ solves Eq. (1.6) and lies on the right hand side from the global minimum of the function defined by the left hand side of this equation. Then Lemma 3.1 implies that $B = \widehat{B}$. Now the inequality (3.6) implies

$$R \le -H_* + \left|\frac{p-n+1}{p}\right|^p \omega_n = \widehat{B} = B$$

and (3.21) holds.

Suppose that $\varepsilon_0 := H_* - \frac{1-n}{p}\Phi\left(\frac{p-n+1}{p}\right)\omega_n > 0$. Then clearly $B < \widehat{B}$. For every $\varepsilon \in (0, \varepsilon_0)$ there exists $t_\varepsilon > a$ such that for every $t \ge t_\varepsilon$

$$\rho(t) \le R + \varepsilon < R + H_* - \frac{1-n}{p}\Phi\left(\frac{p-n+1}{p}\right)\omega_n$$

From the inequality (3.6) we have

$$H_* \le -R + \left|\frac{p-n+1}{p}\right|^p \omega_n.$$

Combining the last two inequalities we get

$$\rho(t) < R + \varepsilon < -\Phi\left(\frac{p-n+1}{p}\right)\omega_n$$

for $t \geq t_\varepsilon$. From here and from Lemma 3.1 it follows that the right hand side of (3.4) can be increased substituting $\rho(s)$ by $R+\varepsilon$, if $\tau \geq t_\varepsilon$. Hence, after integration, we have

$$H(t) < -\rho(t)-(p-1)\omega_n^{-q/p}|R+\varepsilon|^q+(p-n+1)(R+\varepsilon)\left(1-\frac{\tau}{t}\right)+\frac{\tau}{t}\Big[\rho(\tau)+H(\tau)\Big]$$

for every $t \leq t_\varepsilon$. The limit process $\lim_{\varepsilon\to 0^+}\liminf_{t\to\infty}$ gives

$$(p-1)\omega_n^{-q/p}|R|^q+(n-p)R+H_* \leq 0$$

which implies $R \leq B$. The lemma is proved. □

In the following theorem we suppose that only one of the inequalities (3.17) and (3.18) hold.

Theorem 3.4. *Let* (1.4) *holds. Each of the following conditions implies oscillation of Eq.* (1.1):

(i)

$$\text{(3.17) and } H^* > \left|\frac{p-n+1}{p}\right|^p \omega_n - A \text{ hold;} \tag{3.22}$$

(ii)

$$\text{(3.18) and } Q^* > \frac{1}{p-1}\left|\frac{1-n}{p}\right|^p \omega_n + B \text{ hold.} \tag{3.23}$$

Proof of Theorem 3.4. Let us prove (i). Suppose, by contradiction, that (1.1) is nonoscillatory and (3.17) holds. The inequality (3.6) implies

$$H^* \leq -r+\left|\frac{p-n+1}{p}\right|^p \omega_n.$$

From Lemma 3.3 it follows

$$H^* \leq \left|\frac{p-n+1}{p}\right|^p \omega_n - A,$$

a contradiction. The statement (ii) can be proved similarly using inequality (3.6) and Lemma 3.3, which implies

$$Q^* \leq B+\frac{1}{p-1}\left|\frac{n-1}{p}\right|^p \omega_n, \tag{3.24}$$

a contradiction to (3.23). □

If both (3.17) and (3.18) holds, then constants in (3.16), (3.22) and (3.23) can be decreased, as the following theorem shows.

Theorem 3.5. *Let* (1.4)*,* (3.17) *and* (3.18) *holds. Each of the following conditions implies oscillation of Eq.* (1.1):

(i)

$$Q^* > Q_* - A + B; \tag{3.25}$$

(ii)

$$H^* > H_* - A + B;$$

(iii)

$$\limsup_{t\to\infty}[Q(t) + H(t)] > Q_* + H_* - A + B. \tag{3.26}$$

Proof of Theorem 3.5. Suppose that (1.1) is nonoscillatory, (3.17) and (3.18) hold. Suppose $Q_* > \frac{(n-1)-p(p-1)}{p(p-1)}\Phi\big(\frac{n-1}{p}\big)\omega_n$ and $H_* > \frac{1-n}{p}\Phi\big(\frac{p-n+1}{p}\big)\omega_n$. Then $A > \widehat{A}$ and $B < \widehat{B}$. By Lemmas 3.3 and 3.4, for every $\varepsilon \in (0, \min\{A - \widehat{A}, -B + \widehat{B}\})$ there exists t_ε such that

$$\widehat{A} < A - \varepsilon < \rho(t) < B + \varepsilon < \widehat{B}$$

for every $t \geq t_\varepsilon$. Lemma 3.1 implies, that the right hand sides of the inequalities (3.3) and (3.4) can be for $t \geq \tau \geq t_\varepsilon$ increased substituting $\rho(t)$ by $A-\varepsilon$, $B+\varepsilon$, respectively. Hence

$$Q(t) \leq \rho(t) - \omega_n^{-q/p}\Big|A - \varepsilon\Big|^q + \frac{1-n}{p-1}(A - \varepsilon) \tag{3.27}$$

and

$$\begin{aligned} H(t) \leq & -\rho(t) - \Big[(p-1)\omega_n^{-q/p}|B + \varepsilon|^q - (p-n+1)(B+\varepsilon)\Big]\Big[1 - \frac{\tau}{t}\Big] \\ & + \frac{\tau}{t}\Big[\rho(\tau) + H(\tau)\Big] \end{aligned} \tag{3.28}$$

hold for large t and τ. From (3.27) using the limit process $\lim_{\varepsilon\to 0+}\limsup_{t\to\infty}$ and Lemma 3.4 we obtain

$$Q^* \leq B - \omega_n^{-q/p}|A|^q + \frac{1-n}{p-1}A. \tag{3.29}$$

Combining this inequality and Eq. (1.5) we obtain

$$Q^* \leq Q_* - A + B$$

which contradicts (3.25). The condition (i) is proved. The condition (ii) follows in a similar way from (3.28), Lemma 3.3 and Eq. (1.6). The sum of (3.27) and (3.28) gives

$$\begin{aligned} Q(t) + H(t) \leq -\omega_n^{-q/p}\Big|A - \varepsilon\Big|^q + \frac{1-n}{p-1}(A-\varepsilon) - \Big[(p-1)\omega_n^{-q/p}|B+\varepsilon|^q \\ - (p-n+1)(B+\varepsilon)\Big]\Big[1 - \frac{\tau}{t}\Big] + \frac{\tau}{t}\Big[\rho(\tau) + H(\tau)\Big] \end{aligned}$$

The limit process $\lim_{\varepsilon\to 0+}\limsup_{t\to\infty}$, (1.5) and (1.6) imply

$$\limsup_{t\to\infty}[Q(t) + H(t)] \leq Q_* + H_* - A + B,$$

which contradicts (3.26).

If $Q_* = \frac{(n-1)-p(p-1)}{p(p-1)}\Phi\big(\frac{n-1}{p}\big)\omega_n$ ($H_* = \frac{1-n}{p}\Phi\big(\frac{p-n+1}{p}\big)\omega_n$) then $A = \widehat{A}$ ($B = \widehat{B}$) and (3.27) ((3.28)) with $\varepsilon = 0$ follows from the statement (ii) of Lemma 3.1 for every $t \geq a$. The rest of the proof is identical with those one given above. □

The fact that constants from Theorem 3.5 are smaller than those in Theorem 3.4 is explained in Remark 3.5.

Remark 3.4. The right-hand sides in (3.7)–(3.12), (3.25)–(3.26) are optimal and they cannot be increased. This follows from the example of the equation with radial function $c(x) = \left|\frac{p-n}{p}\right|^p \frac{1}{\|x\|^p}$. This equation is nonoscillatory, since $\|x\|^{\frac{p-n}{p}}$ is its solution, and the function $c(x)$ produces equality in the above mentioned criteria.

Remark 3.5. The constants in Theorem 3.5 are smaller than the corresponding ones in Theorems 3.3, 3.4. This follows from the proof of these theorems. Let us show this fact in the case of oscillation criterion Q^*. Lemma 3.1 implies that the constant in inequality (3.29), which was used in proof of Theorem 3.5, is less than or equal to the constant in inequality (3.24) used in Theorem 3.4. If, in addition, the first inequality in (3.17) is sharp, then $A > \widehat{A}$ and the constant in (3.29) is strictly less than that one in (3.24).

Remark 3.6. Recall that the criteria expressed in terms of the functions $C_p(t)$, $Q(t)$ and $H(t)$ are *radial* in the sense of the classification introduced in Remark 2.3 on page 4 – the functions $C_p(t)$, $Q(t)$ and $H(t)$ depend on $\int_{S(r)} c(x)\,\mathrm{d}\sigma$ only and the first step in the proofs is integration of Riccati equation over ball $S(r)$. Preferring integration over the balls in $\mathbb{R}^n$ we loose the information about the distribution of the potential $c(x)$ over the sphere $S(r)$. This makes many things easier and computations more comfortable, but the distribution of potential over spheres may be substantial in cases when the mean value of the function $c(x)$ over spheres is not sufficiently large.

This disadvantage can be removed using integral averaging technique with the so called H-function, as shown in the remaining part of this chapter. In the proofs of the corresponding oscillation criteria we multiply the Riccati equation by a nonradial function first and then integrate this equation over spheres. As a consequence we get nonradial oscillation criteria. Moreover, the function $H(t,x)$ which is used to multiply the Riccati equation is allowed to contain parts where this function is identically zero and a convenient choice of this function allows to formulate oscillation criteria for different (but simple) unbounded domains than exterior of the ball (see for example Theorem 5.1).

4 Oscillation and weighted integral averages

Notation: Let D and D_0 are the sets in $\mathbb{R} \times \mathbb{R}^n$ defined as follows:

$$D = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t \geq \|x\| \geq t_0\},$$
$$D_0 = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t > \|x\| \geq t_0\}.$$

Philos (1989) used a class of functions $H(t,s)$ defined on $D \in \mathbb{R} \times \mathbb{R}$ to obtain oscillation criteria for linear second order Sturm–Liouville differential equation. This technique, usually referred as averaging technique, has been elaborated and extended also for other types of differential equations, see e.g. [Li, 1995; Kong, 1999; Wang, 2001]. Let us point out especially the paper [Wang, 2001], where the usual condition $\frac{\partial H(t,s)}{\partial s} \leq 0$ is relaxed.

In the remaining part of this chapter we extend the averaging technique also to our Eq. (1.1) and obtain new oscillation criteria which are nonradial in their nature and thus different from usual oscillation criteria published in the literature. It is also shown, that this technique allows to get oscillation criteria not only for the exterior of a ball, but also for different types of unbounded domains. Let us start with a direct extension of [Wang, 2001, Theorem 1] to Eq. (1.1) (see also Theorem D on page 69).

Theorem 4.1. *Let* $H(t,x) \in C(D,[0,\infty))$*, and* $\rho(x) \in C^1(\Omega(t_0),(0,\infty))$ *be such that the function* $H(t,x)$ *has continuous partial derivative with respect to* x_i *(*$i = 1..n$*) on* D_0 *and the following conditions hold*

(i) $H(t,x) = 0$ *if and only if* $t = \|x\|$

(ii) *There exists function* $k(s) \in C([t_0,\infty),(0,\infty))$ *such that the function* $f(t,s) := k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma$ *is nonincreasing with respect to s for every* $t \geq s \geq t_0$.

(iii) *The vector–valued function* $\vec{h}(t,x)$ *defined on* D_0 *by*

$$\vec{h}(t,x) = \nabla H(t,x) + \frac{H(t,x)}{\rho(x)}\nabla\rho(x) \tag{4.1}$$

satisfies

$$\int_{\Omega(t_0,t)} H^{1-p}(t,x)\|\vec{h}(t,x)\|^p\rho(x)\,\mathrm{d}x < \infty \tag{4.2}$$

for $t > t_0$.

If

$$\limsup_{t\to\infty}\left(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\right)^{-1} \times \int_{\Omega(t_0,t)}\left[H(t,x)\rho(x)c(x) - \frac{\|\vec{h}(t,x)\|^p\rho(x)}{p^pH^{p-1}(t,x)}\right]\mathrm{d}x = \infty, \tag{4.3}$$

then Eq. (1.1) *is oscillatory.*

Proof. Suppose that (1.1) is not oscillatory. There exists $T \geq t_0$, such that (1.1) has a solution u positive on $\Omega(T)$. We use a modified Riccati substitution. The vector variable

$$\vec{w}(x) := \rho(x)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{\Phi(u(x))} \tag{4.4}$$

is well defined on $\Omega(T)$ and satisfies

$$\operatorname{div}\vec{w}(x) = \rho(x)\frac{\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right)}{\Phi(u)} + \frac{\|\nabla u\|^{p-2}}{\Phi(u)}\langle\nabla u,\nabla\rho(x)\rangle - (p-1)\rho(x)\frac{\|\nabla u\|^p}{|u|^p}.$$

An application of (1.1) and (4.4) to this equality gives

$$\operatorname{div}\vec{w}(x) = -\rho(x)c(x) + \frac{1}{\rho(x)}\langle\vec{w}(x),\nabla\rho(x)\rangle - (p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^q \tag{4.5}$$

and equivalently

$$\rho(x)c(x) = -\operatorname{div}\vec{w}(x) + \frac{1}{\rho(x)}\langle\vec{w}(x),\nabla\rho(x)\rangle - (p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^q$$

for $x \in \Omega(T)$. Multiplication of this equality by the factor $H(t,x)$ and integration over $\Omega(T,t)$ for $t > T$ yields

$$\begin{aligned}\int_{\Omega(T,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x \;&= -\int_{\Omega(T,t)} H(t,x)\,\mathrm{div}\,\vec{w}(x)\,\mathrm{d}x\\&+\int_{\Omega(T,t)} H(t,x)\frac{1}{\rho(x)}\left\langle \vec{w}(x),\nabla\rho(x)\right\rangle\,\mathrm{d}x\\&-\int_{\Omega(T,t)} H(t,x)(p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^{q}\,\mathrm{d}x\,.\end{aligned}$$

From here we conclude that

$$\begin{aligned}\int_{\Omega(T,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x \;&= -\int_{\Omega(T,t)} \mathrm{div}(H(t,x)\vec{w}(x))\,\mathrm{d}x\\&+\int_{\Omega(T,t)} \left\langle\nabla H(t,x),\vec{w}(x)\right\rangle\,\mathrm{d}x\;+\int_{\Omega(T,t)} H(t,x)\frac{1}{\rho(x)}\left\langle \vec{w}(w),\nabla\rho(x)\right\rangle\,\mathrm{d}x\\&-\int_{\Omega(T,t)} H(t,x)(p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^{q}\,\mathrm{d}x\,.\end{aligned}$$

Application of Gauss-Ostrogradski theorem, the property (i) of the function $H(t,x)$ and (4.1) give

$$\begin{aligned}\int_{\Omega(T,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x \;&= \int_{S(T)} H(t,x)\left\langle \vec{w}(x),\vec{\nu}\right\rangle\,\mathrm{d}x\\&+\int_{\Omega(T,t)} \left\langle\vec{h}(t,x),\vec{w}(x)\right\rangle\,\mathrm{d}x\;-\int_{\Omega(T,t)} H(t,x)(p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^{q}\,\mathrm{d}x\,.\end{aligned}\tag{4.6}$$

From here and from the Young inequality

$$(p-1)\|\vec{X}\|^{q}-p\left\langle\vec{X},\vec{Y}\right\rangle+\|\vec{Y}\|^{p}\geq 0 \tag{4.7}$$

for $\vec{X}=\vec{w}(x)H^{\frac{1}{q}}(t,x)\rho^{-\frac{1}{p}}(x)$ and $\vec{Y}=\vec{h}(t,x)\rho^{\frac{1}{p}}(x)p^{-1}H^{\frac{1-p}{p}}(t,x)$ it follows

$$\begin{aligned}&\int_{\Omega(T,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x\\&\qquad\leq\int_{S(T)} H(t,x)\left\langle \vec{w}(x),\vec{\nu}\right\rangle\,\mathrm{d}x\;+\int_{\Omega(T,t)}\frac{\|\vec{h}(t,x)\|^{p}\rho(x)}{p^{p}H^{p-1}(t,x)}\,\mathrm{d}x\,.\end{aligned}$$

which is equivalent to

$$\int_{\Omega(T,t)}\left[H(t,x)\rho(x)c(x)-\frac{\|\vec{h}(t,x)\|^{p}\rho(x)}{p^{p}H^{p-1}(t,x)}\right]\,\mathrm{d}x\;\leq\int_{S(T)} H(t,x)\left\langle \vec{w}(x),\vec{\nu}\right\rangle\,\mathrm{d}x$$

Hence

$$\int_{\Omega(T,t)}\left[H(t,x)\rho(x)c(x)-\frac{\|\vec{h}(t,x)\|^{p}\rho(x)}{p^{p}H^{p-1}(t,x)}\right]\,\mathrm{d}x\;\leq w^{*}(T)\int_{S(T)} H(t,x)\,\mathrm{d}\sigma\,, \tag{4.8}$$

where $w^*(T) = \max_{x\in S(T)}\{\|\vec{w}(x)\|\}$. Using (4.8) we are able to estimate the integral from the condition (4.3)

$$\begin{aligned}
&\int_{\Omega(t_0,t)} \left[H(t,x)\rho(x)c(x) - \frac{\|\vec{h}(t,x)\|^p\rho(x)}{p^p H^{p-1}(t,x)}\right] \mathrm{d}x \\
&\qquad\le \int_{\Omega(t_0,T)} H(t,x)\rho(x)c(x)\,\mathrm{d}x + w^*(T)\int_{S(T)} H(t,x)\,\mathrm{d}\sigma \\
&\qquad\le \int_{t_0}^{T} \left[\int_{S(s)} H(t,x)\,\mathrm{d}\sigma\right] k(s)\frac{\rho^*(s)c^*(s)}{k(s)}\,\mathrm{d}s + \frac{w^*(T)}{k(T)}k(T)\int_{S(T)} H(t,x)\,\mathrm{d}\sigma
\end{aligned}$$

for $t > T$ where $\rho^*(s) = \max_{x\in S(s)}\{\rho(x)\}$ and $c^*(s) = \max_{x\in S(s)}\{|c(x)|\}$. Since $f(t,s) := k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma$ is a nonincreasing function with respect to s, the above inequality implies

$$\begin{aligned}
&\int_{\Omega(t_0,t)} \left[H(t,x)\rho(x)c(x) - \frac{\|\vec{h}(t,x)\|^p\rho(x)}{p^p H^{p-1}(t,x)}\right] \mathrm{d}x \\
&\qquad\le k(t_0)\left[\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\right]\left[\int_{t_0}^{T} \frac{\rho^*(s)c^*(s)}{k(s)}\,\mathrm{d}s + \frac{w^*(T)}{k(T)}\right]
\end{aligned}$$

and hence

$$\begin{aligned}
\left(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\right)^{-1} &\int_{\Omega(t_0,t)} \left[H(t,x)\rho(x)c(x) - \frac{\|\vec{h}(t,x)\|^p\rho(x)}{p^p H^{p-1}(t,x)}\right] \mathrm{d}x \\
&\le k(t_0)\int_{t_0}^{T} \frac{\rho^*(s)c^*(s)}{k(s)}\,\mathrm{d}s + \frac{k(t_0)w^*(T)}{k(T)}
\end{aligned}$$

for large t, which contradicts (4.3). □

Remark 4.1. If both $H(t,x)$ and $\rho(x)$ in Theorem 4.1 depend on t and $\|x\|$ only, then Theorem 4.1 reduces to [Wang, 2001, Theorem 1].

5 Oscillation in general domains

The following theorem is a variant of the preceding one. In contrast to Theorem 4.1, the function $H(t,x)$ need not to be positive for $t_0 \le \|x\| < t$ in theorems below, but can attain also zero values. This allows to eliminate "bad parts" of the potential $c(x)$ from our considerations. We will use the following additional notation:

$$\begin{aligned}
\Omega_{0,t}(a,b) &= \{x\in\mathbb{R}^n : a\le\|x\|\le b, H(t,x)\ne 0\}, \\
S_{0,t}(a) &= \{x\in\mathbb{R}^n : \|x\| = a, H(t,x)\ne 0\}.
\end{aligned}$$

These sets are used to exclude the parts of the sets $\Omega(a,b)$ and $S(a)$ where the function $H(t,x)$ equals zero from the area of integration.

Theorem 5.1. *Let $H(t,x)\in C(D,[0,\infty))$, and $\rho(x)\in C^1(\Omega(t_0),(0,\infty))$ be such that the function $H(t,x)$ has continuous partial derivative with respect to x_i ($i = 1..n$) on D_0 and the following conditions hold*

(i) *If* $\|x\| = t \geq t_0$*, then* $H(t,x) = 0$*.*

(ii) *If* $H(t,x) = 0$ *for some* $(t,x) \in D_0$*, then* $\|\nabla H(t,x)\| = 0$*.*

(iii) *There exists function* $k(s) \in C([t_0,\infty),(0,\infty))$ *such that the function* $f(t,s) := k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma = k(s)\int_{S_{0,t}(s)} H(t,x)\,\mathrm{d}\sigma$ *is positive and nonincreasing with respect to* s *for every* $t > s \geq t_0$*.*

(iv) *The vector–valued function* $\vec{h}(t,x)$ *defined on* D_0 *by* (4.1) *satisfies*

$$\int_{\Omega_{0,t}(t_0,t)} H^{1-p}(t,x)\|\vec{h}(t,x)\|^p \rho(x)\,\mathrm{d}x < \infty \tag{5.1}$$

for $t > t_0$*.*

If

$$\limsup_{t\to\infty} \left(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\right)^{-1} \times \int_{\Omega_{0,t}(t_0,t)} \left[H(t,x)\rho(x)c(x) - \frac{\|\vec{h}(t,x)\|^p\rho(x)}{p^p H^{p-1}(t,x)}\right]\mathrm{d}x = \infty \tag{5.2}$$

then Eq. (1.1) *is oscillatory.*

Proof. Assume the contradiction. As in the proof of Theorem 4.1 we conclude (4.6) for $t > T$, where $\vec{w}$ is the solution of Riccati–type equation (4.5), defined on $\Omega(T)$ by (4.4). Since $H(t,x) = \|\vec{h}(t,x)\| = 0$ for $x \in \Omega(T,t)\setminus\Omega_{0,t}(T,t)$, we have

$$\begin{aligned}
\int_{\Omega(T,t)} \left\langle \vec{h}(t,x), \vec{w}(x)\right\rangle \mathrm{d}x &- \int_{\Omega(T,t)} H(t,x)(p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}x \\
&= \int_{\Omega_{0,t}(T,t)} \Big[\left\langle \vec{h}(t,x), \vec{w}(x)\right\rangle \\
&\qquad - H(t,x)(p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^q\Big]\,\mathrm{d}x\,.
\end{aligned}\tag{5.3}$$

The following relation follows from (5.3) and from Hölder inequality

$$\begin{aligned}
\int_{\Omega(T,t)} \left\langle \vec{h}(t,x), \vec{w}(x)\right\rangle \mathrm{d}x &- \int_{\Omega(T,t)} H(t,x)(p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}x \\
&= \int_T^t \Bigg[\int_{S_{0,t}(s)} \left\langle \vec{h}(t,x), \vec{w}(x)\right\rangle \mathrm{d}\sigma \\
&\qquad - \int_{S_{0,t}(s)} H(t,x)(p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}\sigma\Bigg]\mathrm{d}s \\
&\leq \int_T^t \Bigg[\Big(\int_{S_{0,t}(s)} H^{1-p}(t,x)\rho(x)\|\vec{h}(t,x)\|^p\,\mathrm{d}\sigma\Big)^{\frac{1}{p}} \\
&\qquad \times \Big(\int_{S_{0,t}(s)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}\sigma\Big)^{\frac{1}{q}} \\
&\qquad - \int_{S_{0,t}(s)} H(t,x)(p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}\sigma\Bigg]\mathrm{d}s
\end{aligned}$$

Application of Young inequality (4.7) gives

$$\int_{\Omega(T,t)} \left\langle \vec{h}(t,x), \vec{w}(x) \right\rangle \,\mathrm{d}x - \int_{\Omega(T,t)} H(t,x)(p-1)\rho^{1-q}(x)\|\vec{w}(x)\|^q \,\mathrm{d}x$$

$$\leq \int_T^t p^{-p} \int_{S_{0,t}(s)} H^{1-p}(t,x)\rho(x)\|\vec{h}(t,x)\|^p \,\mathrm{d}\sigma \,\mathrm{d}s$$

$$= \int_{\Omega_{0,t}(T,t)} p^{-p} H^{1-p}(t,x)\rho(x)\|\vec{h}(t,x)\|^p \,\mathrm{d}x\,.$$

Combining this inequality with (4.6) we conclude

$$\int_{\Omega_{0,t}(T,t)} \left[H(t,x)\rho(x)c(x) - p^{-p}\rho(x)H^{1-p}(t,x)\|\vec{h}(t,x)\|^p\right] \,\mathrm{d}x$$

$$\leq \int_{S(T)} H(t,x) \left\langle \vec{w}(x), \vec{\nu} \right\rangle \,\mathrm{d}x \tag{5.4}$$

and similarly as in the proof of Theorem 4.1 we obtain

$$\int_{\Omega_{0,t}(t_0,t)} \left[H(t,x)\rho(x)c(x) - p^{-p}\rho(x)H^{1-p}(t,x)\|\vec{h}(t,x)\|^p\right] \,\mathrm{d}x$$

$$\leq \int_{\Omega(t_0,T)} H(t,x)\rho(x)c(x) \,\mathrm{d}x + \int_{S(T)} H(t,x) \left\langle \vec{w}(x), \vec{\nu} \right\rangle \,\mathrm{d}\sigma$$

$$\leq \int_{t_0}^T \left[\int_{S(s)} H(t,x) \,\mathrm{d}\sigma\right] k(s)\frac{\rho^*(s)c^*(s)}{k(s)} \,\mathrm{d}s + w^*(T) \int_{S(T)} H(t,x) \,\mathrm{d}\sigma$$

$$\leq k(t_0) \left[\int_{S(t_0)} H(t,x) \,\mathrm{d}\sigma\right] \left[\int_{t_0}^T \frac{\rho^*(s)c^*(s)}{k(s)} \,\mathrm{d}s + \frac{w^*(T)}{k(T)}\right],$$

where $w^*(s)$, $\rho^*(s)$ and $c^*(s)$ are the same as in the proof of Theorem 4.1. The last inequality contradicts (5.2). The proof is complete. □

Remark 5.1. Condition (ii) claims that if $H(t,x)$ vanishes, then $\|x\| = t$ or $\nabla H(t,x)$ vanishes as well.

Condition (iii) claims (among others) that the set $S_{0,t}(s)$ is nonempty for every t satisfying $t_0 < s < t$. Hence the function $H(t,x)$ has parts with positive values on every sphere centered in the origin.

Remark 5.2. Under (4.2) we understand that the function $g(t,s)$ defined for $t_0 < s < t$ by

$$g(t,s) := \int_{S_{0,t}(s)} H^{1-p}(t,x)\rho(x)\|\vec{h}(t,x)\|^p \,\mathrm{d}\sigma$$

is integrable with respect to s over the interval (t_0, t). (The point t may be a singular point of the integral, since $H(t,x) = 0$ for $\|x\| = t$.) A similar commentary explains also, how to understand (5.1).

Remark 5.3. Let $\Omega \subset \Omega(t_0)$ be unbounded domain with smooth boundary $\partial\Omega$. If in addition to the conditions of Theorem 5.1 the function $H(t,x)$ vanishes outside Ω and both $H(t,x)$ and $\|\nabla H(t,x)\|$ vanish on $\partial\Omega$ for every $t \geq t_0$, then it follows that Eq. (1.1) is oscillatory in Ω. Hence Theorem 5.1 can be used to formulate explicit oscillation criteria on different types of domains, than exterior of the ball. Examples of the oscillation criteria on half–plane are given on page 35.

The following corollary is an immediate consequence of Theorem 5.1.

Corollary 5.1. *Let the assumptions (i) – (iv) of Theorem 5.1 hold. If*

$$\limsup_{t\to\infty} \left(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma \right)^{-1} \int_{\Omega_{0,t}(t_0,t)} \frac{\|\vec{h}(t,x)\|^p \rho(x)}{H^{p-1}(t,x)}\,\mathrm{d}x < \infty \tag{5.5}$$

and

$$\limsup_{t\to\infty} \left(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma \right)^{-1} \int_{\Omega(t_0,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x = \infty,$$

then Eq. (1.1) *is oscillatory.*

Theorem 5.1 shows that Eq. (1.1) is oscillatory if the expression

$$\left(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma \right)^{-1} \int_{\Omega_{0,t}(t_0,t)} \left[H(t,x)\rho(x)c(x) - \frac{\|\vec{h}(t,x)\|^p \rho(x)}{p^p H^{p-1}(t,x)} \right] \mathrm{d}x$$

is sufficiently large in a neighborhood of infinity (in the sense of infinite limes superior). As a natural continuation we deal with the cases when this condition is broken. In this case the equation still may be oscillatory, if this expression is large in the integral sense. We prove one technical lemma first.

Lemma 5.1. *Let the functions H, h, k and ρ satisfy the hypothesis (i)–(iv) of Theorem 5.1. Suppose that* (5.5)*,* (5.16) *and* (5.17) *hold. Let u be solution of* (1.1) *which is positive on $\Omega(T_0)$ for some $T_0 \ge t_0$ and $\vec{w}(x)$ be the corresponding Riccati variable defined on $\Omega(T_0)$ by* (4.4)*. Then*

$$\liminf_{t\to\infty} \int_{T_0}^{t} \frac{\int_{S(s)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}\sigma}{k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma}\,\mathrm{d}s < \infty. \tag{5.6}$$

Proof. Let us denote

$$F(t) = \left(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma \right)^{-1} \int_{\Omega(T_0,t)} \|\vec{h}(t,x)\| \cdot \|\vec{w}(x)\|\,\mathrm{d}x$$
$$G(t) = \left(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma \right)^{-1} (p-1) \int_{\Omega(T_0,t)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}x$$

for $t > T_0$. As in the proof of Theorem 5.1 we conclude (4.6) and hence

$$\begin{aligned} G(t) - F(t) &\le \Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma \Big)^{-1} \\ &\quad \times \left[\int_{S(T_0)} H(t,x)\|\vec{w}(x)\|\,\mathrm{d}\sigma - \int_{\Omega(T_0,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x \right] \\ &\le w^*(T_0) - \Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma \Big)^{-1} \int_{\Omega(T_0,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x \end{aligned} \tag{5.7}$$

holds for every $t > T_0$, where $w^*(t)$ has been defined in the proof of Theorem 4.1. Hence by (5.17)

$$\liminf_{t\to\infty}[G(t) - F(t)] \le w^*(T_0) - A(T_0) < \infty. \tag{5.8}$$

Suppose that (5.6) does not hold. Then

$$\lim_{t\to\infty} \int_{T_0}^{t} \frac{\int_{S(s)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q \,\mathrm{d}\sigma}{k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma}\,\mathrm{d}s = \infty.$$

According to (5.16) there exists $\eta \in \mathbb{R}$ such that

$$0 < \eta < \inf_{s\ge t_0}\left\{\liminf_{t\to\infty} \frac{k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma}{k(t_0)\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma}\right\} \tag{5.9}$$

and for every $\mu \in \mathbb{R}^+$ there exists $T_1 > T_0$ such that

$$\int_{T_0}^{t} \frac{(p-1)\int_{S(s)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q \,\mathrm{d}\sigma}{k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma}\,\mathrm{d}s \ge \frac{\mu}{\eta k(T_0)} \tag{5.10}$$

for every $t \ge T_1$. Further there exists $T_2 > T_1$ such that

$$\frac{k(T_1)\int_{S(T_1)} H(t,x)\,\mathrm{d}\sigma}{k(t_0)\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma} > \eta \tag{5.11}$$

for all $t \ge T_2$. From the definition of the function $G(t)$ it follows that for $t \ge T_2$

$$\begin{aligned} G(t) =& \Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \int_{T_0}^{t} \left[\left(k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma\right)\right.\\ &\left.\times \frac{(p-1)\int_{S(s)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q \,\mathrm{d}\sigma}{k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma}\right]\mathrm{d}s \end{aligned}$$

holds. Integration by parts and the property (i) of the function $H(t,x)$ imply

$$\begin{aligned} G(t) \ge& \Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \int_{T_0}^{t} \left[-\frac{\partial}{\partial s}\Big(k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma\Big)\right.\\ &\left.\times\Big(\int_{T_0}^{s} \frac{(p-1)\int_{S(\xi)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q \,\mathrm{d}\sigma}{k(\xi)\int_{S(\xi)} H(t,x)\,\mathrm{d}\sigma}\,\mathrm{d}\xi\Big)\right]\mathrm{d}s \end{aligned}$$

and in view of (iii)

$$\begin{aligned} G(t) \ge& \Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \int_{T_1}^{t} \left[-\frac{\partial}{\partial s}\Big(k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma\Big)\right.\\ &\left.\times\Big(\int_{T_0}^{s} \frac{(p-1)\int_{S(\xi)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q \,\mathrm{d}\sigma}{k(\xi)\int_{S(\xi)} H(t,x)\,\mathrm{d}\sigma}\,\mathrm{d}\xi\Big)\right]\mathrm{d}s\,. \end{aligned}$$

Application of (5.10) gives

$$G(t) \geq \Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \frac{\mu}{\eta k(T_0)} \int_{T_1}^{t} -\frac{\partial}{\partial s}\Big(k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma\Big)\,\mathrm{d}s$$
$$\geq \frac{\mu k(T_1)\int_{S(T_1)} H(t,x)\,\mathrm{d}\sigma}{\eta k(T_0)\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma}.$$

In view of (iii)

$$G(t) \geq \frac{\mu k(T_1)\int_{S(T_1)} H(t,x)\,\mathrm{d}\sigma}{\eta k(t_0)\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma}$$

and (5.11) implies

$$G(t) \geq \mu$$

for every $t \geq T_2$. Since μ has been chosen arbitrary, $\lim_{t\to\infty} G(t) = \infty$. Let us consider the sequence $\{t_n\}_{n=1}^{\infty}$ of the points from (T_2, ∞) such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{t\to\infty}[G(t_n) - F(t_n)] = \liminf_{t\to\infty}[G(t) - F(t)]$. In view of (5.8) there exists real constant M with property

$$G(t_n) - F(t_n) \leq M \tag{5.12}$$

for all n. Hence

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} G(t_n) = \infty. \tag{5.13}$$

From (5.12) and (5.13) we obtain

$$\frac{F(t_n)}{G(t_n)} - 1 \geq -\frac{M}{G(t_n)} > -\frac{1}{2}$$

for large n. Hence

$$\frac{F(t_n)}{G(t_n)} > \frac{1}{2}$$

for large n and combination of this inequality with (5.13) yields

$$\lim_{n\to\infty} \frac{F^p(t_n)}{G^{p-1}(t_n)} = \infty. \tag{5.14}$$

However the definition of the function $F(t)$ and the Hölder inequality give

$$F(t) \leq \left[\Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \int_{\Omega(T_0,t)} (p-1)H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}x\right]^{\frac{1}{q}}$$
$$\times \left[\Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \right.$$
$$\left. \times \int_{\Omega(T_0,t)} (p-1)^{1-p} H^{1-p}(t,x)\rho(x)\|\vec{h}(t,x)\|^p\,\mathrm{d}x\right]^{\frac{1}{p}}$$

$$\le [G(t)]^{\frac{1}{q}} \Bigg[\Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma \Big)^{-1} \times \int_{\Omega(T_0,t)} (p-1)^{1-p} H^{1-p}(t,x)\rho(x)\|\vec{h}(t,x)\|^p\,\mathrm{d}x \Bigg]^{\frac{1}{p}}$$

and therefore

$$\frac{F^p(t)}{G^{p-1}(t)} \le (p-1)^{1-p} \Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma \Big)^{-1} \times \int_{\Omega(T_0,t)} (p-1)^{1-p} H^{1-p}(t,x)\rho(x)\|\vec{h}(t,x)\|^p\,\mathrm{d}x\,.$$

Since by (5.9)

$$\frac{k(T_0)\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma}{k(t_0)\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma} \ge \eta$$

for large t, we have

$$\frac{F^p(t)}{G^{p-1}(t)} \le (p-1)^{1-p}\eta^{-1} \Big(k(t_0) \int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma \Big)^{-1} \times k(T_0) \int_{\Omega(t_0,t)} (p-1)^{1-p} H^{1-p}(t,x)\rho(x)\|\vec{h}(t,x)\|^p\,\mathrm{d}x\,. \tag{5.15}$$

If (5.14) would hold we obtain a contradiction with (5.5). This contradiction completes the proof. □

The following theorem extends [Wang, 2001, Theorem 2]. As stated before, it can be applied in some cases when (5.2) fails.

Theorem 5.2. *Let the functions H, h, k and ρ satisfy the hypotheses (i)–(iv) of Theorem 5.1. Suppose also that*

$$0 < \inf_{s \ge t_0} \left\{ \liminf_{t\to\infty} \frac{k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma}{k(t_0)\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma} \right\} \tag{5.16}$$

and (5.5) *holds. If there exists a function $A \in C(\Omega(t_0), \mathbb{R})$ such that*

$$\inf_{t\in(T,\infty)} \left\{ \Big(\int_{S(T)} H(t,x)\,\mathrm{d}\sigma \Big)^{-1} \int_{\Omega_{0,t}(T,t)} \Big[H(t,x)\rho(x)c(x) - \frac{\|\vec{h}(t,x)\|^p \rho(x)}{p^p H^{p-1}(t,x)} \Big]\,\mathrm{d}x \right\} \ge A(T) \tag{5.17}$$

for $T \ge t_0$ and

$$\int_{t_0}^{\infty} (A_+(T))^q \widehat{\rho}^{1-q}(T) k^{-1}(T)\,\mathrm{d}T = \infty, \tag{5.18}$$

where $A_+(T) = \max\{A(T), 0\}$ and

$$\widehat{\rho}(T) = \sup_{t>T} \left\{ \Big(\int_{S(T)} H(t,x)\,\mathrm{d}\sigma \Big)^{-1} \int_{S(T)} \rho(x) H(t,x)\,\mathrm{d}\sigma \right\}, \tag{5.19}$$

then Eq. (1.1) *is oscillatory.*

Proof. Suppose that (1.1) is not oscillatory and u is a solution of (1.1) positive on $\Omega(T_0)$ for some $T_0 \geq t_0$. Let $\vec{w}(x)$ be Riccati variable defined by (4.4). As in the proof of Theorem 5.1 we conclude (5.4) and by (5.17)

$$A(T) \leq \frac{\int_{S(T)} H(t,x)\|\vec{w}(x)\|\,\mathrm{d}\sigma}{\int_{S(T)} H(t,x)\,\mathrm{d}\sigma} \tag{5.20}$$

holds for every $t > T > T_0$. Hence

$$A(T)\int_{S(T)} H(t,x)\,\mathrm{d}\sigma \;\leq \int_{S(T)} H(t,x)\|\vec{w}(x)\|\,\mathrm{d}\sigma$$

for all $t > T$. Hölder inequality gives

$$\begin{aligned} A(T)\int_{S(T)} H(t,x)\,\mathrm{d}\sigma \;\leq &\Bigl(\int_{S(T)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}\sigma\Bigr)^{\frac{1}{q}} \\ &\times \Bigl(\int_{S(T)} H(t,x)\rho(x)\,\mathrm{d}\sigma\Bigr)^{\frac{1}{p}}. \end{aligned}$$

From here we get

$$\begin{aligned} \bigl(A_+(T)\bigr)^q\Bigl(\int_{S(T)} H(t,x)\,\mathrm{d}\sigma\Bigr)^q \leq &\int_{S(T)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}\sigma \\ &\times \Bigl(\int_{S(T)} H(t,x)\rho(x)\,\mathrm{d}\sigma\Bigr)^{q-1} \end{aligned}$$

and the definition of the function $\widehat{\rho}$ yields

$$\bigl(A_+(T)\bigr)^q(\widehat{\rho}(T))^{1-q} \leq \Bigl(\int_{S(T)} H(t,x)\,\mathrm{d}\sigma\Bigr)^{-1}\int_{S(T)} H(t,x)\rho^{1-q}(x)\|\vec{w}(x)\|^q\,\mathrm{d}\sigma$$

for $t > T > T_0$. This inequality combined with (5.6) contradicts (5.18). The proof is complete. □

Remark 5.4. The supremum in (5.19) always exists, since

$$\Bigl(\int_{S(T)} H(t,x)\,\mathrm{d}\sigma\Bigr)^{-1}\int_{S(T)} \rho(x)H(t,x)\,\mathrm{d}\sigma \;\leq \max_{x\in S(T)}\{\rho(x)\}.$$

Remark 5.5. Comparing Theorem 5.2 with [Wang, 2001, Theorem 2] we see that condition (5.17) is in the case of ordinary differential equations replaced by a weaker condition where $\limsup_{t\to\infty}$ stays instead of $\inf_{t\in(T,\infty)}$. The reason, why we need the stronger condition (5.17) is the following. In the proof of Theorem 5.2 we estimate the function $A(T)$ from above with an expression involving solution of Riccati equation — see (5.20). This bound does not depend on the value of t in the case of ODE (integrals are missing and terms $H(t,x)$ cancel), however it does depend on t in the case of Eq. (1.1).

Lemma 5.2. *Let the functions H, h, k and ρ satisfy the hypotheses (i)–(iv) and of Theorem 5.1. Suppose that* (5.16)*,* (5.17) *and*

$$\liminf_{t\to\infty}\Bigl(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\Bigr)^{-1}\int_{\Omega(t_0,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x \;<\infty. \tag{5.21}$$

hold. Let u and $\vec{w}$ be the same as in Lemma 5.1. Then (5.6) *holds.*

Proof. As in the proof of Theorem 5.1 we see that (4.6) holds. With the notation of Lemma 5.1, inequality (5.7) holds. Hence

$$\begin{aligned}\limsup_{t\to\infty}[G(t)-F(t)] \le & w^*(T_0) - \liminf_{t\to\infty}\Big(\int_{S(T_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \\ & \times \int_{\Omega(t_0,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x \\ \le & w^*(T_0) - A(T_0) < \infty. \end{aligned} \tag{5.22}$$

By (5.17)

$$A(t_0) \le \Big(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \int_{\Omega_{0,t}(t_0,t)} \left[H(t,x)\rho(x)c(x) - \frac{\|\vec{h}(t,x)\|^p \rho(x)}{p^p H^{p-1}(t,x)}\right] \mathrm{d}x$$

for $t \ge t_0$. Hence by (5.21)

$$\begin{aligned}&\liminf_{t\to\infty}\Big(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \int_{\Omega_{0,t}(t_0,t)} \frac{\|\vec{h}(t,x)\|^p \rho(x)}{p^p H^{p-1}(t,x)}\,\mathrm{d}x \\ &\quad \le \liminf_{t\to\infty}\Big(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \int_{\Omega_{0,t}(t_0,t)} H(t,x)\rho(x)c(x)\,\mathrm{d}x \, - A(t_0) \\ &\quad < \infty. \end{aligned} \tag{5.23}$$

Let us consider the sequence $\{t_n\}_{n=1}^{\infty}$ in (T_0,∞) satisfying $\lim_{n\to\infty} t_n = \infty$ and

$$\begin{aligned}&\lim_{n\to\infty}\Big(\int_{S(t_0)} H(t_n,x)\,\mathrm{d}\sigma\Big)^{-1} \int_{\Omega_{0,t_n}(t_0,t_n)} \frac{\|\vec{h}(t_n,x)\|^p \rho(x)}{p^p H^{p-1}(t_n,x)}\,\mathrm{d}x \\ &\quad = \liminf_{t\to\infty}\Big(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1} \int_{\Omega_{0,t}(t_0,t)} \frac{\|\vec{h}(t,x)\|^p \rho(x)}{p^p H^{p-1}(t,x)}\,\mathrm{d}x\,. \end{aligned}$$

Now suppose by contradiction that (5.6) fails. As in the proof of Lemma 5.1 and using (5.22) we conclude (5.13). Using the same procedure as in Lemma 5.1 we obtain (5.14) and (5.15), which contradicts to (5.23). Hence (5.6) holds. □

The following theorem extends [Wang, 2001, Theorem 3]. It is a variant of Theorem 5.2 with (5.5) replaced by (5.21).

Theorem 5.3. *Let the functions H, h, k and ρ satisfy the hypotheses (i)–(iv) of Theorem 5.1. Suppose also that* (5.16) *and* (5.21) *hold. If there exists a function $A \in C(\Omega(t_0),\mathbb{R})$ such that* (5.17) *and* (5.18) *hold, then Eq.* (1.1) *is oscillatory.*

Proof. The proof is almost the same as the proof of Theorem 5.2. Lemma 5.2 is applied instead of Lemma 5.1. □

6 Oscillation criteria on half-plane

In the remaining part of this chapter we specify general ideas introduced on previous pages and derive oscillation criteria on half-plane $x_2 \ge 0$.

Example 6.1. Consider the Schrödinger partial differential equation (1.3) in $\mathbb{R}^2$, i.e., $n = p = 2$. For $\lambda > 1$ define the functions H, k and ρ as follows:

$$\rho(x) \equiv 1 \qquad \text{for } x \in \mathbb{R}^2$$
$$k(s) = \frac{1}{s} \qquad \text{for } s > 1$$
$$H(t,x) = \begin{cases} (t-r)^\lambda \sin^2\varphi & \varphi \in [0,\pi), \\ 0 & \varphi \in [\pi, 2\pi), \end{cases}$$

where r and φ are the radial and the polar coordinates of the point $x \in \mathbb{R}^2$. It is easy to see that $S_{t,0}(s)$ is the top half-circle with radius $s < t$ and $\int_{S(s)} H(t,x)\,\mathrm{d}\sigma = \frac{\pi}{2}(t-s)^\lambda s = O(t^\lambda)$. Since $\rho(x) \equiv 1$, $\vec{h}(t,x) = \nabla H(t,x)$ holds and consequently

$$\|\vec{h}(t,x)\|^2 = \begin{cases} \lambda^2 (t-r)^{2\lambda-2}\sin^4\varphi + 4\frac{(t-r)^{2\lambda}}{r^2}\sin^2\varphi\cos^2\varphi & \varphi \in [0,\pi), \\ 0 & \varphi \in [\pi,2\pi). \end{cases}$$

Direct computation shows

$$H^{-1}(t,x)\|\vec{h}(t,x)\|^2 = \lambda^2(t-r)^{\lambda-2}\sin^2\varphi + 4\frac{(t-r)^\lambda}{r^2}\cos^2\varphi$$

for $x \in \Omega_{0,t}(t_0)$ and (5.1) clearly holds. Further (5.2) has the form

$$\limsup_{t\to\infty} t^{-\lambda}\int_{M(t)} \Big[c(x(r,\varphi))(t-r)^\lambda \sin^2\varphi \\ - \frac{\lambda^2}{4}(t-r)^{\lambda-2}\sin^2\varphi - \frac{(t-r)^\lambda}{r^2}\cos^2\varphi\Big]\,\mathrm{d}x = \infty, \tag{6.1}$$

where $M(t) = \{(x_1,x_2) \in \mathbb{R}^2 : 1 \le x_1^2 + x_2^2 \le t^2, x_2 > 0\}$. Since

$$\begin{aligned} \lim_{t\to\infty} t^{-\lambda}\int_{M(t)} (t-r)^{\lambda-2}\sin^2\varphi\,\mathrm{d}x &= \lim_{t\to\infty} t^{-\lambda}\frac{\pi}{2}\int_1^t r(t-r)^{\lambda-2}\,\mathrm{d}r \\ &\le \lim_{t\to\infty} t^{-\lambda}\frac{\pi}{2}\int_1^t t(t-r)^{\lambda-2}\,\mathrm{d}r \\ &= \frac{\pi}{2}\frac{1}{\lambda-1}\lim_{t\to\infty} t^{1-\lambda}(t-1)^{\lambda-1} < \infty, \end{aligned}$$

is (6.1) equivalent to

$$\limsup_{t\to\infty} t^{-\lambda}\int_{M(t)} \Big[c(x(r,\varphi))(t-r)^\lambda\sin^2\varphi - \frac{(t-r)^\lambda}{r^2}\cos^2\varphi\Big]\,\mathrm{d}x\,(r,\varphi) = \infty.$$

Hence (6.1) is sufficient for Eq. (1.3) to be oscillatory on the half-plane $x_2 \ge 0$.

Example 6.2. Let us consider the same equation as in Example 6.1. Let us change the function $\rho(x)$ into $\rho(x) = \frac{1}{\|x\|} = \frac{1}{r}$. The computation in polar coordinates yields

$$\begin{aligned} \|\vec{h}(t,x)\|^2 =& \lambda^2(t-r)^{2\lambda-2}\sin^4\varphi + 2\lambda(t-r)^{2\lambda-1}r^{-1}\sin^4\varphi \\ &+ (t-r)^{2\lambda}r^{-2}\sin^4\varphi + 4(t-r)^{2\lambda}r^{-2}\sin^2\varphi\cos^2\varphi \end{aligned}$$

for $\varphi \in [0, \pi)$ and $\|\vec{h}(t,x)\|^2 = 0$ otherwise. As in the preceding example, (5.1) holds. Further integrating in polar coordinates we ensure that (5.5) holds. Then the condition

$$\limsup_{t\to\infty} t^{-\lambda} \int_{M(t)} c(x(r,\varphi))(t-r)^{\lambda} r^{-1} \sin^2\varphi \, \mathrm{d}x\,(r,\varphi) = \infty$$

is a sufficient condition for oscillation of Eq. (1.1) on the half-plane $x_2 \geq 0$.

Remark 6.1. In contrast to common results in the literature, the conditions in Examples 6.1 and 6.2 are not affected by the behavior of the function $c(x)$ on the half-plane $x_2 \leq 0$, which may be "relatively bad".

Chapter 3

Three terms PDE with p-Laplacian

1 Introduction

In this chapter we study the half-linear partial differential equation with p-Laplacian and damping term in the form

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) + \left\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \right\rangle + c(x)\Phi(u) = 0. \tag{1.1}$$

The functions $c(x)$ and $\vec{b}(x)$ are assumed to be Hölder continuous functions on the domain $\Omega(1)$. The solution of (1.1) is every function defined on $\Omega(1)$ which satisfies (1.1) everywhere on $\Omega(1)$.

Some of the results are formulated for simplicity also for the linear equation

$$\Delta u + \left\langle \vec{b}(x), \nabla u \right\rangle + c(x)u = 0 \tag{1.2}$$

which can be obtained from (1.1) by putting $p = 2$, for the Schrödinger equation

$$\Delta u + c(x)u = 0 \tag{1.3}$$

obtained for $p = 2$ and $\vec{b} \equiv 0$ and also for the undamped half-linear equation

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) + c(x)\Phi(u) = 0 \tag{1.4}$$

which has been studied in Chapters 1 and 2 and can be obtained from (1.1) by putting $\vec{b} \equiv 0$.

The main difference between the results from this chapter and similar results in the literature lies in the fact, that our criteria are not “radial” in the sense of the classification from Remark 2.3 on page 4. See also the discussion in Section 3.1 of the current chapter.

2 Riccati inequality

We start this chapter by investigating the partial Riccati-type differential inequality

$$\operatorname{div}\vec{w} + \|\vec{w}\|^q + c(x) \le 0$$

and some generalizations of this inequality in the forms

$$\operatorname{div}\left(\alpha(x)\vec{w}\right) + K\alpha(x)\|\vec{w}\|^q + \alpha(x)c(x) \le 0 \tag{2.1}$$

and

$$\operatorname{div}\vec{w} + K\|\vec{w}\|^q + c(x) + \left\langle \vec{w}, \vec{b} \right\rangle \le 0, \tag{2.2}$$

where $K \in \mathbb{R}$, $q > 1$. The assumptions on the function α are stated below.

We consider the Riccati inequality on two types of unbounded domains in $\mathbb{R}^n$: The exterior of a ball, centered in the origin, and a general unbounded domain Ω. In the latter case we use the assumption:

(A1) The set Ω is an unbounded domain in $\mathbb{R}^n$, simply connected with a piecewise smooth boundary $\partial\Omega$ and $\operatorname{meas}\big(\Omega \cap S(t)\big) > 0$ for $t > 1$.

Theorem 2.1. *Let Ω satisfy* (A1) *and $c \in C(\Omega, \mathbb{R})$. Suppose that α satisfies*

$$\begin{aligned} &\alpha \in C^1(\Omega \cap \Omega(a_0), \mathbb{R}^+) \cap C_0(\overline{\Omega}, \mathbb{R}), \\ &\int_{a_0}^{\infty} \left(\int_{\Omega \cap S(t)} \alpha(x)\, \mathrm{d}\sigma \right)^{1-q} \mathrm{d}t = \infty\,. \end{aligned} \tag{2.3}$$

Further suppose that there exist $a \ge a_0$, a real constant $K > 0$ and a real-valued differentiable vector function $\vec{w}(x)$ which is bounded (in the sense of the continuous extension, if necessary) on every compact subset of $\overline{\Omega \cap \Omega(a)}$ and satisfies the differential inequality (2.1) *on $\Omega \cap \Omega(a)$. Then*

$$\liminf_{t \to \infty} \int_{\Omega \cap \Omega(a_0, t)} \alpha(x) c(x)\, \mathrm{d}x < \infty. \tag{2.4}$$

Proof. For simplicity let us denote $\widetilde{\Omega}(a) = \Omega(a) \cap \Omega$, $\widetilde{S}(a) = S(a) \cap \Omega$, $\widetilde{\Omega}(a,b) = \Omega(a,b) \cap \Omega$. Suppose, by contradiction, that (2.1) and (2.3) are fulfilled and

$$\lim_{t \to \infty} \int_{\widetilde{\Omega}(a_0, t)} \alpha(x) c(x)\, \mathrm{d}x = \infty. \tag{2.5}$$

Integrating (2.1) over the domain $\widetilde{\Omega}(a,t)$ and applying the Gauss-Ostrogradski divergence theorem we get

$$\begin{aligned} \int_{\widetilde{S}(t)} \alpha(x) \left\langle \vec{w}(x), \vec{\nu}(x) \right\rangle \mathrm{d}\sigma &- \int_{\widetilde{S}(a)} \alpha(x) \left\langle \vec{w}(x), \vec{\nu}(x) \right\rangle \mathrm{d}\sigma \\ &+ \int_{\widetilde{\Omega}(a,t)} \alpha(x) c(x)\, \mathrm{d}x + K \int_{\widetilde{\Omega}(a,t)} \alpha(x) \|\vec{w}(x)\|^q\, \mathrm{d}x \le 0, \end{aligned} \tag{2.6}$$

where $\vec{\nu}(x)$ is the outside normal unit vector to the sphere $S(\|x\|)$ in the point x (note that the product $\alpha(x)\vec{w}(x)$ vanishes on the boundary $\partial\Omega$ since $\alpha \in C_0(\overline{\Omega}, \mathbb{R})$ and $\vec{w}$ is bounded near the boundary). In view of (2.5) there exists $t_0 \ge a$ such that

$$\int_{\widetilde{\Omega}(a,t)} \alpha(x) c(x)\, \mathrm{d}x - \int_{\widetilde{S}(a)} \alpha(x) \left\langle \vec{w}(x), \vec{\nu}(x) \right\rangle \mathrm{d}\sigma \ge 0 \tag{2.7}$$

for every $t \ge t_0$. Further, Schwarz and Hölder inequalities give

$$\begin{aligned} -\int_{\widetilde{S}(t)} \alpha(x) \left\langle \vec{w}(x), \vec{\nu}(x) \right\rangle \mathrm{d}\sigma &\le \int_{\widetilde{S}(t)} \alpha(x) \|w(x)\|\, \mathrm{d}\sigma \\ &\le \Big(\int_{\widetilde{S}(t)} \alpha(x) \|w(x)\|^q\, \mathrm{d}\sigma \Big)^{1/q} \Big(\int_{\widetilde{S}(t)} \alpha(x)\, \mathrm{d}\sigma \Big)^{1/p}. \end{aligned} \tag{2.8}$$

Combination of inequalities (2.6), (2.7), and (2.8) gives

$$K \int_{\widetilde{\Omega}(a,t)} \alpha(x)\|\vec{w}(x)\|^q \,\mathrm{d}x \le \Big(\int_{\widetilde{S}(t)} \alpha(x)\|w(x)\|^q \,\mathrm{d}\sigma\Big)^{1/q}\Big(\int_{\widetilde{S}(t)} \alpha(x)\,\mathrm{d}\sigma\Big)^{1/p}$$

for every $t \ge t_0$. Denote

$$g(t) = \int_{\widetilde{\Omega}(a,t)} \alpha(x)\|w(x)\|^q \,\mathrm{d}x\,.$$

Then the last inequality can be written in the form

$$Kg(t) \le \Big(g'(t)\Big)^{1/q}\Big(\int_{\widetilde{S}(t)} \alpha(x)\,\mathrm{d}\sigma\Big)^{1/p}.$$

From here we conclude for every $t \ge t_0$

$$K^q g^q(t) \le g'(t)\Big(\int_{\widetilde{S}(t)} \alpha(x)\,\mathrm{d}\sigma\Big)^{q/p}$$

hold and equivalently

$$K^q\Big(\int_{\widetilde{S}(t)} \alpha(x)\,\mathrm{d}\sigma\Big)^{1-q} \le \frac{g'(t)}{g^q(t)}.$$

This inequality shows that the integral on the left-hand side of (2.3) has an integrable majorant on $[t_0, \infty)$ and hence it is convergent as well, a contradiction to (2.3). □

Frequently considered cases are $\Omega = \mathbb{R}^n$ and $\Omega = \Omega(a_0)$. In these cases the preceding theorem gives:

Corollary 2.1. *Let $\alpha \in C^1(\Omega(a_0), \mathbb{R}^+)$, $c \in C(\Omega(a_0), \mathbb{R})$. Suppose that*

$$\int_{a_0}^{\infty} \left(\int_{S(t)} \alpha(x)\,\mathrm{d}\sigma\right)^{1-q} \mathrm{d}t = \infty. \tag{2.9}$$

Further suppose, that there exists $a \ge a_0$, real constant $K > 0$ and real–valued differentiable vector function $\vec{w}(x)$ defined on $\Omega(a)$ which satisfies the differential inequality (2.1) *on $\Omega(a)$. Then*

$$\liminf_{t\to\infty} \int_{\Omega(a_0,t)} \alpha(x)c(x)\,\mathrm{d}x < \infty. \tag{2.10}$$

Proof. The proof is a simple modification and simplification of the proof of Theorem 2.1 and therefore it is omitted here. □

In the following theorem we use the integral averaging technique which is for second order linear ordinary differential equation due to [Philos, 1989] and has been used for two terms equation already in Chapter 2 on page 23. Consider two-parametric weight function $H(t, x)$ defined on the closed domain

$$D = \big\{(t, x) \in \mathbb{R} \times \mathbb{R}^n : a_0 \le \|x\| \le t\big\} \tag{2.11}$$

Denote $D_0 = \big\{(t, x) \in \mathbb{R} \times \mathbb{R}^n : a_0 < \|x\| < t\big\}$ and suppose that the function $H(t, x)$ satisfies the hypothesis

(A2) $H(t,x) \in C(D, \mathbb{R}_0^+) \cap C^1(D_0, \mathbb{R}_0^+)$.

Some additional assumptions on the function H are stated below.

Theorem 2.2. *Let Ω be an unbounded domain in $\mathbb{R}^n$ which satisfies assumption* (A1)*, $c \in C(\Omega, \mathbb{R})$ and $\vec{b} \in C(\Omega, \mathbb{R}^n)$. Suppose the function $H(t,x)$ satisfies* (A2) *and the following conditions:*

(i) *$H(t,x) \equiv 0$ for $x \notin \overline{\Omega}$.*

(ii) *If $x \in \partial\Omega$, then $H(t,x) = 0$ and $\|\nabla H(t,x)\| = 0$ for every $t \geq x$.*

(iii) *If $x \in \Omega^0$, then $H(t,x) = 0$ if and only if $\|x\| = t$.*

(iv) *The vector function $\vec{h}(x)$ defined on D_0 with the relation*

$$\vec{h}(t,x) = -\nabla H(t,x) + \vec{b}(x) H(t,x) \tag{2.12}$$

satisfies

$$\int_{\Omega(a_0,t)\cap\Omega} H^{1-p}(t,x) \big\|\vec{h}(t,x)\big\|^p \, \mathrm{d}x < \infty. \tag{2.13}$$

(v) *There exists a continuous function $k(r) \in C([a_0,\infty), \mathbb{R}^+)$ such that the function $f(t,r) := k(r) \int_{S(r)\cap\Omega} H(t,x)\,\mathrm{d}x$ is positive and nonincreasing on $[a_0, t)$ with respect to the variable r for every t, $t > r$.*

Further suppose that there exist real numbers $a \geq a_0$, $K > 0$ and differentiable vector function $\vec{w}(x)$ defined on Ω which is bounded on every compact subset of $\overline{\Omega \cap \Omega(a)}$ and satisfies Riccati inequality (2.2) *on $\Omega \cap \Omega(a)$. Then*

$$\limsup_{t\to\infty} \left(\int_{S(a_0)} H(t,x)\,\mathrm{d}\sigma \right)^{-1} \int\limits_{\Omega(a_0,t)\cap\Omega} \left[H(t,x)c(x) - \frac{\big\|\vec{h}(t,x)\big\|^p}{(Kq)^{p-1} p H^{p-1}(t,x)} \right] \mathrm{d}x < \infty. \tag{2.14}$$

Remark 2.1. Let us emphasize that nabla operator $\nabla H(t,x)$ relates only to the components of x, i.e. $\nabla H(t,x) = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) H(t,x)$, and does not relate to the variable t.

Proof of Theorem 2.2. For simplicity let us introduce the notation $\widetilde{\Omega}(a)$, $\widetilde{S}(a)$ and $\widetilde{\Omega}(a,b)$ as in the proof of Theorem 2.1. Suppose that the assumptions of theorem are fulfilled. Multiplication of (2.2) by the function $H(t,x)$ gives

$$H(t,x)\,\mathrm{div}\,\vec{w}(x) + H(t,x)c(x) + KH(t,x)\|\vec{w}(x)\|^q + H(t,x)\left\langle \vec{w}(x), \vec{b}(x) \right\rangle \leq 0$$

and equivalently

$$\begin{aligned} &\mathrm{div}(H(t,x)\vec{w}(x)) + H(t,x)c(x) \\ &\qquad + KH(t,x)\|\vec{w}(x)\|^q + \left\langle \vec{w}(x), H(t,x)\vec{b}(x) - \nabla H(t,x) \right\rangle \leq 0 \end{aligned}$$

for $x \in \widetilde{\Omega}(a)$ and $t \geq \|x\|$. This and Young inequality

$$\frac{\|\vec{a}\|^p}{p} \pm \left\langle \vec{a}, \vec{b} \right\rangle + \frac{\|\vec{b}\|^q}{q} \geq 0\,. \tag{2.15}$$

imply

$$\operatorname{div}(H(t,x)\vec{w}(x)) + H(t,x)c(x) - \frac{\|H(t,x)\vec{b}(x) - \nabla H(t,x)\|^p}{(Kq)^{p-1}pH^{p-1}(t,x)} \leq 0.$$

Integration of this inequality over the domain $\widetilde{\Omega}(a,t)$ and the Gauss-Ostrogradski divergence theorem give

$$-\int_{\widetilde{S}(a)} H(t,x)\left\langle \vec{w}(x), \vec{\nu}(x) \right\rangle \,\mathrm{d}\sigma + \int_{\widetilde{\Omega}(a,t)} \Big[H(t,x)c(x) - \frac{\|\vec{h}(t,x)\|^p}{(Kq)^{p-1}pH^{p-1}(t,x)} \Big] \,\mathrm{d}x \,\leq 0$$

and hence

$$\int_{\widetilde{\Omega}(a,t)} \Big[H(t,x)c(x) - \frac{\|\vec{h}(t,x)\|^p}{(Kq)^{p-1}pH^{p-1}(t,x)} \Big] \,\mathrm{d}x \,\leq \int_{\widetilde{S}(a)} H(t,x)\|w(x)\| \,\mathrm{d}\sigma$$

holds for $t > a$. We will use this bound to estimate the integral from condition (2.14)

$$\begin{aligned}
&\int_{\widetilde{\Omega}(a_0,t)} \Big[H(t,x)c(x) - \frac{\|\vec{h}(t,x)\|^p}{(Kq)^{p-1}pH^{p-1}(t,x)} \Big] \,\mathrm{d}x \\
&\qquad = \int_{\widetilde{\Omega}(a_0,a)} \Big[H(t,x)c(x) - \frac{\|\vec{h}(t,x)\|^p}{(Kq)^{p-1}pH^{p-1}(t,x)} \Big] \,\mathrm{d}x \\
&\qquad\quad + \int_{\widetilde{\Omega}(a,t)} \Big[H(t,x)c(x) - \frac{\|\vec{h}(t,x)\|^p}{(Kq)^{p-1}pH^{p-1}(t,x)} \Big] \,\mathrm{d}x \\
&\qquad \leq \int_{\widetilde{\Omega}(a_0,a)} H(t,x)c(x) \,\mathrm{d}x \,+ \int_{\widetilde{S}(a)} H(t,x)\|w(x)\| \,\mathrm{d}\sigma\,.
\end{aligned}$$

Denote the maximal functions $c^*(r) = \max\big\{|c(x)| : x \in S(r)\big\}$ and $w^*(r) = \max\big\{\|w(x)\| : x \in S(r)\big\}$. Then

$$\begin{aligned}
&\int_{\widetilde{\Omega}(a_0,t)} \Big[H(t,x)c(x) - \frac{\|\vec{h}(t,x)\|^p}{(Kq)^{p-1}pH^{p-1}(t,x)} \Big] \,\mathrm{d}x \\
&\qquad \leq \int_{a_0}^{a} \Big[k(r) \int_{\widetilde{S}(r)} H(t,x) \,\mathrm{d}\sigma \Big] \frac{c^*(r)}{k(r)} \,\mathrm{d}r \,+ k(a)\frac{w^*(a)}{k(a)} \int_{\widetilde{S}(a)} H(t,x) \,\mathrm{d}\sigma \\
&\qquad \leq k(a_0) \int_{\widetilde{S}(a_0)} H(t,x) \,\mathrm{d}\sigma \,\Big[\int_{a_0}^{a} \frac{c^*(r)}{k(r)} \,\mathrm{d}r \,+ \frac{w^*(a)}{k(a)} \Big]
\end{aligned}$$

holds for every $t \geq a_0$. From here we conclude that the expression

$$\Big(\int_{\widetilde{S}(a_0)} H(t,x) \,\mathrm{d}\sigma \Big)^{-1} \int_{\widetilde{\Omega}(a_0,t)} \Big[H(t,x)c(x) - \frac{\|\vec{h}(t,x)\|^p}{(Kq)^{p-1}pH^{p-1}(t,x)} \Big] \,\mathrm{d}x$$

is bounded for all $t \geq a_0$. Hence (2.14) follows. The proof is complete. □

As in Corollary 2.1, we restate the result of Theorem 2.2 also for $\Omega = \mathbb{R}^n$.

Corollary 2.2. *Let $c \in C(\Omega(a_0))$, $\vec{b} \in C(\Omega(a_0), \mathbb{R}^n)$. Suppose that the function $H(t,x)$ satisfies assumption* (A2) *and the following conditions:*

(i) $H(t,x) = 0$ *if and only if* $\|x\| = t$

(ii) *The vector function $\vec{h}(x)$ defined on D_0 with the relation* (2.12) *satisfies*

$$\int_{\Omega(a_0,t)} H^{1-p}(t,x)\|\vec{h}(t,x)\|^p \,\mathrm{d}x < \infty$$

(iii) *There exists a continuous function $k(r) \in C([a_0,\infty), \mathbb{R}^+)$ such that the function $f(r,t) := k(r)\int_{S(r)} H(t,x)\,\mathrm{d}x$ is positive and nonincreasing on $[a_0,t)$ with respect to the variable r for every t, $t > r$.*

Further suppose that there exist real numbers $a \geq a_0$, $K > 0$ and differentiable vector function $\vec{w}(x)$ defined on $\Omega(a)$ which satisfies the Riccati inequality (2.2) *on $\Omega(a)$. Then*

$$\limsup_{t\to\infty}\Big(\int_{S(a_0)} H(t,x)\,\mathrm{d}\sigma\Big)^{-1}\int_{\Omega(a_0,t)}\Big[H(t,x)c(x) - \frac{\|\vec{h}(t,x)\|^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x < \infty.$$

The proof of this theorem is a simplification of the proof of Theorem 2.2.

3 Oscillation of three terms half-linear equation

Recall that in the current chapter we study three terms half-linear equation (1.1)

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \Big\rangle + c(x)\Phi(u) = 0.$$

Our main tool will be a modification of Lemma 3.1 from page 6 which presents the relationship between positive solution of (1.1) and a solution of the Riccati–type equation:

Lemma 3.1. *Let u be a solution of Eq.* (1.1) *which has no zero on the domain $\Omega \subseteq \mathbb{R}^n$. Then the vector variable $\vec{w}(x)$ defined on the domain Ω by*

$$\vec{w}(x) = \frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)} \tag{3.1}$$

is well defined on Ω and satisfies the Riccati-type equation

$$\operatorname{div}\vec{w} + c(x) + (p-1)\|\vec{w}\|^q + \Big\langle \vec{w}, \vec{b}(x) \Big\rangle = 0 \tag{3.2}$$

for every $x \in \Omega$.

Proof. From (3.1) it follows (the dependence on the variable x is suppressed in the notation)

$$\operatorname{div}\vec{w} = \frac{\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right)}{|u|^{p-2}u} - (p-1)\frac{\|\nabla u\|^p}{|u|^p}$$

on the domain Ω. Since u is a positive solution of (1.1) on Ω it follows

$$\begin{aligned}\operatorname{div}\vec{w} &= -c - \left\langle \vec{b}, \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u} \right\rangle - (p-1)\frac{\|\nabla u\|^p}{|u|^p} \\ &= -c - (p-1)\frac{\|\nabla u\|^p}{|u|^p} - \left\langle \vec{b}, \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u} \right\rangle .\end{aligned}$$

Application of (3.1) gives $\operatorname{div}\vec{w} = -c-(p-1)\|\vec{w}\|^q - \left\langle \vec{b}, \vec{w} \right\rangle$ on Ω. Hence (3.2) holds. □

The first theorem concerns the case in which left-hand sides of (3.2) and (2.1) differ at most in a multiple by the function α.

Theorem 3.1. *Suppose that there exists function $\alpha \in C^1(\Omega(a_0), \mathbb{R}^+)$ which satisfies*

(i) *for $x \in \Omega(a_0)$*

$$\nabla\alpha(x) = \vec{b}(x)\alpha(x) \tag{3.3}$$

(ii) *condition* (2.9) *holds and*

(iii)

$$\lim_{t\to\infty} \int_{\Omega(a_0,t)} \alpha(x)c(x)\,\mathrm{d}x = \infty. \tag{3.4}$$

Then Eq. (1.1) *is oscillatory in* $\Omega(a_0)$.

Proof. Suppose, by contradiction, that (2.9), (3.3) and (3.4) hold and (1.1) is not oscillatory in $\Omega(a_0)$. Then there exists a real number $a \geq a_0$ such that Eq. (1.1) possesses a solution u positive on $\overline{\Omega}(a)$. The function $\vec{w}(x)$ defined on $\Omega(a)$ by (3.1) is well-defined, satisfies (3.2) on $\Omega(a)$ and is bounded on every compact subset of $\overline{\Omega}(a)$. In view of condition (3.3), Eq. (3.2) can be written in the form

$$\alpha\operatorname{div}\vec{w} + \alpha c + (p-1)\alpha\|\vec{w}\|^q + \langle\vec{w}, \nabla\alpha\rangle = 0$$

which implies (2.1) with $K = p-1$. Corollary 2.1 shows that (2.10) holds, a contradiction to (3.4). □

The following theorem concerns the linear case $p = 2$.

Theorem 3.2. *Let $\alpha \in C(\Omega(a_0), \mathbb{R}^+)$ Denote*

$$C_1(x) = c(x) - \frac{1}{4\alpha^2(x)}\left\|\alpha(x)\vec{b}(x) - \nabla\alpha(x)\right\|^2 - \frac{1}{2\alpha(x)}\operatorname{div}\Big(\alpha(x)\vec{b}(x) - \nabla\alpha(x)\Big).$$

Suppose that

$$\begin{aligned}&\int_{a_0}^{\infty}\Big(\int_{S(t)}\alpha(x)\,\mathrm{d}\sigma\Big)^{-1}\mathrm{d}t = \infty\,, \\ &\lim_{t\to\infty}\int_{\Omega(a_0,t)}\alpha(x)C_1(x)\,\mathrm{d}x = \infty.\end{aligned} \tag{3.5}$$

Then the linear damped PDE (1.2) *is oscillatory in* $\Omega(a_0)$.

Proof. Suppose, by contradiction, that (1.2) is nonoscillatory. As in the proof of Theorem 3.1, there exists $a \geq a_0$ such that (3.2) with $p = 2$ has a solution $\vec{w}(x)$ defined on $\Omega(a)$. Denote $\vec{W}(x) = \vec{w}(x) + \frac{1}{2}\left(\vec{b} - \frac{\nabla\alpha}{\alpha}\right)$. Direct computation shows

$$\begin{aligned}
\operatorname{div}\bigl(\alpha\vec{W}\bigr) &= \langle\nabla\alpha, \vec{w}\rangle + \alpha \operatorname{div}\vec{w} + \frac{1}{2}\operatorname{div}\bigl(\alpha\vec{b} - \nabla\alpha\bigr) \\
&= \langle\nabla\alpha, \vec{w}\rangle - \alpha c - \alpha\|\vec{w}\|^2 - \left\langle \alpha\vec{b}, \vec{w}\right\rangle + \frac{1}{2}\operatorname{div}\bigl(\alpha\vec{b} - \nabla\alpha\bigr) \\
&= -\alpha\left(c - \frac{1}{2\alpha}\operatorname{div}\bigl(\alpha\vec{b} - \nabla\alpha\bigr) + \|\vec{w}\|^2 + 2\left\langle \vec{w}, \frac{1}{2}\Bigl(\vec{b} - \frac{\nabla\alpha}{\alpha}\Bigr)\right\rangle\right) \\
&= -\alpha\left(c - \frac{1}{2\alpha}\operatorname{div}\bigl(\alpha\vec{b} - \nabla\alpha\bigr) + \left\|\vec{w} + \frac{1}{2}\Bigl(\vec{b} - \frac{\nabla\alpha}{\alpha}\Bigr)\right\|^2 - \frac{1}{4}\left\|\vec{b} - \frac{\nabla\alpha}{\alpha}\right\|^2\right) \\
&= -\alpha\left(c - \frac{1}{2\alpha}\operatorname{div}\bigl(\alpha\vec{b} - \nabla\alpha\bigr) - \frac{1}{4}\left\|\vec{b} - \frac{\nabla\alpha}{\alpha}\right\|^2\right) - \alpha\left\|\vec{w} + \frac{1}{2}\Bigl(\vec{b} - \frac{\nabla\alpha}{\alpha}\Bigr)\right\|^2
\end{aligned}$$

and the function $\vec{W}$ satisfies

$$\operatorname{div}\left(\alpha\vec{W}\right) + C_1\alpha + \alpha\|\vec{W}\|^2 = 0$$

on $\Omega(a)$. However by Corollary 2.1 inequality (2.10) with C_1 instead of c holds, a contradiction to (3.5). $\square$

The next theorem deals with the general case $p > 1$. In this case we allow also another types of unbounded domains, than $\Omega(a_0)$.

Theorem 3.3. *Let Ω be an unbounded domain which satisfies* (A1). *Suppose that $k \in (1, \infty)$ is a real number and $\alpha \in C^1(\Omega(a_0), \mathbb{R}_0^+)$ is a function defined on $\Omega(a_0)$ such that*

(i) $\alpha(x) = 0$ *if and only if* $x \notin \Omega \cap \Omega(a_0)$,

(ii) (2.3) *holds.*

For $x \in \Omega \cap \Omega(a_0)$ denote

$$C_2(x) = c(x) - \frac{k}{(p\alpha(x))^p}\left\|\alpha(x)\vec{b}(x) - \nabla\alpha(x)\right\|^p.$$

If

$$\lim_{t\to\infty}\int_{\Omega\cap\Omega(a_0,t)} \alpha(x)C_2(x)\,\mathrm{d}x = \infty \tag{3.6}$$

holds, then Eq. (1.1) *is oscillatory in Ω.*

Remark 3.1. Under (3.6) we understand that the integral

$$f(t) = \int_{\Omega\cap S(t)} \alpha(x)C_2(x)\,\mathrm{d}\sigma$$

which may have singularity near the boundary $\partial\Omega$ is convergent for large t's and the function f satisfies $\int^{\infty} f(t)\,\mathrm{d}t = \infty$.

Proof of Theorem 3.3. Suppose, by contradiction, that (1.1) is not oscillatory. Then there exists a number $a \geq a_0$ and a function u defined on $\Omega(a)$ which is positive on $\overline{\Omega \cap \Omega(a)}$ and satisfies (1.1) on $\Omega \cap \Omega(a)$. The vector function $\vec{w}(x)$ defined by (3.1) satisfies (3.2) on $\Omega \cap \Omega(a)$ and is bounded on every compact subset of $\overline{\Omega \cap \Omega(a)}$. Denote $l = k^{\frac{1}{p-1}}$ and let l^* be a conjugate number to the number l, i.e. $\frac{1}{l} + \frac{1}{l^*} = 1$ holds. Clearly $l > 1$ and $l^* > 1$. Riccati equation (3.2) can be written in the form

$$\operatorname{div} \vec{w} + c(x) + \frac{p-1}{l}\|\vec{w}\|^q + \left\langle \vec{w}, \vec{b}(x) - \frac{\nabla\alpha}{\alpha} \right\rangle + \frac{p-1}{l^*}\|\vec{w}\|^q + \left\langle \vec{w}, \frac{\nabla\alpha}{\alpha} \right\rangle = 0$$

for $x \in \Omega \cap \Omega(a)$. From inequality (2.15) it follows

$$\begin{aligned} \frac{p-1}{l}\|\vec{w}\|^q + \left\langle \vec{w}, \vec{b} - \frac{\nabla\alpha}{\alpha} \right\rangle &= \frac{(p-1)q}{l} \left\{ \frac{\|\vec{w}\|^q}{q} + \left\langle \vec{w}, \frac{l}{(p-1)q}\Big(\vec{b} - \frac{\nabla\alpha}{\alpha}\Big) \right\rangle \right\} \\ &\geq -\frac{(p-1)q}{l}\, \frac{l^p}{[(p-1)q]^p} \left\| \vec{b} - \frac{\nabla\alpha}{\alpha} \right\|^p \frac{1}{p} \\ &= -\frac{l^{p-1}}{p^p} \left\| \vec{b} - \frac{\nabla\alpha}{\alpha} \right\|^p \\ &= -\frac{k}{p^p} \left\| \vec{b} - \frac{\nabla\alpha}{\alpha} \right\|^p \end{aligned}$$

Hence the function $\vec{w}$ is a solution of the inequality

$$\operatorname{div} \vec{w} + C_2(x) + \frac{p-1}{l^*}\|\vec{w}\|^q + \left\langle \vec{w}, \frac{\nabla\alpha}{\alpha} \right\rangle \leq 0$$

on $\Omega \cap \Omega(a)$. This last inequality is equivalent to

$$\operatorname{div}\big(\alpha\vec{w}\big) + \alpha C_2 + \frac{p-1}{l^*}\alpha\|\vec{w}\|^q \leq 0.$$

By Theorem 2.1, inequality (2.4) with C_2 instead of c holds, a contradiction to (3.6). The proof is complete. □

The last theorem makes use of the two-parametric weight function $H(t,x)$ from Theorem 2.2 to prove the nonexistence of the solution of Riccati equation.

Theorem 3.4. *Let Ω be an unbounded domain in $\mathbb{R}^n$ which satisfy* (A1)*. Let $H(t,x)$ be the function which satisfies hypothesis* (A2) *and has the properties* (i)–(v) *of Theorem 2.2. If*

$$\limsup_{t\to\infty} \Big(\int_{S(a_0)} H(t,x)\,\mathrm{d}\sigma \Big)^{-1} \int_{\Omega(a_0,t)\cap\Omega} \left[H(t,x)c(x) - \frac{\|\vec{h}(t,x)\|^p}{p^p H^{p-1}(t,x)} \right] \mathrm{d}x = \infty, \tag{3.7}$$

then Eq. (1.1) *is oscillatory in Ω.*

Proof. Suppose, by contradiction, that (1.1) is nonoscillatory. Then Riccati equation (3.2) has a solution defined on $\Omega \cap \Omega(T)$ for some $T > 1$, which is bounded near the boundary $\partial\Omega$. Hence (2.14) of Theorem 2.2 with $K = p - 1$ holds, a contradiction to (3.7). Hence the theorem follows. □

3.1 Examples

Let us illustrate the ideas from the preceding text on examples. The specification of the function α in Theorem 3.3 leads to the following oscillation criterion for a conic domain on the plane. The function α is function of a polar coordinate φ only.

Corollary 3.1. *Let us consider Eq.* (1.4) *on the plane (i.e. $n = 2$) with polar coordinates (r, φ) and let*

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : \varphi_1 < \varphi(x, y) < \varphi_2\right\}, \tag{3.8}$$

where $0 \le \varphi_1 < \varphi_2 \le 2\pi$ and $\varphi(x, y)$ is a polar coordinate of the point $(x, y) \in \mathbb{R}^2$. Further suppose that the smooth function $\alpha \in C^1(\Omega(1), \mathbb{R}_0^+)$ does not depend on r, i.e. $\alpha = \alpha(\varphi)$. Also, suppose that

(i) $\alpha(\varphi) \neq 0$ *if and only if* $\varphi \in (\varphi_1, \varphi_2)$

(ii)

$$I_1 := \int_{\varphi_1}^{\varphi_2} \frac{\left|\alpha'_\varphi(\varphi)\right|^p}{4\alpha^{p-1}(\varphi)} \,\mathrm{d}\varphi \, < \infty, \tag{3.9}$$

where $\alpha'_\varphi = \frac{\partial \alpha}{\partial \varphi}$.

Each one of the following conditions is sufficient for oscillation of (1.4) *on the domain* Ω*:*

(i) $p > 2$ *and*

$$\lim_{t\to\infty} \int_1^t r \int_{\varphi_1}^{\varphi_2} c(r, \varphi)\, \alpha(\varphi) \,\mathrm{d}\varphi \;\mathrm{d}r \, = \infty \tag{3.10}$$

(ii) $p = 2$ *and*

$$\liminf_{t\to\infty} \frac{1}{\ln t} \int_1^t r \int_{\varphi_1}^{\varphi_2} c(r, \varphi)\, \alpha(\varphi) \,\mathrm{d}\varphi \;\mathrm{d}r \, > I_1, \tag{3.11}$$

where $c(r, \varphi)$ is the potential $c(x)$ transformed into polar coordinates and I_1 is defined by (3.9)*.*

Proof. First let us remind that in the polar coordinates $\mathrm{d}x = r\,\mathrm{d}r\,\mathrm{d}\varphi$ and $\mathrm{d}\sigma = r\,\mathrm{d}\varphi$ holds. Direct computation shows that

$$\int^{\infty} \Bigl(\int_{\Omega\cap S(t)} \alpha(x)\,\mathrm{d}\sigma\Bigr)^{1-q} \mathrm{d}t \, = \int_{\varphi_1}^{\varphi_2} \alpha(\varphi)\,\mathrm{d}\varphi \cdot \int^{\infty} t^{1-q}\,\mathrm{d}t\,.$$

and the integral diverges, since $p \ge 2$ is equivalent to $q \le 2$. Hence (2.3) holds. Transforming the nabla operator to the polar coordinates gives $\nabla\alpha = \bigl(0, r^{-1}\alpha'_\varphi(\varphi)\bigr)$. Hence, according to Theorem 3.3, it is sufficient to show that there exists $k > 1$ such that

$$\lim_{t\to\infty} \int_{\Omega\cap\Omega(1,t)} \left[c(r, \varphi)\alpha(\varphi) - \frac{k}{p^p} \frac{|\alpha'_\varphi(\varphi)|^p}{r^p \alpha^{p-1}(\varphi)}\right] \mathrm{d}x \, = \infty. \tag{3.12}$$

Since for $p > 2$

$$\lim_{t\to\infty}\int_{\Omega\cap\Omega(1,t)}\frac{|\alpha'_\varphi(\varphi)|^p}{r^p\alpha^{p-1}(\varphi)}\,\mathrm{d}x \;=\int_{\varphi_1}^{\varphi_2}\frac{|\alpha'_\varphi(\varphi)|^p}{\alpha^{p-1}(\varphi)}\,\mathrm{d}\varphi\,\lim_{t\to\infty}\int_1^t r^{1-p}\,\mathrm{d}r\;<\infty,$$

conditions (3.12) and (3.10) are equivalent.

Finally, suppose $p = 2$. From (3.11) it follows that there exists $t_0 > 1$ and $\epsilon > 0$ such that

$$\frac{1}{\ln t}\int_{\Omega\cap\Omega(1,t)} c(r,\varphi)\,\alpha(\varphi)\,\mathrm{d}x\;> I_1 + 2\epsilon$$

for all $t \ge t_0$ and hence

$$\int_{\Omega\cap\Omega(1,t)} c(r,\varphi)\,\alpha(\varphi)\,\mathrm{d}x\;>\Big[kI_1+\epsilon\Big]\ln t$$

where $k = 1 + \epsilon I_1^{-1}$ holds for $t \ge t_0$. Since

$$\begin{aligned}
kI_1\ln t &= \frac{k\ln t}{4}\int_{\varphi_1}^{\varphi_2}|\alpha'_\varphi(\varphi)|^2\alpha^{-1}(\varphi)\,\mathrm{d}\varphi\\
&=\int_1^t\frac{k}{4r}\Big(\int_{\varphi_1}^{\varphi_2}|\alpha'_\varphi(\varphi)|^2\alpha^{-1}(\varphi)\,\mathrm{d}\varphi\Big)\,\mathrm{d}r\\
&=\int_{\Omega\cap\Omega(1,t)}\frac{k}{4r^2}|\alpha'_\varphi(\varphi)|^2\alpha^{-1}(\varphi)\,\mathrm{d}x
\end{aligned}$$

holds, the last inequality can be written in the form

$$\int_{\Omega\cap\Omega(1,t)}\left[c(r,\varphi)\alpha(\varphi)-\frac{k}{4}\,\frac{|\alpha'_\varphi(\varphi)|^2}{r^2\alpha(\varphi)}\right]\,\mathrm{d}x\;>\epsilon\ln t$$

and the limit process $t \to \infty$ shows that (3.12) holds also for $p = 2$. The proof is complete. □

Example 3.1. For $n = 2$ consider Schrödinger equation (1.3), which in polar coordinates (r, φ) reads as

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big)+\frac{1}{r^2}\,\frac{\partial^2 u}{\partial\varphi^2}+c(r,\varphi)u=0. \tag{3.13}$$

In Corollary 3.1, let us choose $\varphi_1 = 0$, $\varphi_2 = \pi$, $\alpha(\varphi) = \sin^2\varphi$ for $\varphi \in [0, \pi]$ and $\alpha(\varphi) = 0$ otherwise. In this case the direct computation shows that the oscillation constant I_1 in (3.11) is $\frac{\pi}{2}$, i.e. the equation is oscillatory on the half-plane $\Omega = \big\{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\big\}$ if

$$\lim_{t\to\infty}\frac{1}{\ln t}\int_1^t r\int_0^\pi c(r,\varphi)\,\sin^2(\varphi)\,\mathrm{d}\varphi\;\mathrm{d}r\;>\frac{\pi}{2}. \tag{3.14}$$

Similarly, the choice $\alpha(\varphi) = \sin^3\varphi$ gives an oscillation constant $3/2$.

Remark 3.2. It is easy to see that condition (3.14) can be fulfilled also for the function c which satisfy $\int_0^{2\pi} c(r,\varphi)\,\mathrm{d}\varphi\;=0$. Usual radial criteria fail to detect the oscillation in this case.

Another specification of the function $\alpha(x)$ leads to the following corollary.

Corollary 3.2. *Let Ω be an unbounded domain in $\mathbb{R}^2$ specified in Corollary 3.1. Let $A \in C^1([0,2\pi],\mathbb{R}_0^+)$ be a smooth function satisfying*

(i) $A(\varphi) \neq 0$ *if and only in* $\varphi \in (\varphi_1, \varphi_2)$

(ii) $A(0) = A(2\pi)$ *and* $A'(0+) = A'(2\pi-)$

(iii) *the following integral converges*

$$I_2 := \int_{\varphi_1}^{\varphi_2} \frac{\left[A^2(\varphi)(p-2)^2 + (A'(\varphi))^2\right]^{\frac{p}{2}}}{p^p A^{p-1}(\varphi)}\,\mathrm{d}\varphi < \infty. \tag{3.15}$$

If

$$\liminf_{t\to\infty} \frac{1}{\ln t}\int_1^t r^{p-1}\int_{\varphi_1}^{\varphi_2} c(r,\varphi)A(\varphi)\,\mathrm{d}\varphi\;\mathrm{d}r > I_2, \tag{3.16}$$

then (1.4) *is oscillatory in* Ω*.*

Proof. Let α be defined in polar coordinates by the relation

$$\alpha\big(x(r,\varphi)\big) = r^{p-2}A(\varphi).$$

Computation in polar coordinates gives

$$\begin{aligned}\int^{\infty}\Big(\int_{\Omega\cap S(t)}\alpha(x)\,\mathrm{d}\sigma\Big)^{1-q}\mathrm{d}t &= \int^{\infty}\Big(r^{p-1}\Big)^{1-q}\mathrm{d}r\int_{\varphi_1}^{\varphi_2}A(\varphi)\,\mathrm{d}\varphi\\ &= \int^{\infty}\frac{1}{r}\,\mathrm{d}r\int_{\varphi_1}^{\varphi_2}A(\varphi)\,\mathrm{d}\varphi = \infty\end{aligned}$$

and hence (2.3) holds. An application of nabla operator in polar coordinates yields

$$\nabla\alpha(x(r,\varphi)) = \Big(\frac{\partial\alpha(x(r,\varphi))}{\partial r}, \frac{1}{r}\,\frac{\partial\alpha(x(r,\varphi))}{\partial\varphi}\Big) = r^{p-3}\Big((p-2)A(\varphi), A'(\varphi)\Big)$$

and hence

$$\begin{aligned}\frac{\|\nabla\alpha(x(r,\varphi))\|^p}{\alpha^{p-1}(x(r,\varphi))} &= \frac{r^{p(p-3)}\left[(p-2)^2A^2(\varphi)+A'^2(\varphi)\right]^{p/2}}{r^{(p-1)(p-2)}A^{p-1}(\varphi)}\\ &= r^{-2}\frac{\left[(p-2)^2A^2(\varphi)+A'^2(\varphi)\right]^{p/2}}{A^{p-1}(\varphi)}\end{aligned}$$

holds on Ω. Integration over the part $\Omega\cap S(r)$ of the sphere $S(r)$ in polar coordinates gives (in view of (3.15))

$$\int_{\Omega\cap S(r)}\frac{\|\nabla\alpha(x(r,\varphi))\|^p}{p^p\alpha^{p-1}(x(r,\varphi))}\,\mathrm{d}\sigma = r^{-1}I_2.$$

From (3.16) it follows that there exist real numbers $\epsilon > 0$ and $t_0 > 1$ such that

$$\frac{1}{\ln t}\int_1^t r^{p-1}\int_{\varphi_1}^{\varphi_2} c(r,\varphi)A(\varphi)\,\mathrm{d}\varphi\;\mathrm{d}r > I_2 + 2\epsilon = I_2\left(1+\epsilon I_2^{-1}\right)+\epsilon \tag{3.17}$$

holds for $t > t_0$. Denote $k = 1 + \epsilon I_2^{-1}$. Clearly $k > 1$. From (3.17) it follows that for $t > t_0$

$$\int_1^t r^{p-1} \int_{\varphi_1}^{\varphi_2} c(r,\varphi)A(\varphi)\,\mathrm{d}\varphi\ \mathrm{d}r\ > kI_2 \ln t + \epsilon \ln t$$

holds. This inequality can be written in the form

$$\int_1^t \Big[r^{p-1} \int_{\varphi_1}^{\varphi_2} c(r,\varphi)A(\varphi)\,\mathrm{d}\varphi\ - r^{-1}kI_2\Big]\,\mathrm{d}r\ > \epsilon \ln t$$

which is equivalent to

$$\int_{\Omega\cap\Omega(1,t)} \left[c(r,\varphi)\alpha(r,\varphi) - k\frac{\|\nabla\alpha(r,\varphi)\|^p}{p^p\alpha^{p-1}(r,\varphi)}\right]\,\mathrm{d}x\ > \epsilon \ln t,$$

where $\mathrm{d}x\ = r\,\mathrm{d}r\ \mathrm{d}\varphi$. Now the limit process $t \to \infty$ shows that (3.6) holds and hence (1.4) is oscillatory in Ω by Theorem 3.3. □

Example 3.2. An example of the function A which for $p > 1$, $\varphi_1 = 0$ and $\varphi_2 = \pi$ satisfies the conditions from Corollary 3.2 is $A(\varphi) = \sin^p \varphi$ for $\varphi \in (0, \pi)$ and $A(\varphi) = 0$ otherwise. In this case the condition

$$\liminf_{t\to\infty} \frac{1}{\ln t} \int_1^t r^{p-1}\Big(\int_0^\pi c(r,\varphi)\sin^p\varphi\,\mathrm{d}\varphi\Big)\,\mathrm{d}r$$
$$> \int_0^\pi \frac{\big[(p-2)^2\sin^{2p}\varphi + p^2\sin^{2p-2}\varphi\cos^2\varphi\big]^{p/2}}{p^p\sin^{p(p-1)}\varphi}\,\mathrm{d}\varphi$$

is sufficient for oscillation of (1.4) (with $n = 2$) over the domain Ω specified in (3.8). Here $c(r,\varphi)$ is the potential $c(x)$ transformed into polar coordinates (r,φ), i.e. $c(r,\varphi) = c\,(x(r,\varphi))$.

Corollary 3.3. *Let us consider Schrödinger equation* (3.13) *in polar coordinates. Every of the following two conditions is sufficient for the oscillation of this equation over the half-plane*

$$\Omega = \big\{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0\big\}\,.$$

(i) *There exists $\lambda > 1$ such that*

$$\limsup_{t\to\infty} t^{-\lambda} \int_1^t (t-r)^\lambda\Big(r\int_0^\pi c(r,\varphi)\sin^2\varphi\,\mathrm{d}\varphi\ - \frac{\pi}{2r}\Big)\,\mathrm{d}r\ = \infty. \tag{3.18}$$

(ii) *There exists $\lambda > 1$ and $\gamma < 0$ such that*

$$\limsup_{t\to\infty} t^{-\lambda} \int_1^t r^{\gamma+1}(t-r)^\lambda \int_0^\pi c(r,\varphi)\sin^2\varphi\,\mathrm{d}\varphi\ \mathrm{d}r\ = \infty. \tag{3.19}$$

Proof. For $\gamma \le 0$ let us define

$$H(t,x) = \begin{cases} r^\gamma(t-r)^\lambda\sin^2\varphi & \varphi \in (0\pi), \\ 0 & \text{otherwise,} \end{cases}$$

where (r, φ) are polar coordinates of the point $x \in \mathbb{R}^2$. Recall that in polar coordinates $\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}\right)$. Hence

$$\begin{aligned}\vec{h}\big(t, x(r,\varphi)\big) &= -\nabla H\big(t, x(r,\varphi)\big)\\ &= -\Big(r^{\gamma-1}(t-r)^{\lambda-1}(\gamma(t-r)-\lambda r)\sin^2\varphi,\ 2r^{\gamma-1}(t-r)^{\lambda}\sin\varphi\cos\varphi\Big)\end{aligned}$$

and consequently

$$\begin{aligned}\frac{\|\vec{h}(t, x(r,\varphi))\|^2}{H(t, x(r,\varphi))} &= \gamma^2 r^{\gamma-2}(t-r)^{\lambda}\sin^2\varphi - 2\lambda\gamma r^{\gamma-1}(t-r)^{\lambda-1}\sin^2\varphi\\ &\quad + \lambda^2 r^{\gamma}(t-r)^{\lambda-2}\sin^2\varphi + 4r^{\gamma-2}(t-r)^{\lambda}\cos^2\varphi. \qquad (3.20)\end{aligned}$$

Inequality $\lambda - 2 > -1$ holds for $\lambda > 1$. Hence the integral over $\Omega \cap \Omega(1,t)$ converges and (2.13) for $p = 2$ holds. Further

$$\int_{S(r)\cap\Omega} H(t,x)\,\mathrm{d}\sigma = r\int_0^{\pi} r^{\gamma}(t-r)^{\lambda}\sin^2\varphi\,\mathrm{d}\varphi = \frac{\pi}{2}r^{\gamma+1}(t-r)^{\lambda}$$

and the condition (v) of Theorem 2.2 holds with $k(r) = r^{-1-\gamma}$. It remains to prove that conditions (3.18) and (3.19) imply (3.7). Since $\int_0^{\pi}\sin^2\varphi\,\mathrm{d}\varphi = \int_0^{\pi}\cos^2\varphi\,\mathrm{d}\varphi = \frac{\pi}{2}$, it follows from (3.20) that

$$\begin{aligned}\int_{S(r)\cap\Omega}\frac{\|\vec{h}(t, x(r,\varphi))\|^2}{H(t, x(r,\varphi))}\,\mathrm{d}\sigma &= \frac{\pi}{2}(\gamma^2+4)r^{\gamma-1}(t-r)^{\lambda} - \pi\lambda\gamma r^{\gamma}(t-r)^{\lambda-1}\\ &\quad + \frac{\pi}{2}\lambda^2 r^{\gamma+1}(t-r)^{\lambda-2}. \qquad (3.21)\end{aligned}$$

Next we will show that

$$\lim_{t\to\infty} t^{-\lambda}\int_1^t r^{\gamma}(t-r)^{\lambda-1}\,\mathrm{d}r < \infty, \qquad (3.22)$$

$$\lim_{t\to\infty} t^{-\lambda}\int_1^t r^{\gamma+1}(t-r)^{\lambda-2}\,\mathrm{d}r < \infty \qquad (3.23)$$

and for $\gamma < 0$ also

$$\lim_{t\to\infty} t^{-\lambda}\int_1^t r^{\gamma-1}(t-r)^{\lambda}\,\mathrm{d}r < \infty \qquad (3.24)$$

holds. Inequality (3.22) follows from the estimate

$$\int_1^t r^{\gamma}(t-r)^{\lambda-1}\,\mathrm{d}r \le \int_1^t 1^{\gamma}(t-r)^{\lambda-1}\,\mathrm{d}r = \frac{1}{\lambda}(t-1)^{\lambda}.$$

Integration by parts shows

$$\int_1^t r^{\gamma+1}(t-r)^{\lambda-2}\,\mathrm{d}r = \frac{(t-1)^{\lambda-1}}{\lambda-1} + \frac{\gamma+1}{\lambda-1}\int_1^t r^{\gamma}(t-r)^{\lambda-1}\,\mathrm{d}r$$

and in view of (3.22) inequality (3.23) holds as well. Finally, for $\gamma < 0$ integration by parts gives

$$\int_1^t r^{\gamma-1}(t-r)^{\lambda}\,\mathrm{d}r = \frac{(t-1)^{\lambda}}{\gamma} + \frac{\lambda}{\gamma}\int_1^t r^{\gamma}(t-r)^{\lambda-1}\,\mathrm{d}r$$

and inequality (3.24) follows from (3.22). Hence the terms from (3.21) have no influence on the divergence of (3.7) (except the term $r^{-1}(t-r)^{\lambda}$ which appears for $\gamma = 0$) and hence (3.7) follows from (3.18) and (3.19), respectively. Consequently, the equation is oscillatory by Theorem 3.4. □

4 Interval type oscillation criteria

Kong (1999) used the Riccati technique and the two-parametric averaging function $H(t, s)$ (a technique originally due to Philos (1999)) to obtain new conjugacy criteria for linear second order ordinary differential equation

$$\big(p(t)y'\big)' + q(t)y = 0 \tag{4.1}$$

and derived sufficient conditions which guarantee existence of infinitely many intervals with pairs of conjugate points. These conditions allow to eliminate "bad parts" of the interval (t_0, ∞) from the oscillation criteria and are applicable even if the integral of the function $q(t)$ is extremely small, e.g. if $\int_0^\infty q(t)\,\mathrm{d}t = -\infty$. The results from [Kong, 1999] have been extended in [Wang, 2004] for half-linear ODE.

In the remaining part of this chapter we extend results from [Kong, 1999; Wang, 2004] to damped half-linear PDE (1.1). In addition, we offer an improvement of these results (see Remark 4.3 below) which is new even in the case of the half-linear ODE (4.1) and this improvement is closely related to the recent result of Sun (2004).

Oscillation properties of Eq. (1.1) and several (less or more general) similar equations have been studied by Riccati technique in a series of papers by Xu and his colabors, see [Xu, 2005; Xing, Xu, 2003; Xing, Xu, 2005]. In these papers authors, starting with integration of the Riccati equation over spheres in $\mathbb{R}^n$ centered in the origin, convert the n-dimensional problem into a problem in one variable and then employ the corresponding techniques from the oscillation theory of ordinary differential equations. The oscillation criteria obtained in this way are radial[1] and these criteria are able detect the oscillation only if the mean value of the potential function $c(x)$ over the spheres centered in the origin is "sufficiently large".

Here[2] we prefer an advanced approach than that one used in papers by Xu: we use the averaging function which does not need to preserve radial symmetry. As a particular example, we use the $(n+1)$-variable function $H(t, \|x\|)\rho(x)$, where $x \in \mathbb{R}^n$, rather than the function of two variables $H(t, s)k(s)$ with $s \in \mathbb{R}$, used in [Xing, Xu, 2003], where s corresponds to our $\|x\|$.

In the sequel we define two classes of averaging functions: each of them will be used on one of the parts of boundary $\partial\Omega(a, b) = S(a) \cup S(b)$. Recall that the set D is defined on page 39.

Definition 4.1. The function $H(t, s) \in C(D, [0, \infty))$ is said to belong to the class $\mathcal{H}$ if

(i) $H(t, s) = 0$ if and only if $t = s$.

(ii) The partial derivative $\frac{\partial H}{\partial s}(t, s)$ exists.

(iii) Denoting

$$h_2(t, s) = -\frac{\partial H}{\partial s}(t, s)H^{-1}(t, s), \qquad \text{for } (t, s) \in D, t \neq s,$$

[1] in the sense of Remark 2.3 from page 4

[2] like already in part 4 of Chapter 2 on page 23

the function $h_2^p(t,s)H(t,s)$ is locally integrable on each compact subset in D.

Remark 4.1. Remember that the function $h_2(t,s)$ has singularity for $s=t$, since $H(t,t) = 0$. The same is true also for the function $h_1^*(t,s)$ defined below.

Definition 4.2. The function $H^*(t,s) \in C(D,[0,\infty))$ is said to belong to the class $\mathcal{H}^*$ if

(i) $H^*(t,s) = 0$ if and only if $t = s$.

(ii) The partial derivative $\frac{\partial H^*}{\partial t}(t,s)$ exists.

(iii) Denoting

$$h_1^*(t,s) = \frac{\partial H^*}{\partial t}(t,s)\Big[H^*(t,s)\Big]^{-1}, \qquad \text{for } (t,s) \in D,\, t \neq s,$$

the function $\Big[h_1^*(t,s)\Big]^p H^*(t,s)$ is locally integrable on each compact subset in D.

Remark 4.2. Note that the functions h_1^*, h_2 play slightly different role in our results than in the paper [Wang, 2004], where $h_2(t,s) = \frac{\partial H}{\partial s}(t,s)H^{-1/2}(t,s)$ and h_1^* is defined in the similar way. The reason is that we wish to gain simpler formulas in our resulting oscillation criteria.

Our main tool – Riccati-type substitution which converts Eq. (1.1) into first order Riccati-type equation has been introduced in Lemma 3.1 on page 42.

We start with some estimates in a neighborhood of the boundary of the set $\Omega(a,b)$. The first lemma treats the boundary at b and the second one at a.

Lemma 4.1. *Let u be a solution of* (1.1) *such that $u(x) > 0$ for $c \leq \|x\| < b$. Let $\rho \in C^1(\mathbb{R}^n, \mathbb{R}^+)$ be a smooth positive function and H be a function of the class $\mathcal{H}$. The vector variable $\vec{w}(x)$ defined by* (3.1) *satisfies the inequality*

$$\begin{aligned} \int_{\Omega(c,b)} H\big(b,\|x\|\big)c(x)\rho(x)\,\mathrm{d}x \;\leq H(b,c)\int_{S(c)} \rho(x)\,\langle \vec{w}(x)\,,\,\vec{\nu}(x)\rangle\;\mathrm{d}\sigma \\ + \int_{\Omega(c,b)} \Big\|\vec{b}(x) - \frac{\nabla\rho(x)}{\rho(x)} + h_2\big(b,\|x\|\big)\vec{\nu}\Big\|^p \rho(x)H\big(b,\|x\|\big)p^{-p}\,\mathrm{d}x\,. \end{aligned} \tag{4.2}$$

Proof. Suppose that positive solution u of (1.1) exists for $c \leq \|x\| < b$. Multiplying Riccati equation (3.2) by $\rho(x)$ we get

$$c(x)\rho(x) = -\rho(x)\,\mathrm{div}\,\vec{w} - (p-1)\rho(x)\|\vec{w}\|^q - \Big\langle \rho(x)\vec{w}\,,\,\vec{b}(x)\Big\rangle$$

and hence

$$c(x)\rho(x) = -\,\mathrm{div}\big(\rho(x)\vec{w}\big) - (p-1)\rho(x)\|\vec{w}\|^q - \Big\langle \rho(x)\vec{w}\,,\,\vec{b}(x) - \frac{\nabla\rho(x)}{\rho(x)}\Big\rangle. \tag{4.3}$$

Integrating over the sphere $S(s)$ of radius s, multiplying by $H(t,s)$ and integrating with respect to s over the interval (c,t), where $t < b$, we get

$$\begin{aligned} \int_{\Omega(c,t)} H\big(t,\|x\|\big)\rho(x)c(x)\,\mathrm{d}x \;&= -\int_c^t H(t,s)\int_{S(s)} \mathrm{div}\big(\rho(x)\vec{w}\big)\,\mathrm{d}\sigma\;\mathrm{d}s \\ &\quad - (p-1)\int_{\Omega(c,t)} H\big(t,\|x\|\big)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x \\ &\quad - \int_{\Omega(c,t)} H\big(t,\|x\|\big)\Big\langle \rho(x)\vec{w}\,,\,\vec{b}(x) - \frac{\nabla\rho(x)}{\rho(x)}\Big\rangle\;\mathrm{d}x\,. \end{aligned}$$

Integration by parts in the first integral on the right hand side, Gauss-Ostrogradski formula and the definition of the function h_2 give

$$\begin{aligned}\int_{\Omega(c,t)} H\bigl(t,\|x\|\bigr)c(x)\rho(x)\,\mathrm{d}x &= H(t,c)\int_{S(c)}\rho(x)\,\langle\vec{w}\,,\,\vec{\nu}\rangle\,\mathrm{d}\sigma\\ &-\int_{\Omega(c,t)} h_2\bigl(t,\|x\|\bigr)H\bigl(t,\|x\|\bigr)\rho(x)\,\langle\vec{w}\,,\,\vec{\nu}\rangle\,\mathrm{d}x\\ &-(p-1)\int_{\Omega(c,t)} H\bigl(t,\|x\|\bigr)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x\\ &-\int_{\Omega(c,t)} H\bigl(t,\|x\|\bigr)\left\langle\rho(x)\vec{w}\,,\,\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle\,\mathrm{d}x\,.\end{aligned}$$

Using Young inequality (2.15) with

$$\begin{aligned}\vec{a} &= H\bigl(t,\|x\|\bigr)\Bigl(\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+h_2\bigl(t,\|x\|\bigr)\vec{\nu}\Bigr)\Bigl((p-1)H\bigl(t,\|x\|\bigr)\Bigr)^{-\frac{1}{q}}\rho^{1-\frac{1}{q}}(x)q^{-\frac{1}{q}}\\ \vec{b} &= \rho(x)\Bigl((p-1)H\bigl(t,\|x\|\bigr)\Bigr)^{\frac{1}{q}}\rho^{\frac{1}{q}-1}(x)q^{\frac{1}{q}}\vec{w}\end{aligned}$$

we get

$$\begin{aligned}\int_{\Omega(c,t)} H\bigl(t,\|x\|\bigr)c(x)\rho(x)\,\mathrm{d}x &\le H(t,c)\int_{S(c)}\rho(x)\,\langle\vec{w}\,,\,\vec{\nu}\rangle\,\mathrm{d}\sigma\\ &+\int_{\Omega(c,t)}\frac{1}{p}H^p\bigl(t,\|x\|\bigr)\Bigl\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+h_2\bigl(t,\|x\|\bigr)\vec{\nu}\Bigr\|^p\\ &\times\rho(x)\Bigl(pH\bigl(t,\|x\|\bigr)\Bigr)^{-p/q}\,\mathrm{d}x\,.\end{aligned}$$

Now simple algebraic simplifications, identity $\frac{p}{q}=p-1$ and limit process $t\to b^-$ give (4.2). □

Lemma 4.2. *Let u be a solution of* (1.1) *such that $u(x)>0$ for $a<\|x\|\le c$. Let $\rho\in C^1(\mathbb{R}^n,\mathbb{R}^+)$ be a smooth positive function and H^* be a function of the class $\mathcal{H}^*$. The vector variable $\vec{w}(x)$ defined by* (3.1) *satisfies the inequality*

$$\begin{aligned}\int_{\Omega(a,c)} H^*\bigl(\|x\|,a\bigr)\rho(x)c(x)\,\mathrm{d}x &\le -H^*(c,a)\int_{S(c)}\rho(x)\,\langle\vec{w}(x)\,,\,\vec{\nu}\rangle\,\mathrm{d}\sigma\\ +\int_{\Omega_(a,c)}\Bigl\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}&-h_1^*\bigl(\|x\|,a\bigr)\vec{\nu}\Bigr\|^p\rho(x)H^*\bigl(\|x\|,a\bigr)p^{-p}\,\mathrm{d}x\,.\end{aligned}\tag{4.4}$$

Proof. We begin as in the proof of Lemma 4.1 and obtain (4.3). Integrating (4.3) over the sphere $S(s)$ of radius s, multiplying by $H^*(s,t)$ and integrating with respect to s over the interval (t,c), where $a<t$, we get

$$\begin{aligned}\int_{\Omega(t,c)} H^*\bigl(\|x\|,t\bigr)c(x)\rho(x)\,\mathrm{d}x &= -\int_t^c H^*(s,t)\int_{S(s)}\operatorname{div}\bigl(\rho(x)\vec{w}\bigr)\,\mathrm{d}\sigma\;\mathrm{d}s\\ &-(p-1)\int_{\Omega(t,c)} H^*\bigl(\|x\|,t\bigr)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x\\ &-\int_{\Omega(t,c)} H^*\bigl(\|x\|,t\bigr)\left\langle\rho(x)\vec{w}\,,\,\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle\,\mathrm{d}x\,.\end{aligned}$$

As in the proof of Lemma 4.2, the integration by parts in the first integral on the right hand side, Gauss-Ostrogradski formula and the definition of the function h_1^* give

$$\begin{aligned}\int_{\Omega(t,c)} H^*\big(\|x\|,t\big)c(x)\rho(x)\,\mathrm{d}x &= -H^*(c,t)\int_{S(c)}\rho(x)\,\langle\vec{w}\,,\,\vec{\nu}\rangle\,\mathrm{d}\sigma\\ &+\int_{\Omega(t,c)} h_1^*\big(\|x\|,t\big)H^*\big(\|x\|,t\big)\rho(x)\,\langle\vec{w}\,,\,\vec{\nu}\rangle\,\mathrm{d}x\\ &-(p-1)\int_{\Omega(t,c)} H^*\big(\|x\|,t\big)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x\\ &-\int_{\Omega(t,c)} H^*\big(\|x\|,t\big)\left\langle\rho(x)\vec{w}\,,\,\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle\mathrm{d}x\,.\end{aligned}$$

Young inequality (2.15) with

$$\vec{a}=H^*\big(\|x\|,t\big)\Big(\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}-h_1^*\big(\|x\|,t\big)\vec{\nu}\Big)\Big((p-1)H^*\big(\|x\|,t\big)\Big)^{-\frac{1}{q}}\rho^{1-\frac{1}{q}}(x)q^{-\frac{1}{q}}$$

and

$$\vec{b}=\rho(x)\Big((p-1)H^*\big(\|x\|,t\big)\Big)^{\frac{1}{q}}\rho^{\frac{1}{q}-1}(x)q^{\frac{1}{q}}\vec{w},$$

some simplifications and limit process $t\to a^+$ give (4.4), similarly as in the proof of Lemma 4.2. □

4.1 Conjugacy and oscillation criteria

The following theorem is a sufficient condition which ensures that every solution of the equation has zero inside $\Omega(a,b)$. In one-dimensional case this implies that there are conjugate point in the interval (a,b).

Theorem 4.1. *Suppose that there exist real number $c\in(a,b)$, positive smooth function $\rho(x)$ and averaging functions $H(t,s)\in\mathcal{H}$, $H^*(t,s)\in\mathcal{H}^*$, such that*

$$\begin{aligned}\frac{1}{H^*(c,a)}\int_{\Omega(a,c)} H^*\big(\|x\|,a\big)\rho(x)c(x)\,\mathrm{d}x &+\frac{1}{H(b,c)}\int_{\Omega(c,b)} H\big(b,\|x\|\big)\rho(x)c(x)\,\mathrm{d}x\\ >\frac{1}{H^*(c,a)}\int_{\Omega(a,c)}&\Big\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}-h_1^*\big(\|x\|,a\big)\vec{\nu}\Big\|^p\rho(x)H^*\big(\|x\|,a\big)p^{-p}\,\mathrm{d}x\\ +\frac{1}{H(b,c)}\int_{\Omega(c,b)}&\Big\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+h_2\big(b,\|x\|\big)\vec{\nu}\Big\|^p\rho(x)H\big(b,\|x\|\big)p^{-p}\,\mathrm{d}x\,.\end{aligned}\tag{4.5}$$

Then every solution of Eq. (1.1) *has at least one zero inside* $\Omega(a,b)$.

Proof. Suppose, by contradiction, that a solution u with no zero in the interior of $\Omega(a,b)$ exists. Without loss of generality we can suppose that the function c is positive inside $\Omega(a,b)$. Then (4.2) and (4.4) hold. Dividing these inequalities by $H(b,c)$ and $H^*(c,a)$ respectively and summing up we obtain an opposite inequality to (4.5). This contradiction shows that Theorem 4.1 holds. □

Theorem 4.2. *If there exist $t_0 > 0$, $H \in \mathcal{H}$, $H^* \in \mathcal{H}^*$, $\rho \in C^1(\Omega(t_0), \mathbb{R}^+)$ such that for every $\tau > t_0$ the inequalities*

$$\limsup_{t\to\infty} \int_{\Omega(\tau,t)} \Big[H\big(t, \|x\|\big)\rho(x)c(x) - \frac{\rho(x)H\big(t,\|x\|\big)}{p^p} \Big\| \vec{b}(x) - \frac{\nabla\rho(x)}{\rho(x)} + h_2\big(t, \|x\|\big)\vec{\nu} \Big\|^p \Big] \,\mathrm{d}x > 0 \tag{4.6}$$

and

$$\limsup_{t\to\infty} \int_{\Omega(\tau,t)} \Big[H^*\big(\|x\|, \tau\big)\rho(x)c(x) - \frac{\rho(x)H^*\big(\|x\|,\tau\big)}{p^p} \Big\| \vec{b}(x) - \frac{\nabla\rho(x)}{\rho(x)} - h_1^*\big(\|x\|, \tau\big)\vec{\nu} \Big\|^p \Big] \,\mathrm{d}x > 0. \tag{4.7}$$

hold, then Eq. (1.1) *is oscillatory.*

Main idea of the proof. If the assumptions of Theorem 4.2 hold, then for every $T > t_0$ there exist numbers $a < c < b$ such that (4.5) holds and hence the equation has arbitrarily large zeros. Here we omit the details, since the proof is completely analogous to the one-dimensional case, see e.g. [Wang, 2004, Theorem 3]. □

Remark 4.3. If $n = 1$, $\vec{b} = \vec{o}$ and $H(t,s) = H^*(t,s)$, then Theorem 4.1 corresponds to [Wang, 2004, Theorem 3] with $r \equiv 1$. Remark that, as far as the author knows, all relevant results in the literature suppose $H(t,s) = H^*(t,s)$, i.e. the same weight function is used on both ends of the interval (a,b). Hence the possibility $H(t,s) \not\equiv H^*(t,s)$ causes that Theorem 4.2 is new even for linear ODE (4.1).

Another, very similar, approach which allows to use weight function with different growth on both ends of the interval (a,b) has been presented in [Sun, 2004] for $n = 1$ (see Theorem E on page 71) and in [Xu, 2005] for $n \geq 2$. Namely, these authors use the function $\widehat{H}(r,s,l)$ of three variables which corresponds, in some sense, to our product $H(r,s)H^*(s,l)$ (see also Theorem E on page 71). In the following theorem we utilize this idea and use the product $H\big(t_2, \|x\|\big)H^*\big(\|x\|, t_1\big)$ as an averaging function in the procedure from Lemma 4.1. As a result we obtain an oscillation criterion which is simpler than (4.6)–(4.7) in the sense that it consists of one inequality only, but it contains more complicated function in the integral. This theorem is an n-dimensional extension of [Sun, 2004, Theorem 2.5] and non-radial extension of [Xu, 2005, Theorem 2.2] with slightly different meaning of h_1^*, h_2, as mentioned above.

Theorem 4.3. *Suppose that for every $T > t_0$ there exist $t_1 > T$, $H \in \mathcal{H}$ and $H^* \in \mathcal{H}^*$ such that*

$$\limsup_{t\to\infty} \int_{\Omega(t_1,t)} H\big(t, \|x\|\big)H^*\big(\|x\|, t_1\big) \times \Big[\rho(x)c(x) - \frac{\rho(x)}{p^p} \Big\| \vec{b}(x) - \frac{\nabla\rho(x)}{\rho(x)} + \Big[h_2\big(t, \|x\|\big) - h_1^*\big(\|x\|, t_1\big)\Big]\vec{\nu} \Big\|^p \Big] \,\mathrm{d}x > 0 \tag{4.8}$$

Then Eq. (1.1) *is oscillatory.*

Proof. As in the proof of Lemma 4.1 we get (4.3). Integrating over the sphere $S(s)$ of radius s, multiplying by $H(t,s)H^*(s,t_1)$ and integrating with respect to s over the interval (t_1,t) we get

$$\begin{aligned}
\int_{\Omega(t_1,t)} & H\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\rho(x)c(x)\,\mathrm{d}x \\
&= -\int_{t_1}^{t} H(t,s)H^*(s,t_1)\int_{S(s)} \operatorname{div}\big(\rho(x)\vec{w}\big)\,\mathrm{d}\sigma\ \mathrm{d}s \\
&\quad -(p-1)\int_{\Omega(t_1,t)} H\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x \\
&\quad -\int_{\Omega(t_1,t)} H\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\left\langle \rho(x)\vec{w}\,,\,\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle \mathrm{d}x\,.
\end{aligned}$$

Integration by parts in the first integral on the right hand side, Gauss-Ostrogradski formula and the definition of the function h_2 give

$$\begin{aligned}
\int_{\Omega(t_1,t)} & H\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\rho(x)c(x)\,\mathrm{d}x \\
&= -\int_{\Omega(t_1,t)} \Big[h_2\big(t,\|x\|\big)-h_1^*\big(\|x\|,t_1\big)\Big] \\
&\qquad \times H\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\rho(x)\,\langle\vec{w}\,,\,\vec{\nu}\rangle\ \mathrm{d}x \\
&\quad -(p-1)\int_{\Omega(t_1,t)} H\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x \\
&\quad -\int_{\Omega(t_1,t)} H\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\left\langle \rho(x)\vec{w}\,,\,\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle \mathrm{d}x\,.
\end{aligned}$$

The Young inequality yields

$$\begin{aligned}
\int_{\Omega(t_1,t)} H\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\rho(x)c(x)\,\mathrm{d}x &\le \int_{\Omega(t_1,t)} \frac{1}{p}\Big[H\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\Big]^p \\
&\quad \times \left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+\Big[h_2\big(t,\|x\|\big)-h_1^*\big(\|x\|,t_1\big)\Big]\vec{\nu}\right\|^p \\
&\quad \times \rho(x)\Big(pH\big(t,\|x\|\big)H^*\big(\|x\|,t_1\big)\Big)^{-p/q}\mathrm{d}x\,.
\end{aligned}$$

Using some algebraic simplifications we find that the integral from the left hand side of (4.8) is bounded from above by zero for every $t>t_1$ which contradicts the assumption (4.8). Theorem is proved. □

Remark 4.4. The sharpness of the presented method can be shown on examples of radially symmetric equations which follow the corresponding examples for $n=1$ and therefore we omit details.

Several effective criteria can be derived from the above criteria by choosing particular averaging functions. Typical functions of the classes $\mathcal{H}$ and $\mathcal{H}^*$ are

$$H(t,s)=(t-s)^{\alpha}, \qquad \text{and} \qquad H^*(t,s)=(t-s)^{\beta},$$

where $\min\{\alpha,\beta\}>p-1$ (this restriction follows from the condition (iii)). The oscillation criteria with this averaging functions are called Kamenev-type criteria.

Chapter 4

Three terms elliptic half-linear PDE

1 Introduction

In this chapter we study oscillation properties of the half–linear partial differential equation

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u\Big\rangle + c(x)\Phi(u) = 0 \tag{1.1}$$

where $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, $A(x)$ is elliptic $n \times n$ matrix with differentiable components, $c(x)$ is Hölder continuous function and $\vec{b}(x) = \big(b_1(x), \dots, b_n(x)\big)$ is continuous n-vector function. Under a solution of (1.1) in $\Omega \subseteq \mathbb{R}^n$ we understand a differentiable function $u(x)$ such that $A(x)\big\|\nabla u(x)\big\|^{p-2}\nabla u(x)$ is also differentiable and u satisfies (1.1) in Ω.

A special case of (1.1) is the linear partial differential equation which can be obtained from (1.1) for $p = 2$. Another special case of (1.1) is the undamped equation

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + c(x)\Phi(u) = 0 \tag{1.2}$$

which for $p = 2$ reduces to linear equation

$$\operatorname{div}\Big(A(x)\nabla u\Big) + c(x)u = 0. \tag{1.3}$$

If $n = 1$, then Eq. (1.2) reduces to the half–linear ordinary differential equation

$$\Big(a(r)\Phi(u')\Big)' + b(r)\Phi(u) = 0, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}r}. \tag{1.4}$$

The following notation is used: The vector norm $\|\vec{b}\| = \big(\sum_{i=1}^n b_i^2\big)^{\frac{1}{2}}$ is the usual Euclidean norm, $\|A\| = \sup_{\|\vec{b}\|\neq 0} \frac{\|A\vec{b}\|}{\|\vec{b}\|}$ is induced matrix norm and $\lambda_{\min}(x)$, $\lambda_{\max}(x)$ are the smallest and largest eigenvalues of the matrix $A(x)$, respectively. From the fact that $A(x)$ is positive definite symmetric matrix it follows that $\|A(x)\| = \lambda_{\max}(x)$.

For simplicity, if M is matrix and $\vec{k}$ vector, then the product $\vec{k}M$ denotes the matrix product of $1 \times n$ row matrix $\vec{k}$ and $n \times n$ matrix M and the product $M\vec{k}$ denotes the matrix product of the $n \times n$ matrix M and $n \times 1$ column matrix $\vec{k}$.

The results from this chapter are based on a suitable radialization of Eq. (1.1) and conversion of this equation into an ordinary differential equation. This argument has been used very effectively by many authors in various situations, see [Redheffer, 1986; Furusho, 1990; Furusho, 1992; Usami, 1995; Jaroš, Kusano, Yoshida, 2000; Mařík, 2000[3]; Došlý, Mařík, 2001; Naito, Usami, 2001; Mařík, 2004[1]; Xu, 2005; Xing, Xu, 2005; Xu, 2006[1]]. We show that using this argument (particularly, using results from this chapter) it is possible to derive easily sharper results than several recent oscillation criteria which can be found in the literature.

According to the oscillation theory of ordinary differential equations, Eq. (1.4) is said to be oscillatory if every its solution has infinitely many zeros on the interval (r_0, ∞) and nonoscillatory if there exists $r_1 \geq r_0$ such that (1.4) has solution on (r_1, ∞) without zeros. If u is a solution of (1.4) which has no zero on (r_1, ∞), then the function $w(r) = a(r)\frac{|u'(r)|^{p-2}u'(r)}{|u(r)|^{p-2}u(r)}$ is solution of the Riccati equation

$$\mathcal{R}[w] := w' + b(r) + (p-1)a^{1-q}(r)|w|^q = 0.$$

This Riccati equation is frequently used to derive oscillation criteria for Eq. (1.4). More precisely, the following theorem holds.

Theorem A ([Došlý, Řehák, 2005, Theorem 2.2.1])**.** *The following statements are equivalent*

(i) *Equation* (1.4) *is nonoscillatory.*

(ii) *There exists* r_1 *and a continuously differentiable function* $w : [r_1, \infty) \to \mathbb{R}$ *such that*

$$\mathcal{R}[w](r) = 0 \quad \textit{for } r \in [r_1, \infty).$$

(iii) *There exists* r_1 *and a continuously differentiable function* $w : [r_1, \infty) \to \mathbb{R}$ *such that*

$$\mathcal{R}[w](r) \leq 0 \quad \textit{for } r \in [r_1, \infty). \tag{1.5}$$

Thus, if Eq. (1.4) is oscillatory, then Riccati inequality (1.5) has no solution in any neighborhood of infinity.

For the partial differential equation we use the concept of oscillation introduced in Chapter 2.

Many oscillation criteria proved originally for Eq. (1.4) have been extended also to (1.1). The proof of a typical oscillation criterion for (1.1) is usually based on the Riccati type substitution $\vec{w}(x) = A(x)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}$ which converts positive or negative solutions of (1.1) into solution of (partial) Riccati equation. This equation is integrated over balls in n-dimensional space centered in the origin and the problem is converted into problem in one dimension. The rest of the proof usually simply repeats steps from the proof of the corresponding oscillation criterion for (1.4) (neglecting some technical problems which arise for $n \geq 2$).

The disadvantage of this approach is obvious: for every new oscillation criterion derived for ordinary differential equations we have to derive a corresponding criterion for partial differential equations. Since many new oscillation criteria for (1.4) appear in the literature, it turns out to be better to find general theorem which allows to detect oscillation of partial differential equation from oscillation of some ordinary differential equation rather than readjust the proof of every oscillation criterion from (1.4) to (1.1). Some results of this type have been proved in [Jaroš, Kusano, Yoshida, 2000; Došlý, Mařík, 2001; Naito, Usami, 2001]. Let us mention one of the typical results, proved by O. Došlý.

Theorem B ([Došlý, Mařík, 2001, Theorem 3.5]). *Equation*

$$\operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) + c(x)|u|^{p-2}u = 0$$

is oscillatory, if the ordinary differential equation

$$\Big(\omega_n r^{n-1}\Phi(u')\Big)' + \left(\int_{S(r)} c(x)\,\mathrm{d}x\right)\Phi(u) = 0$$

is oscillatory.

Concerning oscillation criteria for (1.4) we refer to the monograph [Došlý, Řehák, 2005], papers [Kusano, Naito, Ogata, 1994; Li, Yeh, 1995; Kusano, Naito, 1997; Hoshino et al, 1998; Došlý, 1998; Kandelaki, Lomtatidze, Ugulava, 2000; Došlý, Lomtatidze, 2006] and the references therein.

The aim of this chapter is to extend Theorem B to Eq. (1.1). The application of this theorem provides a tool to derive oscillation criteria for (1.1) easily from existing oscillation criteria for (1.4). As we show below, this method can be used not only to provide a simple proofs of existing or new oscillation criteria, but it also improves some of already known results.

2 Reduction into ODE

In this section we formulate our main results.

Theorem 2.1. *For a real number $l > 1$ define the functions*

$$\begin{aligned} a(r) &= (l^*)^{p-1}\int_{S(r)} \|A(x)\|^p \lambda_{\text{min}}^{1-p}(x)\,\mathrm{d}\sigma\,, \\ b(r) &= \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{\lambda_{\text{min}}^{p-1}(x)}\frac{\|\vec{b}(x)\|^p}{p^p}\right]\mathrm{d}\sigma\,. \end{aligned} \tag{2.1}$$

where $l^* = \frac{l}{l-1}$ is the conjugate number to the number l if $\|\vec{b}(x)\| \neq 0$ and $l^* = 1$ if $\|\vec{b}(x)\| = 0$. If the equation

$$\Big(a(r)\Phi(u')\Big)' + b(r)\Phi(u) = 0. \tag{2.2}$$

is oscillatory, then Eq. (1.1) *is also oscillatory.*

Proof. Suppose, by contradiction, that (2.2) is oscillatory and (1.1) is not oscillatory. Then there exists a solution u of this equation which is positive on $\Omega(r_1)$ for r_1 sufficiently large. For $x \in \Omega(r_1)$ define n-vector function

$$\vec{w}(x) = A(x)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}. \tag{2.3}$$

The function $\vec{w}$ satisfies

$$\begin{aligned} \operatorname{div}\vec{w} &= \frac{\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)}{|u|^{p-2}u} + (1-p)\left\langle A(x)\|\nabla u\|^{p-2}\nabla u, \nabla u\right\rangle |u|^{-p} \\ &= -c(x) - \left\langle \vec{b}(x), \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u}\right\rangle - (p-1)\frac{\left\langle A(x)\|\nabla u\|^{p-2}\nabla u, \nabla u\right\rangle}{|u|^p}. \end{aligned} \tag{2.4}$$

Further, using the smallest eigenvalue of the matrix A and Young inequality (see (4.7) on page 25) we have

$$\begin{aligned}(p-1)\frac{\langle A(x)\|\nabla u\|^{p-2}\nabla u,\nabla u\rangle}{|u|^p}&+\left\langle \vec{b}(x),\frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u}\right\rangle \\ &\geq (p-1)\left(\frac{1}{l}+\frac{1}{l^*}\right)\lambda_{\min}\frac{\|\nabla u\|^p}{|u|^p}+\left\langle \vec{b},\frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u}\right\rangle \\ &=\frac{p\lambda_{\min}}{l}\Bigg[\left(\frac{\|\nabla u\|^{p-1}}{|u|^{p-1}}\right)^{\frac{p}{p-1}}\frac{p-1}{p} \\ &\quad+\left\langle\frac{l}{p\lambda_{\min}}\vec{b},\frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u}\right\rangle+\frac{1}{p}\left(\frac{l}{p\lambda_{\min}}\right)^p\|\vec{b}\|^p\Bigg] \\ &\quad-\left(\frac{l}{\lambda_{\min}}\right)^{p-1}\frac{1}{p^p}\|\vec{b}\|^p+(p-1)\frac{\lambda_{\min}}{l^*}\frac{\|\nabla u\|^p}{|u|^p} \\ &\geq -\left(\frac{l}{\lambda_{\min}}\right)^{p-1}\frac{1}{p^p}\|\vec{b}\|^p+(p-1)\frac{\lambda_{\min}}{l^*}\frac{\|\nabla u\|^p}{|u|^p}\end{aligned}$$

and this inequality is trivial if $\|\vec{b}(x)\|=0$ and $l^*=1$. Combining this computation and (2.4) we get

$$\operatorname{div}\vec{w}+c(x)-\left(\frac{l}{\lambda_{\min}}\right)^{p-1}\frac{1}{p^p}\|\vec{b}\|^p+(p-1)\lambda_{\min}\frac{1}{l^*}\frac{\|\nabla u\|^p}{|u|^p}\leq 0. \tag{2.5}$$

From the inequality

$$\|\vec{w}\|\leq\|A\|\frac{\|\nabla u\|^{p-1}}{|u|^{p-1}} \tag{2.6}$$

we get

$$\operatorname{div}\vec{w}+c(x)-\left(\frac{l}{\lambda_{\min}}\right)^{p-1}\frac{1}{p^p}\|\vec{b}\|^p+(p-1)\lambda_{\min}\frac{1}{l^*\|A\|^q}\|\vec{w}\|^q\leq 0. \tag{2.7}$$

Define new function

$$W(r)=\int_{S(r)}\langle\vec{w},\vec{\nu}\rangle\,\mathrm{d}\sigma\,. \tag{2.8}$$

The inequality

$$\begin{aligned}|W(r)|&=\left|\int_{S(r)}\left\langle\frac{\lambda_{\min}^{\frac{1}{q}}(x)}{\|A(x)\|}\vec{w},\frac{\|A(x)\|}{\lambda_{\min}^{\frac{1}{q}}(x)}\vec{\nu}\right\rangle\mathrm{d}\sigma\right| \\ &\leq\left(\int_{S(r)}\frac{\lambda_{\min}(x)}{\|A(x)\|^q}\|\vec{w}\|^q\,\mathrm{d}\sigma\right)^{\frac{1}{q}}\left(\int_{S(r)}\|A(x)\|^p\lambda_{\min}^{-\frac{p}{q}}(x)\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\end{aligned}$$

yields

$$\left(\int_{S(r)}\|A(x)\|^p\lambda_{\min}^{-\frac{p}{q}}(x)\,\mathrm{d}\sigma\right)^{-\frac{q}{p}}|W(r)|^q\leq\int_{S(r)}\frac{\lambda_{\min}(x)}{\|A(x)\|^q}\|\vec{w}\|^q\,\mathrm{d}\sigma\,.$$

By Gauss–Ostrogradski divergence theorem we have

$$
\begin{aligned}
W'(r) = \frac{\mathrm{d}}{\mathrm{d}r}\int_{S(r)} \langle \vec{w}, \vec{\nu}\rangle \;\mathrm{d}\sigma \;&= \frac{\mathrm{d}}{\mathrm{d}r}\left[\int_{S(r)} \langle \vec{w}, \vec{\nu}\rangle \;\mathrm{d}\sigma \;-\int_{S(a)} \langle \vec{w}, \vec{\nu}\rangle \;\mathrm{d}\sigma\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\int_{\Omega(a,r)} \operatorname{div}\vec{w}\,\mathrm{d}x \\
&= \int_{S(r)} \operatorname{div}\vec{w}\,\mathrm{d}\sigma\,.
\end{aligned}
\tag{2.9}
$$

The function W satisfies

$$
\begin{aligned}
W'(r) + \int_{S(r)} &\left[c(x) - \left(\frac{l}{\lambda_{\min}(x)}\right)^{p-1} \frac{\|\vec{b}(x)\|^p}{p^p}\right] \mathrm{d}\sigma \\
&+ (p-1)\frac{1}{l^*}\left(\int_{S(r)} \|A(x)\|^p \lambda_{\min}^{-\frac{p}{q}}(x)\,\mathrm{d}\sigma\right)^{1-q} |W(r)|^q \leq 0
\end{aligned}
\tag{2.10}
$$

on $[r_1, \infty)$ and hence the inequality

$$W' + b(r) + (p-1)a^{1-q}(r)|W|^q \leq 0 \tag{2.11}$$

has solution on $\Omega(a)$. By Theorem A, Eq. (2.2) is nonoscillatory, a contradiction. Theorem is proved. □

Remark 2.1. If $\|\vec{b}(x)\| \equiv 0$ and $A(x) = a\big(\|x\|\big)I_n$ where $a(r)$ is smooth function and I_n is $n \times n$ identity matrix, then Theorem 2.1 reduces to [Jaroš, Kusano, Yoshida, 2000, Theorem 3.4].

Remark 2.2. An important step in the proof of Theorem 2.1 is to derive Eq. (2.4). A closer look at the proof shows that it is sufficient to derive (2.4) with equality sign replaced by inequality sign $\leq$. Hence it is possible to use this method to study equations which are in certain sense majorants to (1.1). These equations cover for example

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u\Big\rangle + c(x)f(u) = 0 \tag{2.12}$$

where $f(u)$ is a differentiable function which satisfies $f(0) = 0$, $uf(u) > 0$ for $u \neq 0$ and

$$\frac{f'(u)}{f^{2-q}(u)} \geq p-1. \tag{2.13}$$

Equation (2.12) is sometimes called super-half-linear equation.

If the function $f(u)$ satisfies (2.13) with $p-1$ replaced by $\varepsilon > 0$, it is sufficient to replace $f(u)$ and $c(s)$ by $f^*(u) = \epsilon^* f(u)$ and $c^*(x) = \frac{1}{\epsilon^*}c(x)$, respectively, where $\epsilon^* = \left(\frac{p-1}{\epsilon}\right)^{p-1}$. The function $f^*(u)$ satisfies (2.13) and $f(u)c(x) = f^*(u)c^*(x)$ holds.

Finally, it is possible to use this method also to prove nonexistence of positive solution of the equation

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u\Big\rangle + B(x,u) = 0,$$

where

$$B(x,u) \geq c(x)f(u) \quad \text{ for } u \geq 0$$

and the function $f(u)$ satisfies hypotheses stated above.

Remark 2.3. Many oscillation criteria for the ordinary half-linear differential equation are derived for Eq. (2.2) with $a(r) \equiv 1$. However, if the integral of $a^{1-q}(r)$ is divergent, i.e. if $\int^{\infty} a^{1-q}(r)\,\mathrm{d}r = \infty$, then the transformation of independent variable $s = \varphi(r) := \int_{r_0}^{r} a^{1-q}(t)\,\mathrm{d}t$, $y(s) = u(r)$ transforms (2.2) into

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\Phi\Big(\frac{\mathrm{d}y}{\mathrm{d}s}\Big)\Big) + b(r)a^{1-q}(r)\Phi(y) = 0, \quad r = \varphi^{-1}(s)$$

and interval $[r_0, \infty)$ is transformed into $[0, \infty)$. Using this transformation, an extension of the oscillation criteria derived for $a(r) \equiv 1$ to general case (2.2) used in Theorem 2.1 is straightforward.

Remark 2.4. Several oscillation criteria for (2.2) require $\int^{\infty} a^{1-q}(r)\,\mathrm{d}r = \infty$. If the matrix $A(x)$ is a constant matrix, then the divergence of this integral is equivalent to the condition $p \geq n$. This is a natural phenomenon. The fact that the oscillation properties of (1.1) are different for $p < n$ and $p \geq n$ has been discussed in details in [Došlý, Mařík, 2001].

Some oscillation criteria in the literature contain an additional (and in some sense arbitrary) function (say $\theta(r)$) and thus are more general. A convenient choice of the function θ allows to ensure that the condition from some oscillation criterion (usually divergence or positivity of some integral) holds. A common way to find criteria of this type is to include the function θ into definition of the function $W(r)$. The following lemma is an application of this idea to (2.2). Note that it is sufficient to consider ordinary differential equation only to apply this idea.

Lemma 2.1. *Let $m > 1$ be positive number, $m^* = \frac{m}{m-1}$ be its conjugate number and $\theta(r)$ be smooth positive function. If the equation*

$$\Big((m^*)^{p-1}\theta(r)a(r)\Phi(u')\Big)' + \Big(\theta(r)b(r) - a(r)\frac{m^{p-1}}{p^p}\frac{|\theta'(r)|^p}{\theta^{p-1}(r)}\Big)\Phi(u) = 0 \tag{2.14}$$

is oscillatory, then Eq. (2.2) *is also oscillatory.*

Proof. Suppose that (2.2) is not oscillatory. We prove that (2.14) is also nonoscillatory. If (2.2) is nonoscillatory, then there is a function $w(r)$ which satisfies

$$w'(r) + b(r) + (p-1)a^{1-q}(r)|w(r)|^q = 0$$

on (r_1, ∞) for r_1 sufficiently large. Define the function

$$Z(r) = \theta(r)w(r).$$

The function Z satisfies equation

$$Z'(r) + \theta(r)b(r) + (p-1)\Big(\theta(r)a(r)\Big)^{1-q}|Z(r)|^q - \frac{\theta'(r)}{\theta(r)}Z(r) = 0. \tag{2.15}$$

Using mutually conjugate numbers m, m^* and Young inequality we get

$$(p-1)(\theta a)^{1-q}|Z|^q - \frac{\theta'}{\theta}Z$$

$$
\begin{aligned}
&= p(\theta a)^{1-q}\frac{1}{m}\left[\frac{p-1}{p}|Z|^q - \frac{m\theta'}{p\theta}(\theta a)^{q-1}Z \right.\\
&\quad \left. + \frac{1}{p}\left|\frac{m\theta'}{p\theta}\right|^p (\theta a)^{(q-1)p}\right]\\
&\quad + \frac{1}{m^*}(p-1)\Big(\theta a\Big)^{1-q}|Z|^q - \left|\frac{\theta'}{p\theta}\right|^p \theta a m^{p-1}\\
&\geq \frac{1}{m^*}(p-1)\Big(\theta a\Big)^{1-q}|Z|^q - \frac{1}{p^p}\frac{|\theta'|^p}{\theta^{p-1}} a m^{p-1}
\end{aligned}
$$

This inequality combined with (2.15) shows that the inequality

$$
Z' + \theta(r)b(r) - a(r)\frac{m^{p-1}}{p^p}\frac{|\theta'(r)|^p}{\theta^{p-1}(r)} + \frac{p-1}{m^*}\Big(\theta(r)a(r)\Big)^{1-q}|Z|^q \leq 0
$$

has solution on (r_1, ∞) and (2.14) is nonoscillatory by Theorem A. The proof of the lemma is complete. □

The following corollary is based on a similar idea as Lemma 2.1. The difference is that it makes use of a function $\rho(x)$ of n variables rather than the function $\theta(r)$ of one variable and the proof is more complicated since it is not sufficient to work with ordinary differential equations but we have to return in the proof to partial Riccati equation. However, it is sufficient to simply repeat the steps from the proof of Theorem 2.1 with modified functions. From this reason we proved the simpler version of this theorem first and now we sketch the extension to more general case.

Corollary 2.1. *Let $\rho \in C^1(\Omega(1), \mathbb{R}^+)$. Theorem 2.1 remains valid, if Eqs.* (2.1) *are replaced by*

$$
\begin{aligned}
a(r) &= (l^*)^{p-1}\int_{S(r)} \rho(x)\|A(x)\|^p \lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma\,,\\
b(r) &= \int_{S(r)} \rho(x)\left[c(x) - \frac{l^{p-1}}{p^p\lambda_{\min}^{p-1}(x)}\left\|\vec{b}(x) - \frac{\nabla\rho(x)}{\rho(x)}A(x)\right\|^p\right]\,\mathrm{d}\sigma\,,
\end{aligned}
\tag{2.16}
$$

and $l^* = 1$ if $\left\|\rho(x)\vec{b}(x) - \nabla\rho(x)A(x)\right\| = 0$ and $l^* = \frac{l}{l-1}$ otherwise.

Proof. Suppose by contradiction that (2.2) with $a(r)$ and $b(r)$ defined by (2.16) is oscillatory and (1.1) is nonoscillatory. Define vector, matrix and scalar functions $\vec{b}_\rho(x) = \rho(x)\vec{b}(x) - \nabla\rho(x)A(x)$, $\vec{w}_\rho(x) = \rho(x)\vec{w}(x)$, $A_\rho(x) = \rho(x)A(x)$ and $c_\rho(x) = \rho(x)c(x)$. Further, let $\lambda_{\min,\rho}(x) = \rho(x)\lambda_{\min}(x)$ and $\|A_\rho(x)\| = \rho(x)\|A(x)\|$ be minimal eigenvalue and norm of the matrix $A_\rho(x)$ respectively. It is sufficient to prove that the conclusion of Theorem 2.1 remains valid if the functions $\vec{b}(x)$, $A(x)$, $\vec{w}(x)$, $c(x)$, $\lambda_{\min}(x)$ and $\|A(x)\|$ are replaced by $\vec{b}_\rho(x)$, $A_\rho(x)$, $\vec{w}_\rho(x)$, $c_\rho(x)$, $\lambda_{\min,\rho}(x)$ and $\|A_\rho(x)\|$ respectively, since these replacements convert (2.1) into (2.16).

We start as in the proof of Theorem 2.1 and derive (2.4). Multiplying (2.4) by the function $\rho(x)$ we find that (2.4) is equivalent to the equation

$$
\begin{aligned}
\operatorname{div}\Big(\rho(x)\vec{w}(x)\Big) + \rho(x)c(x) + \left\langle \rho(x)\vec{b}(x) - \nabla\rho(x)A(x), \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u}\right\rangle\\
+ (p-1)\frac{\left\langle \rho(x)A(x)\|\nabla u\|^{p-2}\nabla u, \nabla u\right\rangle}{|u|^p} = 0.
\end{aligned}
$$

Note that this equation also arises from (2.4) by using the above mentioned replacements. Naturally, using the steps from Theorem 2.1 we conclude inequality which arises from (2.10) by using the same replacements. Hence inequality (2.11) with $a(r)$, $b(r)$ defined by (2.16) has a solution on $[r_1, \infty)$. By Theorem A, Eq. (2.2) with $a(r)$, $b(r)$ defined by (2.16) is nonoscillatory, a contradiction. □

Remark 2.5. In general, it is not easy to find the norm $\|A(x)\|$. From this reason we provide some upper estimates for this norm:

$$\begin{aligned} \|A\| &\leq \|A\|_F := \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} \\ \|A\| &\leq n \max_{1\leq i,j\leq n} |a_{ij}| \\ \frac{1}{\sqrt{n}}\|A\| &\leq \|A\|_\infty := \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| \\ \frac{1}{\sqrt{n}}\|A\| &\leq \|A\|_1 := \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| \end{aligned} \tag{2.17}$$

These estimates can be used together with the following simple corollary.

Corollary 2.2. *Let l be a real number, $l > 1$, $\widetilde{b}(r)$ be continuous function and $\widetilde{a}(r)$ be smooth function such that*

$$\widetilde{a}(r) \geq (l^*)^{p-1} \int_{S(r)} \|A(x)\|^p \lambda_{\text{min}}^{1-p}(x)\, \mathrm{d}\sigma\,,$$

$$\widetilde{b}(r) \leq \int_{S(r)} \left[c(x) - \left(\frac{l}{\lambda_{\text{min}}(x)} \right)^{p-1} \frac{\|\vec{b}(x)\|^p}{p^p} \right] \mathrm{d}x\,,$$

where $l^* = \frac{l}{l-1}$ is the conjugate number to the number l if $\|\vec{b}(x)\| \not\equiv 0$ and $l^* = 1$ if $\|\vec{b}(x)\| \equiv 0$. If the ordinary differential equation

$$\left(\widetilde{a}(r)\Phi(u')\right)' + \widetilde{b}(r)\Phi(u) = 0 \tag{2.18}$$

is oscillatory, then Eq. (1.1) *is also oscillatory.*

Proof. Suppose that (2.18) is oscillatory. From the assumptions it follows that (2.2) is a Sturmian majorant to (2.18) and hence (2.2) is also oscillatory. Now the statement follows from Theorem 2.1. □

Obviously, the equality signs in (2.16) can be replaced by inequality signs in the same way as in Corollary 2.2. The following Theorem 2.2 is a variant of Theorem 2.1 and presents sharper result, but covers the case $1 < p \leq 2$ only.

Theorem 2.2. *Let $1 < p \leq 2$. For a real number $l > 1$ define the functions*

$$\begin{aligned} \widehat{a}(r) &= (l^*)^{p-1} \int_{S(r)} \lambda_{\text{max}}(x)\, \mathrm{d}\sigma\,, \\ \widehat{b}(r) &= \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{p^p} \lambda_{\text{max}}(x) \|\vec{b}(x) A^{-1}(x)\|^p \right] \mathrm{d}\sigma\,, \end{aligned} \tag{2.19}$$

where $l^* = \frac{l}{l-1}$ *is the conjugate number to the number* l *if* $\|\vec{b}(x)\| \neq 0$ *and* $l^* = 1$ *if* $\|\vec{b}(x)\| = 0$. *Here* $\vec{b}(x)A^{-1}(x)$ *denotes the matrix product of row matrix* $\big(b_1(x), \dots, b_n(x)\big)$ *and the inverse* $A^{-1}(x)$. *If the equation*

$$\Big(\widehat{a}(r)\Phi(u')\Big)' + \widehat{b}(r)\Phi(u) = 0 \tag{2.20}$$

is oscillatory, then Eq. (1.1) *is also oscillatory.*

Proof. Suppose, by contradiction, that (2.20) is oscillatory and (1.1) is nonoscillatory. We start as in the proof of Theorem 2.1 and derive (2.4) which can be written in the form

$$\operatorname{div}\vec{w} + c + \left\langle \vec{b}, A^{-1}\vec{w} \right\rangle + (p-1)\left\langle \vec{w}, A^{-1}\vec{w} \right\rangle \frac{\|\nabla u\|^{2-p}}{|u|^{2-p}} = 0. \tag{2.21}$$

If $\lambda_{\max}$ is the maximal eigenvalue of the matrix A, then the number $\frac{1}{\lambda_{\max}}$ is the minimal eigenvalue of its inverse A^{-1} and hence

$$\left\langle \vec{w}, A^{-1}\vec{w} \right\rangle \geq \|\vec{w}\|^2 \frac{1}{\lambda_{\max}}.$$

From the property of matrix norm we have (2.6) which is for $p \leq 2$ equivalent to the inequality

$$\frac{\|\nabla u\|^{2-p}}{|u|^{2-p}} \geq \frac{\|\vec{w}\|^{(2-p)/(p-1)}}{\|A\|^{(2-p)/(p-1)}} = \frac{\|\vec{w}\|^{(2-p)/(p-1)}}{\lambda_{\max}^{(2-p)/(p-1)}}.$$

Combining these computation we have the following estimate for the last term on the left hand side of (2.21)

$$\left\langle \vec{w}, A^{-1}\vec{w} \right\rangle \frac{\|\nabla u\|^{2-p}}{|u|^{2-p}} \geq \|\vec{w}\|^{2+(2-p)/(p-1)} \lambda_{\max}^{-1+(p-2)/(p-1)} = \|\vec{w}\|^q \lambda_{\max}^{1-q}.$$

From these estimates and from Eq. (2.21) we get inequality

$$\operatorname{div}\vec{w} + c + \left\langle \vec{b}A^{-1}, \vec{w} \right\rangle + (p-1)\lambda_{\max}^{1-q}\|\vec{w}\|^q \leq 0. \tag{2.22}$$

Using essentially the same method as in the proof of Theorem 2.1 we use mutually conjugate numbers l and l^* to split the last term into two terms and use the Young inequality to remove the term $\left\langle \vec{b}A^{-1}, \vec{w} \right\rangle$:

$$\begin{aligned}
\left\langle \vec{b}A^{-1}, \vec{w} \right\rangle + (p-1)\lambda_{\max}^{1-q}\|\vec{w}\|^q &= \left\langle \vec{b}A^{-1}, \vec{w} \right\rangle + (p-1)\left(\frac{1}{l} + \frac{1}{l^*}\right)\lambda_{\max}^{1-q}\|\vec{w}\|^q \\
&= \frac{p}{l}\lambda_{\max}^{1-q}\left[\frac{p-1}{p}\|\vec{w}\|^q + \left\langle \frac{\lambda_{\max}^{q-1} l}{p}\vec{b}A^{-1}, \vec{w} \right\rangle + \frac{1}{p}\lambda_{\max}^{p(q-1)}\frac{l^p}{p^p}\|\vec{b}A^{-1}\|^p\right] \\
&\quad + (p-1)\frac{1}{l^*}\lambda_{\max}^{1-q}\|\vec{w}\|^q - \frac{l^{p-1}\lambda_{\max}}{p^p}\|\vec{b}A^{-1}\|^p \\
&\geq (p-1)\frac{1}{l^*}\lambda_{\max}^{1-q}\|\vec{w}\|^q - \frac{l^{p-1}\lambda_{\max}}{p^p}\|\vec{b}A^{-1}\|^p.
\end{aligned}$$

This computation remains valid if $\|\vec{b}\| = 0$ and $l^* = 1$. In this case l disappears. Inequality (2.22) now yields

$$\operatorname{div} w + c(x) - \frac{l^{p-1}}{p^p}\lambda_{\max}\|\vec{b}A^{-1}\|^p + (p-1)\frac{1}{l^*}\lambda_{\max}^{1-q}\|\vec{w}\|^q \le 0. \tag{2.23}$$

Define the function $W(r)$ by (2.8). Hölder inequality yields

$$\begin{aligned}|W(r)| &= \left|\int_{S(r)} \left\langle \lambda_{\max}^{(1-q)/q}\vec{w}, \lambda_{\max}^{(q-1)/q}\vec{\nu}\right\rangle \mathrm{d}\sigma\right| \\ &\le \left(\int_{S(r)} \lambda_{\max}^{1-q}\|\vec{w}\|^q \,\mathrm{d}\sigma\right)^{\frac{1}{q}} \left(\int_{S(r)} \lambda_{\max}\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\end{aligned}$$

and

$$\left(\int_{S(r)} \lambda_{\max}\,\mathrm{d}\sigma\right)^{1-q} |W(r)|^q \le \int_{S(r)} \lambda_{\max}^{1-q}\|\vec{w}\|^q \,\mathrm{d}\sigma\,.$$

This inequality, inequality (2.23) and equality (2.9) show that the function $W(r)$ satisfies

$$\begin{aligned}W' + \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{p^p}\lambda_{\max}\|\vec{b}A^{-1}\|^p\right] \mathrm{d}\sigma \\ + (p-1)\left(l^{*p-1}\int_{S(r)} \lambda_{\max}\,\mathrm{d}\sigma\right)^{1-q} |W|^q \le 0.\end{aligned}$$

Thus, the inequality

$$W' + \widehat{b}(r) + (p-1)\widehat{a}^{1-q}(r)|W|^q \le 0 \tag{2.24}$$

has solution on (r_1, ∞) and (2.20) is not oscillatory by Theorem A. This contradiction proves the theorem. □

Remark 2.6. Similarly to Theorem 2.1 and Corollary 2.2, the functions $\widehat{a}(r)$ and $\widehat{b}(r)$ can be replaced by any smooth bigger and continuous smaller functions, respectively.

The following corollary is a version of Corollary 2.1.

Corollary 2.3. *Let $\rho \in C^1(\Omega(1), \mathbb{R}^+)$. Theorem 2.2 remains valid, if Eqs.* (2.19) *are replaced by*

$$\begin{aligned}\widehat{a}(r) &= (l^*)^{p-1}\int_{S(r)} \rho(x)\lambda_{\max}(x)\,\mathrm{d}\sigma\,, \\ \widehat{b}(r) &= \int_{S(r)} \rho(x)\left[c(x) - \frac{l^{p-1}}{p^p}\lambda_{\max}(x)\left\|\vec{b}(x)A^{-1}(x) - \frac{\nabla\rho(x)}{\rho(x)}\right\|^p\right] \mathrm{d}\sigma\,,\end{aligned} \tag{2.25}$$

and $l^* = 1$ if $\left\|\rho(x)\vec{b}(x)A^{-1}(x) - \nabla\rho(x)\right\| = 0$ and $l^* = \frac{l}{l-1}$ otherwise.

Proof. The proof is analogical to the proof of Corollary 2.1. We suppose that (1.1) is not oscillatory and prove that (2.20) is also nonoscillatory. Using the same method as in the proof of Theorem 2.2 we derive inequality (2.21) which can be written in the form

$$\operatorname{div}\Big(\rho\vec{w}\Big) + \rho c + \Big\langle \rho\vec{b} - \nabla\rho A, A^{-1}\vec{w}\Big\rangle + (p-1)\rho\left\langle \vec{w}, A^{-1}\vec{w}\right\rangle \frac{\|\nabla u\|^{2-p}}{|u|^{1-p}} = 0. \quad (2.26)$$

With the notation $A_\rho(x) = \rho(x)A(x), \vec{b}_\rho(x) = \rho(x)\vec{b}(x) - \nabla\rho(x)A(x), \vec{w}_\rho(x) = \rho(x)\vec{w}(x)$ Eq. (2.26) can be written in the form

$$\operatorname{div}\Big(\vec{w}_\rho\Big) + c_\rho + \Big\langle \vec{b}_\rho, A_\rho^{-1}\vec{w}_\rho\Big\rangle + (p-1)\left\langle \vec{w}_\rho, A_\rho^{-1}\vec{w}_\rho\right\rangle \frac{\|\nabla u\|^{2-p}}{|u|^{1-p}} = 0$$

where $A_\rho^{-1}(x) = \rho^{-1}(x)A^{-1}(x)$ is the inverse matrix to $A_\rho(x)$. This equation has the same form as (2.26). Thus using the same steps as in the proof of Theorem 2.2 we prove that there exists a function $W(r)$ which satisfies

$$W' + \int_{S(r)} \left[c_\rho - \frac{l^{p-1}}{p^p}\lambda_{\max,\rho}\|\vec{b}_\rho A_\rho^{-1}\|^p\right] \mathrm{d}\sigma \\ + (p-1)\left(l^{*p-1}\int_{S(r)} \lambda_{\max,\rho}\,\mathrm{d}\sigma\right)^{1-q} \|W\|^q \le 0,$$

where $\lambda_{\max,\rho}(x) = \rho(x)\lambda_{\max}(x)$ is the largest eigenvalue of the matrix $A_\rho(x)$. This shows that the Riccati inequality (2.24) with $\widehat{a}(r)$ and $\widehat{b}(r)$ defined by (2.25) has a solution. Thus (2.20) is nonoscillatory by Theorem A and the corollary is proved. □

Remark 2.7. As we said, Theorem 2.2 produces sharper results than Theorem 2.1. Really, consider for simplicity the undamped case $\|\vec{b}\| = 0$. Since $\|A(x)\| = \lambda_{\max}(x) \ge \lambda_{\min}(x)$, we have

$$\|A(x)\|^p\lambda_{\min}^{1-p}(x) = \|\lambda_{\max}(x)\|^p\lambda_{\min}^{1-p}(x) = \left(\frac{\lambda_{\max}(x)}{\lambda_{\min}(x)}\right)^{p-1}\lambda_{\max}(x) \ge \lambda_{\max}(x),$$

where the quotient $\frac{\lambda_{\max}(x)}{\lambda_{\min}(x)} \ge 1$ is the conditioned number of the matrix $A(x)$. Hence $a(r) \ge \widehat{a}(r)$ and (2.19) is Sturmian majorant to (2.1) and Theorem 2.2 is sharper. Really, if $1 < p \le 2$, (2.1) is oscillatory and Theorem 2.1 applies, then (2.19) is also oscillatory and Theorem 2.2 applies as well. The converse is not true, in general.

Since both proofs of Theorem 2.1 and 2.2 are very similar, the fact that the latter theorem is sharper deserves closer explanation. Let us compare proofs of both theorems. In the proof of Theorem 2.1 we derive (2.7) from (2.5). In order to do this we have to power both sides of inequality (2.6) to the power q. Similarly, in the proof of Theorem 2.2 we have to conclude (2.22) from (2.21) and (2.6) by powering both sides of (2.6) to the power $\frac{2-p}{p-1}$ (from here we have the restriction $p \le 2$). Since

$$\frac{2-p}{p-1} < \frac{1}{p-1} < \frac{p}{p-1} = q$$

for $p > 1$, it follows that we use smaller power in this crucial step in Theorem 2.2 and thus the relative error between the left and right hand side of inequality (2.6) does not increase so much in Theorem 2.2 (comparing to Theorem 2.1).

3 Applications

This section shows application of general theorems on some examples. These examples are of a different kind than examples accompanying usual oscillation criteria in literature. We will not prove oscillation of an equation for which other oscillation criteria fail, but we show that several recent oscillation criteria can be improved and derived in few simple steps using results from the preceding section.

The following theorem has been proved originally for damped linear equation by Xu. However, we reformulate this theorem for undamped equation only in order to obtain results which can be compared to the results from the preceding section and which are extensible to half-linear case.

Theorem C ([Xu, 2006[1], Theorem 3.1])**.** *Let* $\theta \in C([r_0,\infty],\mathbb{R}^+)$*,* $m>1$*,* $\lambda \in C([r_0,\infty),\mathbb{R}^+)$*,* $\lambda(r) \geq \max_{\|x\|=r} \lambda_{\max}(x)$ *for* $r \geq r_0$*. If*

$$\lim_{r\to\infty}\int_{\Omega(r_0,r)}\left[\theta(\|x\|)c(x)-\lambda(\|x\|)\frac{m}{4}\frac{\theta'^2(\|x\|)}{\theta(\|x\|)}\right]\mathrm{d}x = \infty$$

and

$$\lim_{r\to\infty}\int_{\Omega(r_0,r)}\frac{1}{\theta(\|x\|)\lambda(\|x\|)}\,\mathrm{d}x = \infty,$$

then Eq. (1.3) *is oscillatory.*

The classical Leighton–Wintner criterion states that the equation

$$\Big(\alpha(r)u'\Big)' + \beta(r)u = 0$$

is oscillatory if

$$\int^{\infty}\alpha^{-1}(s)\,\mathrm{d}s = \infty = \int^{\infty}\beta(s)\,\mathrm{d}s\,.$$

For Eq. (1.3) the functions $\widehat{a}(r)$, $\widehat{b}(r)$ from Theorem 2.2 become

$$\widehat{a}(r) = \int_{S(r)}\lambda_{\max}(x)\,\mathrm{d}\sigma\,,$$
$$\widehat{b}(r) = \int_{S(r)}c(x)\,\mathrm{d}\sigma\,.$$

Using Theorem 2.2, Lemma 2.1 and the Leighton–Wintner oscillation criterion we conclude that the maximum from the definition of the function $\lambda(r)$ can be removed and the function $\lambda(\|x\|)$ can be replaced by (smaller) function $\lambda_{\max}(x)$.

Corollary 3.1. *The statement of Theorem C remains valid if the function* $\lambda(\|x\|)$ *is replaced by* $\lambda_{\max}(x)$*.*

Since a half-linear version of Leighton–Wintner criterion also exists, a half-linear extension of Theorem C is straightforward. (For another half-linear extension of the Leighton–Wintner criterion see Corollary 3.3 below and Corollary 2.1 on page 13 which deals with $A(x)=I$, i.e. with p-Laplace operator.)

Corollary 3.2. *Let $\theta \in C([r_0,\infty], \mathbb{R}^+)$, $m > 1$ and $q = \frac{p}{p-1}$ be conjugate number to the number p. If*

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} \left[\theta(\|x\|)c(x) - \lambda_{\max}(x) \frac{m^{p-1}}{p^p} \frac{\theta'^p(\|x\|)}{\theta^{p-1}(\|x\|)} \right] \mathrm{d}x = \infty$$

and

$$\lim_{r\to\infty} \int^r \theta^{1-q}(s) \left(\int_{S(s)} \lambda_{\max}(x)\, \mathrm{d}\sigma \right)^{1-q} \mathrm{d}s = \infty,$$

then Eq. (1.2) *is oscillatory.*

Proof. Equation (1.4) is oscillatory if

$$\int^\infty b(r)\,\mathrm{d}r = \infty = \int^\infty a^{1-q}(r)\,\mathrm{d}r\,.$$

Thus the statement is an immediate consequence of Theorem 2.2 and Lemma 2.1. □

An application of the half-linear Leighton–Wintner criterion to Corollary 2.1 gives the following oscillation criterion.

Corollary 3.3. *Let $\rho \in C^1(\Omega(r_0), \mathbb{R}^+)$ and $k > 1$. If*

$$\lim_{r\to\infty} \int_{r_0}^r \left(\int_{S(t)} \rho(x)\|A(x)\|^p \lambda_{\min}^{1-p}(x)\, \mathrm{d}\sigma \right)^{1-q} \mathrm{d}t = \infty$$

and

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} \rho(x) \left[c(x) - \frac{k}{p^p \lambda_{\min}^{p-1}(x)} \left\| \vec{b}(x) - \frac{\nabla\rho(x)}{\rho(x)} A(x) \right\|^p \right] \mathrm{d}x = \infty,$$

then Eq. (1.1) *is oscillatory.*

Proof. The proof is similar to the proof of Corollary 3.2 and thus omitted. □

Corollary 3.3 is closely related to the results from Chapter 3 where we consider $A(x) = I_n$ and detect oscillation in more general domains than exterior of a ball. However, in the case which is covered by both Corollary 3.3 and Chapter 3 the conclusion of Corollary 3.3 is identical to Theorem 3.3 on page 44.

The method of weighted integral averages is frequently used to obtain various extensions of Kamenev type oscillation criteria and also interval oscillation criteria. In the sequel we introduce two results based on this method, Theorems D and E.

Theorem D ([Wang, 2001, Theorem 1])**.** *Let $D_0 = \{(t,s) : t > s \geq t_0\}$ and $D = \{(t,s) : t \geq s \geq t_0\}$. Let functions $H \in C(D;\mathbb{R})$, $h \in C(D_0;\mathbb{R})$, k, $\rho \in C^1([t_0,\infty);(0,\infty))$ satisfy the following three conditions:*

(i) $H(t,t) = 0$ *for* $t \geq t_0$, $H(t,s) > 0$ *on* D_0

(ii) *H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable*

(iii)

$$-\frac{\partial}{\partial s}\big(H(t,s)k(s)\big) - H(t,s)k(s)\frac{\rho'(s)}{\rho(s)} = h(t,s) \quad \forall (t,s) \in D_0$$

and

$$\int_{t_0}^{t} H^{1-p}(t,s)|h(t,s)|^p \,\mathrm{d}s < \infty$$

for every t.

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)k(s)\rho(s)b(s) - \frac{\rho(s)a(s)|h(t,s)|^p}{p^p[H(t,s)k(s)]^{p-1}} \right] = \infty,$$

then Eq. (1.4) *is oscillatory.*

An application of Theorems 2.1 and 2.2 to this result gives the following corollary.

Corollary 3.4. *Let φ, $k \in C^1([r_0,\infty),\mathbb{R}^+)$ be real functions. Suppose that there exists continuous function $H(r,s)$ defined for $r \geq s \geq r_0$ such that*

(i) $H(r,r) = 0$ *and* $H(r,s) > 0$ *for* $r > s \geq r_0$,

(ii) *the function H has continuous nonpositive partial derivative with respect to the second variable,*

(iii) *the function $h(r,s)$ defined by the relation*

$$-\frac{\partial}{\partial s}\Big[H(r,s)k(s)\Big] - H(r,s)k(s)\frac{\varphi'(s)}{\varphi(s)} = h(r,s)$$

satisfies

$$\int_{r_0}^{r} H^{1-p}(r,s)|h(r,s)|^p \,\mathrm{d}s < \infty$$

for every r

(iv)

$$\limsup_{r\to\infty} \frac{1}{H(r,r_0)} \int_{r_0}^{r} \Bigg\{ H(r,s)k(s)\varphi(s) \int_{S(s)} c(x)\,\mathrm{d}\sigma - \frac{1}{p^p}\big[H(r,s)k(s)\big]^{1-p}\Theta(s)\varphi(s)|h(r,s)|^p \Bigg\} \mathrm{d}s = \infty, \quad (3.1)$$

where

$$\Theta(s) = \begin{cases} \displaystyle\int_{S(s)} \|A(x)\|^p \lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma & \text{if } p > 2 \\ \displaystyle\int_{S(s)} \lambda_{\max}(x)\,\mathrm{d}\sigma & \text{if } 1 < p \leq 2 \end{cases} \quad (3.2)$$

Then Eq. (1.2) *is oscillatory.*

Corollary 3.4 improves [Xing, Xu, 2005, Theorem 2.1] in several aspects. First, we use the norm consistent with Euclidean vector norm rather than the Frobenius norm used in [Xing, Xu, 2005] and thus obtain sharper result (see inequality (2.17)).

Second, the term

$$\Theta_{\mathrm{Xu}}(s) := \rho^{1-p}(s)\omega_n s^{n-1} \quad \text{where} \quad \rho(s) \leq \min_{x\in S(s)} \frac{\lambda_{\min}(x)}{\|A(x)\|_F^q}. \tag{3.3}$$

appears in [Xing, Xu, 2005, Theorem 2.1] in condition (3.1) instead of $\Theta(s)$. In Corollary 3.4 we have shown that this term $\Theta_{\mathrm{Xu}}(s)$ can be replaced by smaller term $\Theta(s)$. In other words, the maximum of the function $\|A(x)\|^p\lambda_{\min}^{1-p}(x)$ over the sphere $S(s)$ (which corresponds to the minimum of the function $\frac{\lambda_{\min}(x)}{\|A(x)\|^q}$ from (3.3)) can be replaced by its integral mean value and if $p \leq 2$ we can further decrease this term as (3.2) shows. In this sense, the Corollary 3.4 not only provides a simple alternative proof of [Xing, Xu, 2005, Theorem 2.1], but yields sharper result.

The following Theorem E is an example of interval type oscillation criterion for damped linear differential equation.

Theorem E ([Sun, 2004, Theorem 2.1])**.** *Consider equation*

$$\Big(r(t)y'\Big)' + p(t)y' + q(t)f(y) = 0,$$

where $r(t) \in C([a,\infty),(0,\infty)$*,* $p(t)$*,* $q(t) \in C([a,\infty),\mathbb{R})$*,* $f(u) \in C(\mathbb{R},\mathbb{R})$*,* $uf(u) > 0$ *and* $f'(u) \geq \mu > 0$ *for* $u \neq 0$*. This equation is oscillatory provided that for each* $l \geq a$ *there exists a function* H *with properties*

(i) $H \in C(E,\mathbb{R})$*, where* $E = \{(t,s,l); a \leq l \leq s \leq t < \infty\}$

(ii) $H(t,t,l) = 0 = H(t,l,l)$*,* $H(t,s,l) \neq 0$ *for* $l < s < t$*,*

(iii) *the function* $h(t,s,l)$ *defined by relation*

$$\frac{\partial H}{\partial s}(t,s,l) = h(t,s,l)H(t,s,l)$$

is such that $h^2(t,s,l)H(t,s,l)$ *is locally integrable with respect to* s *on the set* $t \geq s \geq l \geq a$*,*

(iv)

$$\limsup_{t\to\infty} \int_l^t H(t,s,l)\left[q(s) - \frac{r(s)}{4\mu}\left(\frac{p(s)}{r(s)} - h(t,s,l)\right)^2\right] \mathrm{d}s > 0.$$

As an application of Theorem 2.2 to this result we get the following oscillation criterion.

Corollary 3.5. *Suppose that for each* $l \geq a$ *there exist a function* $H(r,s,l)$ *defined for* $r \geq s \geq l \geq a$ *such that*

(i) $H(r,r,l) = 0 = H(r,l,l)$ *for* $r > l \geq a$ *and* $H(r,s,l) > 0$ *for* $r > s > l$*,*

(ii) *$H(r,s,l)$ has continuous partial derivative with respect to s for $r>s>l$,*

(iii) *the function $h(r,s,l)$ defined by relation*

$$\frac{\partial H}{\partial s}(r,s,l)=h(r,s,l)H(r,s,l)$$

is such that $h^2(r,s,l)H(r,s,l)$ is locally integrable with respect to s on the set $r\geq s\geq l\geq a$,

(iv)

$$\limsup_{t\to\infty}\int_l^t H(r,s,l)\biggl\{\int_{S(s)}\Bigl[c(x)-\frac{l}{4}\lambda_{\max}(x)\|\vec{b}(x)A^{-1}(x)\|^2\Bigr]\,\mathrm{d}\sigma \\ -\frac{l^*}{4}\Psi_M(s)h^2(r,s,l)\biggr\}\,\mathrm{d}s>0 \tag{3.4}$$

where $\Psi_M(r)=\int_{S(r)}\lambda_{\max}(x)\,\mathrm{d}\sigma$ and $l>1$, $l^*=\frac{p}{p-1}$ are mutually conjugate numbers. If $\|\vec{b}(x)\|=0$ we can put $l^*=1$.

Then equation

$$\operatorname{div}\Bigl(A(x)\nabla u\Bigr)+\Bigl\langle\vec{b}(x),\nabla u\Bigr\rangle+c(x)u=0$$

is oscillatory.

Corollary 3.5 improves [Xu, 2005, Theorem 3.1] which has been proved for slightly more general equation (covered by Remark 2.2, nevertheless). The condition (3.4) is in [Xu, 2005] replaced by

$$\limsup_{t\to\infty}\int_l^t H(r,s,l)\biggl\{\int_{S(s)}\Bigl[c(x)-\frac{1}{2}\lambda_{\max}(x)\|\vec{b}(x)A^{-1}(x)\|^2\Bigr]\,\mathrm{d}\sigma \\ -\frac{1}{2}\Psi_{\mathrm{Xu}}(s)h^2(r,s,l)\biggr\}\,\mathrm{d}s>0 \tag{3.5}$$

where $\Psi_{\mathrm{Xu}}(r)=\lambda(r)\omega_n r^{n-1}$ and $\lambda(r)\geq\max_{x\in S(r)}\lambda_{\max}(x)$. It is easy to see that oscillation criterion involving condition (3.4) is sharper than the criterion involving (3.5). Really, the maximum of the eigenvalue $\lambda_{\max}(x)$ over the sphere of diameter r which appears in the definition of Ψ_{Xu} is replaced by the integral mean value of this eigenvalue in $\Psi_M(r)$ and thus $\Psi_M(r)$ is smaller than $\Psi_{\mathrm{Xu}}(r)$. Another difference between (3.4) and (3.5) is in the fact that fixed values $1/2$ in (3.5) are replaced by $l/4$ and $l^*/4$ with arbitrary conjugate numbers l, l^* in (3.4).

Chapter 5

Related equations and inequalities

1 Inequality with p-Laplacian

In the first part of this chapter we study positive solutions of the partial differential inequality

$$\operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) + B(x,u) \le 0, \tag{1.1}$$

where $B(x,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Inequality (1.1) covers several equations and inequalities studied in literature and also in this thesis. If $p = 2$ then (1.1) reduces to the semilinear Schrödinger inequality

$$\Delta u + B(x,u) \le 0, \tag{1.2}$$

studied in [Swanson, 1979; Noussair, Swanson, 1980]. Another important special case of (1.1) is the half–linear differential equation

$$\operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) + c(x)\Phi(u) = 0, \tag{1.3}$$

studied in Chapters 1 and 2.

We will introduce sufficient conditions for nonexistence of a solution which would be eventually positive (i.e., positive outside of some ball in $\mathbb{R}^n$). Remark that in a similar way one can study also negative solutions of the inequality

$$\operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) + B(x,u) \ge 0,$$

and a combination of these results produces criteria for nonexistence of a solution of the inequality

$$u\Big[\operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) + B(x,u)\Big] \le 0 \tag{1.4}$$

which would have no zero outside of some ball in $\mathbb{R}^n$, the so called weak oscillation criteria. A simple version of this procedure is used in Corollary 1.5. A more elaborated version of this procedure can be found in [Noussair, Swanson, 1980].

1.1 Riccati transformation

The main tool used for the study of positive solutions is the generalized Riccati transformation. Various forms of Riccati transformation have been used in [Noussair, Swanson, 1980] and [Došlý, Mařík, 2001], where inequality (1.2) and Eq. (1.3) were studied, respectively. Our approach combines both these methods and covers both these substitutions as special cases. We use the transformation

$$\vec{w}(x) = -\alpha(\|x\|)\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{\varphi(u(x))} \qquad (1.5)$$
$$\alpha \in C^1([a_0,\infty),\mathbb{R}^+), \qquad \varphi \in C^1(\mathbb{R}^+,\mathbb{R}^+)$$

which maps a positive C^1 function $u(x)$ into an n-vector function $\vec{w}(x)$.

Lemma 1.1. *Let u be a positive solution of* (1.1) *on* $\Omega(a_0)$. *Then the n-vector function $\vec{w}(x)$ is well-defined by* (1.5) *and satisfies the Riccati–type inequality*

$$\operatorname{div}\vec{w}(x) \geq \frac{\alpha(\|x\|)B\big(x,u(x)\big)}{\varphi\big(u(x)\big)} + \frac{\alpha'(\|x\|)}{\alpha(\|x\|)}\langle\vec{\nu}(x),\vec{w}(x)\rangle + \alpha^{1-q}(\|x\|)\varphi^{q-2}\big(u(x)\big)\varphi'\big(u(x)\big)\|\vec{w}(x)\|^q. \quad (1.6)$$

Proof. Let $u(x) \geq 0$ be a solution of (1.1) on $\Omega(a_0)$ and let $\vec{w}(x)$ be defined by (1.5). From (1.5) it follows that

$$\operatorname{div}\vec{w} = \frac{\alpha}{\varphi(u)}\operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) - \|\nabla u\|^{p-2}\left\langle\nabla u, \nabla\Big(\frac{\alpha}{\varphi(u)}\Big)\right\rangle$$

and in view of (1.1)

$$\operatorname{div}\vec{w} \geq \frac{\alpha B(x,u)}{\varphi(u)} - \frac{\alpha'\|\nabla u\|^{p-2}}{\varphi(u)}\langle\nabla u,\vec{\nu}\rangle + \frac{\alpha\varphi'(u)}{\varphi^2(u)}\|\nabla u\|^p$$

holds (the dependence on $x \in \Omega(a_0)$ is suppressed in the notation). In view of (1.5), this inequality is equivalent to (1.6). □

1.2 Nonexistence of positive solution

Our main result concerning inequality (1.1) is the following

Theorem 1.1. *Let $a_0 \geq 0$. Suppose that there exist functions*

$$\alpha \in C^1([a_0,\infty),\mathbb{R}^+), \qquad \varphi \in C^1(\mathbb{R}^+,\mathbb{R}^+), \qquad c \in C(\mathbb{R}^n,\mathbb{R}),$$

and numbers k, l, $k > 0$, $l > 1$, such that

(i) $B(x,u) \geq c(x)\varphi(u)$ *for* $x \in \mathbb{R}^n$, $u > 0$,

(ii) $\varphi'(u)\varphi^{q-2}(u) \geq k$ *for* $u > 0$,

(iii) $\lim_{r\to\infty}\int_{\Omega(a_0,r)}\left[\alpha(\|x\|)c(x) - \frac{1}{p}\Big(\frac{l}{kq}\Big)^{p-1}\Big|\alpha'(\|x\|)\Big|^p\alpha^{1-p}(\|x\|)\right]\mathrm{d}x = +\infty,$

(iv) $\lim_{r\to\infty}\int_{a_0}^{r}\alpha^{\frac{1}{1-p}}(r)\,r^{\frac{1-n}{p-1}}\,\mathrm{d}r = +\infty.$

Then (1.1) *has no positive solution on* $\Omega(a)$ *for arbitrary* $a > 0$.

Proof. Suppose, by contradiction, that u is a solution of (1.1) positive on $\Omega(a)$ for some $a > a_0$. Lemma 1.1 and the assumptions (i), (ii) imply

$$\operatorname{div}\vec{w} \geq \alpha c + \frac{\alpha'}{\alpha}\langle \vec{\nu}, \vec{w}\rangle + \alpha^{1-q}k\|\vec{w}\|^q \\ = \alpha c + \alpha^{1-q}\frac{kq}{l}\left[\frac{\|w\|^q}{q} + \left\langle \vec{w}, \frac{l\alpha^{q-2}\alpha'}{kq}\vec{\nu}\right\rangle\right] + \alpha^{1-q}\frac{k}{l^*}\|\vec{w}\|^q,$$

where $l^* = \frac{l}{l-1}$ is the conjugate number to the number l. The Young inequality implies

$$\frac{\|\vec{w}\|^q}{q} + \left\langle \vec{w}, \frac{l\alpha^{q-2}\alpha'}{qk}\vec{\nu}\right\rangle + \frac{1}{p}\left(\frac{l\alpha^{q-2}|\alpha'|}{qk}\right)^p \geq 0.$$

Combining both these inequalities we obtain

$$\operatorname{div}\vec{w} \geq \alpha c - \alpha^{1-q}\frac{kq}{lp}\left(\frac{l\alpha^{q-2}|\alpha'|}{qk}\right)^p + \alpha^{1-q}\frac{k}{l^*}\|\vec{w}\|^q \\ = \alpha c - \frac{1}{p}\left(\frac{l}{qk}\right)^{p-1}|\alpha'|^p\alpha^{1-p} + \alpha^{1-q}\frac{k}{l^*}\|\vec{w}\|^q.$$

Integration of the last inequality over $\Omega(a,r)$ and the Gauss–Ostrogradski divergence theorem give

$$\int_{S_r}\langle \vec{w}, \vec{\nu}\rangle \, \mathrm{d}s - \int_{S_a}\langle \vec{w}, \vec{\nu}\rangle \, \mathrm{d}s \\ \geq \frac{k}{l^*}\int_{\Omega(a,r)}\alpha^{1-q}\|\vec{w}\|^q \, \mathrm{d}x + \int_{\Omega(a,r)}\left[\alpha c - \frac{1}{p}\left(\frac{l}{qk}\right)^{p-1} p|\alpha'|^p\alpha^{1-p}\right]\mathrm{d}x\,.$$

By assumption (iii), there exists r_0, $r_0 > a$, such that

$$\int_{\Omega(a,r)}\left[\alpha c - \frac{1}{p}\left(\frac{l}{qk}\right)^{p-1}|\alpha'|^p\alpha^{1-p}\right]\mathrm{d}x + \int_{S_a}\langle \vec{w}, \vec{\nu}\rangle \, \mathrm{d}s \geq 0 \quad \text{for } r > r_0.$$

Hence

$$\int_{S_r}\langle \vec{w}, \vec{\nu}\rangle \, \mathrm{d}s \geq \frac{k}{l^*}g(r) \tag{1.7}$$

holds for $r > r_0$, where

$$g(r) = \int_{\Omega(a,r)}\alpha^{1-q}\big(\|x\|\big)\|\vec{w}(x)\|^q \, \mathrm{d}x\,.$$

The Hölder inequality gives

$$\int_{S_r}\langle \vec{w}, \vec{\nu}\rangle \, \mathrm{d}s \leq \left(\int_{S_r}\|w\|^q \, \mathrm{d}s\right)^{\frac{1}{q}}\left(\int_{S_r} 1 \, \mathrm{d}s\right)^{\frac{1}{p}} = \alpha^{\frac{1}{p}}(r)\Big(g'(r)\Big)^{\frac{1}{q}}\omega_n^{\frac{1}{p}}r^{\frac{n-1}{p}}. \tag{1.8}$$

From (1.7) and (1.8) we obtain

$$\left(g'(r)\right)^{\frac{1}{q}}\alpha^{\frac{1}{p}}(r)\omega_n^{\frac{1}{p}}r^{\frac{n-1}{p}} \geq \frac{k}{l^*}g(r) \qquad \text{for } r \geq r_0$$

and equivalently

$$\frac{g'(r)}{g^q(r)}\omega_n^{\frac{q}{p}} \geq \left(\frac{k}{l^*}\right)^q \alpha^{-\frac{q}{p}}(r)r^{(1-n)\frac{q}{p}} = \left(\frac{k}{l^*}\right)^q \alpha^{\frac{1}{1-p}}(r)r^{\frac{1-n}{p-1}} \qquad \text{for } r \geq r_0.$$

Integration of this inequality over the interval (r_0, ∞) gives a convergent integral on the left–hand side and a divergent integral on the right–hand side of this inequality, by virtue of the assumption (iv). This contradiction completes the proof. □

Remark 1.1. For $\varphi(u) = \Phi(u)$ we have $\varphi'(u)\varphi^{q-2}(u) = p - 1$ and the assumption (ii) holds with $k = p - 1$. Conversely, $\varphi(u) \geq \left(\frac{k}{p-1}\right)^{p-1}u^{p-1}$ is necessary for (ii) to be satisfied. Remark also that neither sign restrictions, nor radial symmetry, are supposed for the function $c(x)$ in (i).

Corollary 1.1 (Leighton type criterion)**.** *Let $p \geq n$. Suppose that there exists a continuous function $c(x)$ such that*

$$B(x, u) \geq c(x)\Phi(u) \qquad \textit{for } u > 0 \tag{1.9}$$

and

$$\lim_{r\to\infty}\int_{\Omega(1,r)} c(x)\,\mathrm{d}x = +\infty. \tag{1.10}$$

Then Eq. (1.1) *has no positive solution on $\Omega(a)$ for arbitrary $a > 0$.*

Proof. Follows from Theorem 1.1 for $\alpha(r) \equiv 1$ and $\varphi(u) = u^{p-1}$. □

Remark 1.2. Remark that (1.10) is known to be a sufficient condition for oscillation of (1.3) provided $p \geq n$, see [Došlý, Mařík, 2001]. It is also known that the condition $p \geq n$ in this criterion cannot be omitted.

Corollary 1.2. *Suppose that* (1.9) *holds and there exists $m > 1$ such that*

$$\lim_{r\to\infty}\int_{\Omega(1,r)}\left[\|x\|^{p-n}c(x) - m\left|\frac{p-n}{p}\right|^p\frac{1}{\|x\|^n}\right]\mathrm{d}x = +\infty. \tag{1.11}$$

Then Eq. (1.1) *has no positive solution on $\Omega(a)$ for arbitrary $a > 0$.*

Proof. Follows from Theorem 1.1 for $\alpha(r) = r^{p-n}$ and $\varphi(u) = u^{p-1}$, $m = l^{p-1}$. □

Remark 1.3. If the limit $\lim_{r\to\infty}\frac{1}{\ln r}\int_{\Omega(1,r)}\|x\|^{p-n}c(x)\,\mathrm{d}x$ exists, or if this limit equals $+\infty$, then (1.11) is equivalent to the condition

$$\lim_{r\to\infty}\frac{1}{\ln r}\int_{\Omega(1,r)}\|x\|^{p-n}c(x)\,\mathrm{d}x > \omega_n\left|\frac{p-n}{p}\right|^p.$$

This condition is very close to the criterion (3.9) from Theorem 3.1 on page 16 for oscillation of the half–linear equation, which contains "$\limsup$" instead of "$\lim$" and one additional condition

$$\liminf_{r\to\infty}\Big[r^{p-1}\Big(C_0-\int_{\Omega(1,r)}\|x\|^{1-n}c(x)\,\mathrm{d}x\Big)\Big]>-\infty,$$

where

$$C_0=\lim_{r\to\infty}\frac{p-1}{r^{p-1}}\int_1^r t^{p-2}\int_{\Omega(1,t)}\|x\|^{1-n}c(x)\,\mathrm{d}x\;\mathrm{d}t\,.$$

Among others, the constant $\left|\frac{p-n}{p}\right|^p$ in (3.9) is optimal and cannot be improved.

Corollary 1.3. *Let* $p\geq n$*,* $p>2$*,* (1.9) *and*

$$\lim_{r\to\infty}\int_{\Omega(\cdot,r)}\ln\big(\|x\|\big)c(x)\,\mathrm{d}x\;=+\infty. \tag{1.12}$$

Then Eq. (1.1) *has no positive solution on* $\Omega(a)$ *for arbitrary* $a>0$*.*

Proof. Let $a>e$, $p\geq n$, $p>2$, $\alpha(r)=\ln r$. Since

$$\lim_{r\to\infty}\frac{\alpha^{\frac{1}{1-p}}(r)r^{\frac{1-n}{p-1}}}{\frac{1}{r\ln r}}=\lim_{r\to\infty}r^{\frac{p-n}{p-1}}\ln^{\frac{p-2}{p-1}}r\geq 1,$$

the condition (iv) of Theorem 1.1 holds. Further,

$$\begin{aligned}\int_{\Omega(e,r)}|\alpha'(\|x\|)|^p\alpha^{1-p}(\|x\|)\,\mathrm{d}x\;&=\omega_n\int_e^r\xi^{n-1-p}\ln^{1-p}\xi\,\mathrm{d}\xi\\&\leq\omega_n\int_e^r\xi^{-1}\ln^{1-p}\xi\,\mathrm{d}\xi\;=\omega_n\frac{1}{p-2}\left[1-\ln^{2-p}r\right].\end{aligned}$$

Hence $\lim_{r\to\infty}\int_{\Omega(e,r)}\big|\alpha'(\|x\|)\big|^p\alpha^{1-p}(\|x\|)\,\mathrm{d}x$ exists and (1.12) is equivalent to the condition (iii) of Theorem 1.1. Now Theorem 1.1 implies the conclusion. □

The choice $\alpha(r)=\ln^\beta r$ leads to

Corollary 1.4. *Let* $p\geq n$*, let* (1.9) *hold and suppose that there exists* β*,* $\beta\in(0,p-1)$ *such that*

$$\lim_{r\to\infty}\int_{\Omega(\cdot,r)}\ln^\beta\big(\|x\|\big)c(x)\,\mathrm{d}x\;=+\infty.$$

Then (1.1) *has no positive solution on* $\Omega(a)$ *for arbitrary* $a>0$*.*

Proof. The proof is analogical to the proof of Corollary 1.3. □

Following terminology in [Noussair, Swanson, 1980] and in Chapter 1, inequality (1.4) is called *weakly oscillatory* in Ω whenever every solution u of the inequality is oscillatory in Ω.

Corollary 1.5. *Let* $B(x,u):\mathbb{R}^{n+1}\to\mathbb{R}$ *be a continuous function which is odd with respect to the variable* u*, i.e. let* $B(x,-u)=-B(x,u)$*. Let the assumptions of Theorem 1.1 be satisfied. Then inequality* (1.4) *is weakly oscillatory in* $\mathbb{R}^n$*.*

Proof. Suppose that there exists $a>0$ such that inequality (1.4) has a solution u without zeros on $\Omega(a)$. If u is a positive function, then Theorem 1.1 yields a contradiction. Further, if u is a negative solution on $\Omega(a)$, then $v(x):=-u(x)$ is a positive solution of (1.4) on $\Omega(a)$ and the same argument as in the first part of this proof leads to a contradiction. □

1.3 Perturbed half-linear differential inequality

Let us consider a perturbed half–linear differential inequality

$$\operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) + c(x)\Phi(u) + \sum_{i=1}^{m} q_i(x)\psi_i(u) \le 0, \tag{1.13}$$

where $c(x)$, $q_i(x)$ are continuous functions, $\psi_i(u)$ are continuously differentiable, positive and nondecreasing for $u > 0$. Define

$$q(x) = \min\big\{c(x), q_1(x), q_2(x), \dots, q_m(x)\big\}$$

and

$$\varphi(u) = u^{p-1} + \sum_{i=1}^{m} \psi_i(u).$$

Then

$$c(x)|u|^{p-1}\operatorname{sgn} u + \sum_{i=1}^{m} q_i(x)\psi_i(u) \ge q(x)\varphi(u) \qquad \varphi'(u)\varphi^{q-2}(u) \ge p-1$$

and hence Theorem 1.1 can be applied. Remark that since q_i may change sign, a standard argument based on the Sturmian majorant and a comparison with half–linear differential equation (1.3) cannot be applied (as has been explained for $p = 2$ already in [Noussair, Swanson, 1980]).

2 Equation with degenerated p-Laplacian

In the second part of this chapter we will study the partial differential equation with pseudo-Laplacian in the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Phi\Big(\frac{\partial u}{\partial x_i}\Big) + B(x,u) = 0. \tag{2.1}$$

The nonlinearity $B(x,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is supposed to be a continuous function odd with respect to the second variable, i.e.

(i) $B(x,-u) = -B(x,u)$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$.

Hence if the function $u(x)$ solves (2.1), then the function $-u(x)$ is also solution of (2.1).

Furthermore we suppose that there exist real-valued functions $c(x) \in C(\mathbb{R}^n)$, $\varphi(u) \in C^1(\mathbb{R})$ such that the following conditions hold

(ii) $B(x,u) \ge c(x)\varphi(u)$ for all $u > 0$

(iii) $\varphi(u) > 0$ for $u > 0$,

(iv) there exists $k > 0$ such that $\varphi^{q-2}(u)\varphi'(u) \ge k$ for $u > 0$.

A significant particular case of (2.1) we obtain for $B(x, u) = c(x)\Phi(u)$. In this case $k = p - 1$ holds in (iv) and (2.1) has the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi\Big(\frac{\partial u}{\partial x_i}\Big) + c(x)\Phi(u) = 0. \tag{2.2}$$

An important property of Eq. (2.2) is that a constant multiple of every solution is also a solution of this equation. The study of this equation is motivated by the fact that it is Euler–Lagrange equation for the $p-$degree functional

$$\mathcal{F}_p(u; \Omega) := \int_\Omega \left[\sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^p - c(x)|u(x)|^p \right] dx = \int_\Omega \left[\|\nabla u\|_p^p - c(x)|u|^p \right] dx.$$

Equation (2.2) has been investigated in a series of papers of G. Bognár [Bognár, 1993; Bognár, 1995; Bognár, 1997] where the basic properties of the eigenvalue problem have been established. The Picone–type identity and Riccati–type substitution (our main tool) for (2.2) has been recently introduced in [Došlý, 2002].

The following notation will be used throughout this section: $\|\cdot\|_p$ and $\|\cdot\|_q$ are the p and q-norms in $\mathbb{R}^n$

$$\|x\|_p = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p}, \quad \|x\|_q = \Big(\sum_{i=1}^{n} |x_i|^q\Big)^{1/q} \quad \text{for } x \in \mathbb{R}^n,$$

the function $\Phi_q(x)$ is defined similarly to Φ by the relation $\Phi_q(x) = |x|^{q-2}x$ and the sets $\Omega_q(a, b)$, $\Omega_q(a)$ and $S_q(a)$ are defined as follows:

$$\begin{aligned} \Omega_q(a, b) &= \big\{x \in \mathbb{R}^n : a \le \|x\|_q \le b\big\}, \\ \Omega_q(a) &= \lim_{b\to\infty} \Omega_q(a, b) = \big\{x \in \mathbb{R}^n : a \le \|x\|_q\big\}, \\ S_q(a) &= \partial\Omega_q(a) = \big\{x \in \mathbb{R}^n : a = \|x\|_q\big\}. \end{aligned}$$

Finally $\omega_{n,q} := \int_{S_q(1)} \mathrm{d}\sigma$ is the surface area of the unit sphere (with respect to the q-norm).

Motivated by terminology in [Noussair, Swanson, 1980] and Chapter 1, we define an oscillation of (2.1) as follows

Definition 2.1 (weak oscillation)**.** A function $f : \Omega_q \to \mathbb{R}$ is called oscillatory in Ω, if and only if $f(x)$ has zero in $\Omega \cap \Omega_q(a)$ for every $a > 0$. Eq. (2.1) is called *oscillatory* in Ω whenever every solution u of (2.1) is oscillatory in Ω. Eq. (2.1) is *oscillatory*, if it is oscillatory in $\mathbb{R}^n$.

2.1 Modified Riccati transformation

A modification of Riccati substitution from [Došlý, 2002] is presented in the following lemma.

Lemma 2.1. *Let* $a_0 \in \mathbb{R}^+$, $\alpha \in C^1((a_0, \infty), \mathbb{R}^+)$. *If* $u \in C^2(\mathbb{R}^n, \mathbb{R})$ *is a solution of* (2.1) *on* $\Omega_q(a_0)$ *such that* $u(x) \neq 0$ *for* $x \in \Omega_q(a_0)$, *then the vector function* $\vec{w}(x)$ *is well-defined on* $\Omega_q(a_0)$ *by*

$$\vec{w}(x) = \big(w_i(x)\big)_{i=1}^n, \quad w_i(x) = -\frac{\alpha\big(\|x\|_q\big)}{\varphi\big(u(x)\big)} \Phi\Big(\frac{\partial u}{\partial x_i}\Big)$$

and satisfies the inequality

$$\operatorname{div}\vec{w} \geq \alpha\big(\|x\|_q\big)c(x) + k\alpha^{1-q}\big(\|x\|_q\big)\|\vec{w}\|_q^q + \frac{\alpha'\big(\|x\|_q\big)}{\alpha\big(\|x\|_q\big)}\left\langle \vec{w}, \vec{\nu}_q\right\rangle, \tag{2.3}$$

where $\vec{\nu}_q(x)$ *is the following vector:* $\vec{\nu}_q(x) = \left(\Phi_q\Big(\frac{x_1}{\|x\|_q}\Big), \dots, \Phi_q\Big(\frac{x_n}{\|x\|_q}\Big)\right)$

Proof. In view of (i), without loss of generality, consider that $u(x) > 0$ on $\Omega_q(a_0)$. It holds

$$\frac{\partial w_i}{\partial x_i} = -\frac{\alpha\big(\|x\|_q\big)}{\varphi(u)}\frac{\partial}{\partial x_i}\Big(\Phi\Big(\frac{\partial u}{\partial x_i}\Big)\Big) - \Phi\Big(\frac{\partial u}{\partial x_i}\Big)\frac{\alpha'\big(\|x\|_q\big)}{\varphi(u)}\frac{\partial\|x\|_q}{\partial x_i} + \alpha\big(\|x\|_q\big)\Big|\frac{\partial u}{\partial x_i}\Big|^p\frac{\varphi'(u)}{\varphi^2(u)}.$$

Since $\frac{\partial\|x\|_q}{\partial x_i} = \Phi_q\Big(\frac{x_i}{\|x\|_q}\Big) = \nu_i$, we get

$$\frac{\partial w_i}{\partial x_i} = -\frac{\alpha\big(\|x\|_q\big)}{\varphi(u)}\frac{\partial}{\partial x_i}\Big(\Phi\Big(\frac{\partial u}{\partial x_i}\Big)\Big) + \frac{\alpha'\big(\|x\|_q\big)}{\alpha\big(\|x\|_q\big)}w_i\nu_i + \varphi'(u)\varphi^{q-2}(u)\alpha^{1-q}\big(\|x\|_q\big)|w_i|^q.$$

From this equation and from (2.1) it follows

$$\operatorname{div}\vec{w} = \alpha\big(\|x\|_q\big)\frac{B(x,u)}{\varphi(u)} + \varphi'(u)\varphi^{q-2}(u)\alpha^{1-q}\big(\|x\|_q\big)\|\vec{w}\|_q^q + \frac{\alpha'\big(\|x\|_q\big)}{\alpha\big(\|x\|_q\big)}\left\langle \vec{w}, \vec{\nu}_q\right\rangle.$$

Taking into account conditions (ii), (iii) and (iv) we obtain inequality (2.3). □

The following inequality is used in the proof. It replaces Young inequality used in previous chapters.

Lemma 2.2. *It holds*

$$\frac{\|x\|_p^p}{p} + \sum_{i=1}^n x_i y_i + \frac{\|y\|_q^q}{q} \geq 0$$

for every $x, y \in \mathbb{R}^n$, $x = (x_i)_{i=1}^n$, $y = (y_i)_{i=1}^n$.

For the proof of this lemma, see [Došlý, 2002].

2.2 Oscillation criteria

Theorem 2.1. *Let* $a_0 \in \mathbb{R}^+$, $\alpha \in C^1((a_0,\infty),\mathbb{R}^+)$ *and* $l > 1$. *If*

$$\lim_{r\to\infty}\int_{\Omega_q(a_0,r)}\Big[\alpha\big(\|x\|_q\big)c(x) - \frac{1}{p}\Big(\frac{l}{kq}\Big)^{p-1}\alpha^{1-p}\big(\|x\|_q\big)|\alpha'\big(\|x\|_q\big)|^p\Big]\,\mathrm{d}x = +\infty \tag{2.4}$$

and

$$\lim_{r\to\infty}\int_{a_0}^r \frac{1}{\big(r^{n-1}\alpha(r)\big)^{\frac{1}{p-1}}} = +\infty, \tag{2.5}$$

then Eq. (2.1) *is oscillatory in* $\mathbb{R}^n$.

Proof. Suppose, by contradiction, that u is a solution of (2.1) which is positive on $\Omega_q(a_0)$ for some $a_0 > 0$. Then $\vec{w}$ is defined on $\Omega_q(a_0)$. From inequality (2.3), using integration over the domain $\Omega_q(a_0, r)$ and the Gauss–Ostrogradski divergence theorem we get

$$\int_{S_q(r)} \langle \vec{w}, \vec{n} \rangle \, \mathrm{d}\sigma - \int_{S_q(a_0)} \langle \vec{w}, \vec{n} \rangle \, \mathrm{d}\sigma \geq$$
$$\geq \int_{\Omega_q(a_0,r)} \left(\alpha(\|x\|_q) c(x) + k\alpha^{1-q}(\|x\|_q) \|\vec{w}\|_q^q + \frac{\alpha'(\|x\|_q)}{\alpha(\|x\|_q)} \langle \vec{w}, \vec{\nu}_q \rangle \right) \mathrm{d}x \,, \quad (2.6)$$

where $\vec{n}$ is the outward normal unit vector to $\Omega_q(a_0, r)$ i.e. $\vec{n} = \frac{\vec{\nu}_q}{\|\vec{\nu}_q\|}$ and $\vec{\nu}_q$ is defined in Lemma 2.1. Observe that $\|\vec{\nu}_q\|_p = 1$.

Now, let $l^* = \frac{l}{l-1} > 1$ be the conjugate number to the number l. Then

$$k\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q + \frac{\alpha'(\|x\|_q)}{\alpha(\|x\|_q)} \langle \vec{w}, \vec{\nu}_q \rangle$$
$$= \frac{kq}{l} \alpha^{1-q}(\|x\|_q) \left(\frac{\|\vec{w}\|_q^q}{q} + \frac{l\alpha'(\|x\|_q)\alpha^{q-2}(\|x\|_q)}{qk} \langle \vec{w}, \vec{\nu}_q \rangle \right) + \frac{k}{l^*} \alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q.$$

Using Lemma 2.2 we obtain

$$k\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q + \frac{\alpha'(\|x\|_q)}{\alpha(\|x\|_q)} \langle \vec{w}, \vec{\nu}_q \rangle$$
$$\geq -\frac{qk}{lp} \alpha^{1-q}(\|x\|_q) \left\| \frac{l\alpha'(\|x\|_q)\alpha^{q-2}(\|x\|_q)}{qk} \vec{\nu}_q \right\|_p^p + \frac{k}{l^*} \alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q$$
$$= -\frac{1}{p} \left(\frac{l}{kq} \right)^{p-1} \alpha^{1-p}(\|x\|_q) |\alpha'(\|x\|_q)|^p + \frac{k}{l^*} \alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q.$$

This inequality together with (2.6) yields

$$\int_{S_q(r)} \langle \vec{w}, \vec{n} \rangle \, \mathrm{d}\sigma - \int_{S_q(a_0)} \langle \vec{w}, \vec{n} \rangle \, \mathrm{d}\sigma$$
$$\geq \int_{\Omega_q(a_0,r)} \left[\alpha(\|x\|_q) c(x) - \frac{1}{p} \left(\frac{l}{kq} \right)^{p-1} \alpha^{1-p}(\|x\|_q) |\alpha'(\|x\|_q)|^p \right] \mathrm{d}x$$
$$+ \frac{k}{l^*} \int_{\Omega_q(a_0,r)} \alpha^{1-q}(\|x\|_q) \|\vec{w}\|_q^q \, \mathrm{d}x \,. \quad (2.7)$$

In view of (2.4), there exists $r_0 > a_0$ such that

$$\int_{\Omega_q(a_0,r)} \left[\alpha(\|x\|_q) c(x) - \frac{1}{p} \left(\frac{l}{kq} \right)^{p-1} \alpha^{1-p}(\|x\|_q) |\alpha'(\|x\|_q)|^p \right] \mathrm{d}x + \int_{S_q(a_0)} \vec{w}\vec{n} \, \mathrm{d}\sigma \geq 0$$

and now (2.7) implies

$$\int_{S_q(r)} \langle \vec{w}, \vec{n} \rangle \, \mathrm{d}\sigma \geq \frac{k}{l^*} \int_{\Omega_q(a_0,r)} \alpha^{1-q}(\|x\|_q) \|\vec{w}\|_q^q \, \mathrm{d}x \quad (2.8)$$

for $r > r_0$. Application of the Hölder inequality in $\mathbb{R}^n$ yields

$$\int_{S_q(r)} \langle \vec{w}, \vec{n} \rangle \, \mathrm{d}\sigma \leq \int_{S_q(r)} \|\vec{w}\|_q \|\vec{n}\|_p \, \mathrm{d}\sigma \,.$$

Since $\|\cdot\|$ and $\|\cdot\|_p$ are equivalent norms in $\mathbb{R}^n$, there exists $K > 0$ such that $\|\vec{n}\|_p \le K\|\vec{n}\| = K$. This fact and another application of Hölder inequality give

$$\int_{S_q(r)} \langle \vec{w}, \vec{n} \rangle \, \mathrm{d}\sigma \le K\Big(\omega_{n,q} r^{n-1}\Big)^{1/p}\Big(\int_{S_q(r)} \|\vec{w}\|_q^q \, \mathrm{d}\sigma\Big)^{1/q}. \tag{2.9}$$

Denote

$$g(r) = \int_{\Omega_q(a_0,r)} \alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q \, \mathrm{d}x \,.$$

Then it holds

$$g'(r) = \alpha^{1-q}(r) \int_{S_q(r)} \|\vec{w}\|_q^q \, \mathrm{d}\sigma$$

and (2.9) gives

$$\int_{S_q(r)} \langle \vec{w}, \vec{n} \rangle \, \mathrm{d}\sigma \le K\omega_{n,q}^{1/p} r^{\frac{n-1}{p}} \Big(\alpha^{q-1}(r) g'(r)\Big)^{\frac{1}{q}}. \tag{2.10}$$

Combining (2.8) and (2.10) we obtain the inequality

$$\frac{k}{l^*} g(r) \le K\omega_{n,q}^{1/p} r^{\frac{n-1}{p}} \Big(\alpha^{q-1}(r) g'(r)\Big)^{1/q}$$

for $r > r_0$. Hence

$$\Big(\frac{1}{r^{n-1}\alpha(r)}\Big)^{\frac{1}{p-1}} \le \frac{l^* \omega_{n,q}^{\frac{q}{p}}}{kK^q} \frac{g'(r)}{g^q(r)}.$$

Integration of this inequality over $[r_0, \infty)$ gives the divergent integral on the left hand side, according to the assumption (2.5), and the convergent integral on the right hand side. This contradiction completes the proof. □

A suitable choice of the function α in Theorem 2.1 leads to effective oscillation criteria for Eqs. (2.1) and (2.2). This is the content of the following corollaries. The first one is a Leighton–type oscillation criterion (see [Swanson, 1968, Th. 2.24, p. 70]).

Corollary 2.1. *Suppose that* $p \ge n$ *and*

$$\lim_{r\to\infty} \int_{\Omega(1,r)} c(x) \, \mathrm{d}x = +\infty.$$

Then Eq. (2.1) *is oscillatory in* $\mathbb{R}^n$.

Proof. Follows immediately from Theorem 2.1 for $\alpha(r) \equiv 1$. □

Remark that the condition $p \ge n$ cannot be removed. This is known already from the study of Schrödinger equation (for $p = 2$).

Another choice of the function α improves this criterion if $p > 2$.

Corollary 2.2. *Let $p \geq n$, $p > 2$ and*

$$\lim_{r\to\infty} \int_{\Omega_q(1,r)} \ln\big(\|x\|_q\big) c(x)\,\mathrm{d}x \;= +\infty. \tag{2.11}$$

Then Eq. (2.1) *is oscillatory in* $\mathbb{R}^n$.

Proof. Let $a_0 > e$ be arbitrary and $\alpha(r) = \ln(r)$ on $[a_0, \infty)$. Since

$$\lim_{r\to\infty} \frac{\alpha^{\frac{1}{1-p}}(r) r^{\frac{1-n}{p-1}}}{\frac{1}{r\ln r}} = \lim_{r\to\infty} r^{\frac{p-n}{p-1}} \ln^{\frac{p-2}{p-1}} r \geq 1,$$

integral (2.5) diverges by ratio-convergence test. Further, since

$$\begin{aligned}\int_{\Omega_q(a_0,r)} |\alpha'(\|x\|_q)|^p \alpha^{1-p}(\|x\|_q)\,\mathrm{d}x \;&= \omega_{n,q} \int_e^r \xi^{n-1-p} \ln^{1-p} \xi \,\mathrm{d}\xi \\ &\leq \omega_{n,q} \int_{a_0}^r \xi^{-1} \ln^{1-p} \xi\,\mathrm{d}\xi = \omega_{n,q} \frac{1}{p-2}[1 - \ln^{2-p} r],\end{aligned}$$

the limit $\lim_{r\to\infty} \int_{\Omega_q(a_0,r)} |\alpha'(\|x\|)|^p \alpha^{1-p}(\|x\|)\,\mathrm{d}x$ converges and (2.11) is equivalent to the condition (2.4) of Theorem 2.1. All conditions of Theorem 2.1 are satisfied and the proof is complete. □

The following theorem covers also the case when $p < n$.

Corollary 2.3. *Let*

$$\liminf_{r\to\infty} \frac{1}{\ln r} \int_{\Omega_q(1,r)} \|x\|_q^{p-n} c(x)\,\mathrm{d}x \;> \omega_{n,q} \frac{|p-n|^p}{p(kq)^{p-1}}. \tag{2.12}$$

Then Eq. (2.1) *is oscillatory in* $\mathbb{R}^n$.

Proof. Let $\alpha(r) = r^{p-n}$. Then (2.5) holds and it is sufficient to prove that also (2.4) holds, i.e. that there exists $l > 1$ such that

$$\lim_{r\to\infty} \int_{\Omega_s(1,r)} \Big[\|x\|_q^{p-n} c(x) - \frac{1}{p}\Big(\frac{l}{kq}\Big)^{p-1} |p-n|^p \|x\|_q^{-n}\Big]\,\mathrm{d}x \;= +\infty. \tag{2.13}$$

According to (2.12) there exists $m > 1$, $\varepsilon > 0$ and $r_0 > 1$ such that

$$\int_{\Omega_q(1,r)} \|x\|_q^{p-n} c(x)\,\mathrm{d}x \;> (m+\varepsilon)\omega_{n,q} \frac{|p-n|^p}{p(kq)^{p-1}} \ln r \tag{2.14}$$

for $r > r_0$. Since

$$\int_{\Omega_q(1,r)} \|x\|_q^{-n}\,\mathrm{d}x \;= \omega_{n,q} \int_1^r \frac{1}{s}\,\mathrm{d}s \;= \omega_{n,q} \ln r,$$

can (2.14) be written in the form

$$\int_{\Omega_q(1,r)} \Big[\|x\|_q^{p-n} c(x) - m\frac{|p-n|^p}{p(kq)^{p-1}} \|x\|_q^{-n}\Big]\,\mathrm{d}x \;> \varepsilon\omega_{n,q} \frac{|p-n|^p}{p(kq)^{p-1}} \ln r$$

which implies (2.13). The proof is complete. □

Corollary 2.4. *Let*

$$\liminf_{r\to\infty} \frac{1}{\ln r} \int_{\Omega_q(1,r)} \|x\|_q^{p-n} c(x)\,\mathrm{d}x > \omega_{n,q} \left|\frac{p-n}{p}\right|^p. \tag{2.15}$$

Then Eq. (2.2) *is oscillatory in* $\mathbb{R}^n$.

Proof. Follows immediately from Corollary 2.3. □

Remark 2.1. The constant $\omega_{n,q}\left|\frac{p-n}{p}\right|^p$ in (2.15) is optimal and cannot be decreased. This follows from the example of equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi\Big(\frac{\partial u}{\partial x_i}\Big) + \left|\frac{p-n}{p}\right|^p \|x\|_q^{-p} \Phi(u) = 0.$$

This equation is not oscillatory, since it has nonoscillatory solution $u(x) = \|x\|_q^{\frac{p-n}{p}}$ and the function $c(x) = \left|\frac{p-n}{p}\right|^p \|x\|_q^{-p}$ produces equality in condition (2.15).

Remark 2.2. We have already mentioned that the function $\Phi(u) := |u|^{p-1} \operatorname{sgn} u$ satisfies hypothesis (iii) and (iv) with $k = p-1$. On the other hand in most real applications we claim $B(x, 0) = 0$ for all x and consequently $\varphi(0) = 0$. In this case integration of (iv) implies $\varphi(u) \geq \left(\frac{k}{p-1}\right)^{p-1} u^{p-1}$ and the function $\varphi(u)$ must satisfy this growth condition.

Example 2.1. Similarly to (1.13), consider perturbed equation (2.2)

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi\Big(\frac{\partial u}{\partial x_i}\Big) + c(x)\Phi(u) + \sum_{i=1}^{m} q_i(x)\psi_i(u) = 0, \quad p \in (1, 2] \tag{2.16}$$

where $c(x), q_i(x) \in C(\mathbb{R}^n)$, $\psi_i(u) \in C^1(\mathbb{R})$, $\psi_i(-u) = -\psi_i(u)$ for all $i = 1..m$ and all $u \in \mathbb{R}$, and $\psi_i(u)$ are positive and nondecreasing functions for $u > 0$ and all $i = 1..m$. Define

$$q(x) = \min\big\{c(x), q_1(x), q_2(x), \dots, q_m(x)\big\}$$

and

$$\varphi(u) = \Phi(u) + \sum_{i=1}^{m} \psi_i(u).$$

Then

$$c(x)\Phi(u) + \sum_{i=1}^{m} q_i(x)\psi_i(u) \geq q(x)\varphi(u) \quad \varphi'(u)\varphi^{q-2}(u) \geq p-1$$

and hence Theorem 2.1 can be applied. Remark that we do not make any sign restriction to the functions q_i and hence (2.16) needs not to be majorant for (2.2) in the sense of Sturmian theory.

3 Variational technique and nonradial criteria

Oscillation criteria for Schrödinger partial differential equation

$$\Delta u + c(x)u = 0 \tag{3.1}$$

and several its generalizations have been studied by many authors and also in this work, see [Müller–Pfeiffer, 1980; Noussair, Swanson, 1980; Atakarryev, Toraev, 1986; Schminke, 1989; Fiedler, 1988; Jaroš, Kusano, Yoshida, 2000; Mařík, 2000[3]; Došlý, Mařík, 2001] and the references therein. From these works and also from the preceding chapters of this thesis it follows that (3.1) is oscillatory if the potential function $c(x)$ is sufficiently large either in the sense of the inequality containing directly the function $c(x)$ (the so called Kneser type criteria) or in the sense of the integral like $\int_{\|x\|\leq t} c(x)\,\mathrm{d}x$. Hence in the latter case the equation is oscillatory if the integral mean value of the function $c(x)$ over the spheres in $\mathbb{R}^n$ centered in the origin is sufficiently large and the oscillation criteria depend in fact on this integral mean value only. In this sense these criteria preserve a kind of radial symmetry. However, if the function $c(x)$ is sufficiently large in some direction only, then Eq. (3.1) may be oscillatory even if the function $c(x)$ contains also "bad" parts which causes that the integral mean value of the function $c(x)$ over the spheres is small and the criteria containing the integral like $\int_{\|x\|\leq t} c(x)\,\mathrm{d}x$ may fail to detect this oscillation. From this reason we classified the oscillation criteria for PDE's into radial and nonradial (Remark 2.3 on page 4).

A similar phenomenon has been observed also in the case of ordinary differential equation. The equation

$$u'' + c(x)u = 0$$

may be oscillatory even if $\int_1^\infty c(x)\,\mathrm{d}x = -\infty$, see e.g. [Kong, 1999].

An idea how to remove the above mentioned disadvantage is to include only the "good" parts of the function $c(x)$ into the oscillation criterion. We used this approach in Chapters 3 and 4. This approach may also provide oscillation criteria for more general types of unbounded domains than simply an exterior of some ball. For Kneser type oscillation and nonoscillation criteria on various types of unbounded domains see [Atakarryev, Toraev, 1986].

As in Chapters 1 and 2, we will study the second order partial differential equation

$$\operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) + c(x)\Phi(u) = 0, \tag{3.2}$$

where $p \geq 2$. The function $c(x)$ is supposed to be Hölder continuous in $\mathbb{R}^n$. Under solution we understand in a classical sense every function u which satisfies (3.2) everywhere in $\mathbb{R}^n$.

Throughout this chapter we assume that all domains are simply connected with piecewise smooth boundary.

In connection with Eq. (3.2) we will use the same concept of oscillation as defined in Chapter 1.

Our main tools are the following two inequalities (both are easy to check), Lemma 3.3 which follows from [Došlý, Mařík, 2001] and Lemma 3.4 from [Mařík, 1999, Theorem 3.2].

Lemma 3.1. *For $\vec{a}$, $\vec{b} \in \mathbb{R}^n$*

$$\|\vec{a} + \vec{b}\|^2 \leq 2\big(\|\vec{a}\|^2 + \|\vec{b}\|^2\big) \tag{3.3}$$

holds.

Lemma 3.2. *For $a \geq 0$, $b \geq 0$ and $l \geq 1$*

$$(a+b)^l \leq 2^{l-1}\big(a^l + b^l\big) \tag{3.4}$$

holds.

Lemma 3.3. *If there exists a solution u of Eq.* (3.2) *which is positive in the compact domain M and $y \in W_0^{1,p}(M)$, then*

$$\int_M \Big(\|\nabla y(x)\|^p - c(x)|y(x)|^p\Big)\,\mathrm{d}x \geq 0 \tag{3.5}$$

Lemma 3.4. *Let R, $C \in C([a,b])$ be continuous functions, $R(t) > 0$. The p-degree functional*

$$J(\eta;a,b) := \int_a^b \Big(R(t)|\eta'(t)|^p - C(t)|\eta(t)|^p\Big)\,\mathrm{d}t$$

is nonnegative for every function $\eta \in W_0^{1,p}(a,b)$ if and only if the equation

$$\Big(r(t)|y'|^{p-2}y'\Big) + c(t)|y|^{p-2}y = 0, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}t}$$

is disconjugate on (a,b), i.e. every its solution has at most one zero on (a,b).

Remark that Lemma 3.4 holds also on the class of piecewise smooth functions η with boundary conditions $\eta(a) = 0 = \eta(b)$.

Theorem 3.1. *Let $p \geq 2$, $\alpha \in C^1(\mathbb{R}^n)$ and let $\vec{\nu}$ be a normal unit vector to the sphere $S(\|x\|)$ oriented outwards. Let $K = 2^{-1+p/2}$ if $\langle \nabla\alpha(x), \vec{\nu}\rangle = 0$ for all x and $K = 2^{p-1}$ otherwise. Suppose that $\alpha(x)$ vanishes outside Ω and is positive inside Ω. Denote*

$$R(t) = \int_{S(t)\cap\Omega} K\alpha^p(x)\,\mathrm{d}\sigma$$

and

$$C(t) = \int_{S(t)\cap\Omega} \Big(\alpha^p(x)c(x) - K\|\nabla\alpha(x)\|^p\Big)\,\mathrm{d}\sigma\,.$$

If the ordinary differential equation

$$\Big(R(t)\Phi(u')\Big)' + C(t)\Phi(u) = 0, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}t} \tag{3.6}$$

is oscillatory at $+\infty$, then Eq. (3.2) *is oscillatory in Ω.*

Proof. Suppose that (3.6) is oscillatory. In view of Lemma 3.3 it is sufficient to prove that for every $a > 0$ there exists a function $y \in W^{1,p}(\Omega(a)\cap\Omega)$ with compact support $M \subseteq \overline{\Omega(a)\cap\Omega}$ for which (3.5) fails.

According to Lemma 3.4, for every $a > 0$ there exists $b > a$ and a piecewise smooth function z such that $z(a) = 0 = z(b)$ and

$$\int_a^b \Big(R(t)|z'(t)|^p - C(t)|z(t)|^p\Big)\,\mathrm{d}t < 0.$$

Set $y(x) = \alpha(x)z(\|x\|)$ for $a \le \|x\| \le b$ and $y(x) \equiv 0$ otherwise. Clearly $y(x) = 0$ on $\partial(\Omega \cap \Omega(a,b))$. Direct computation shows

$$\nabla y(x) = \nabla \alpha(x) z(\|x\|) + \varphi(x) z'(\|x\|)\vec{\nu}$$

and by (3.3)

$$\|\nabla y(x)\|^2 \le A\Big(\|\nabla \alpha(x)\|^2 z^2(\|x\|) + \alpha^2(x) z'^2(\|x\|)\Big), \tag{3.7}$$

where $A = 2$. Moreover, if $\langle \nabla \alpha(x), \vec{\nu}\rangle = 0$, then (3.7) holds also with $A = 1$ and equality sign. By inequality (3.4) we have for $p \ge 2$ and $l = p/2$

$$\|\nabla y(x)\|^p \le A^{\frac{p}{2}} 2^{\frac{p}{2}-1}\Big(\|\nabla \alpha(x)\|^p |z(\|x\|)|^p + \alpha^p(x)|z'(\|x\|)|^p\Big).$$

Hence

$$\begin{aligned}
&\int_{\Omega(a,b)\cap\Omega} \big(\|\nabla y(x)\|^p - c(x)|y(x)|^p\big)\,\mathrm{d}x \\
&= \int_a^b \int_{S(t)\cap\Omega} \big(\|\nabla y(x)\|^p - c(x)|y(x)|^p\big)\,\mathrm{d}\sigma \ \mathrm{d}t \\
&= \int_a^b |z'(t)|^p \Big(\int_{S(t)\cap\Omega} K\alpha(x)^p\,\mathrm{d}\sigma\Big) \\
&\quad - |z(t)|^p \Big(\int_{S(t)\cap\Omega} \big(\alpha^p(x)c(x) - K\|\nabla\alpha(x)\|^p\big)\,\mathrm{d}\sigma\Big)\,\mathrm{d}t \\
&= \int_a^b \Big(R(t)|z'(t)|^p - C(t)|z(t)|^p\Big)\,\mathrm{d}t < 0.
\end{aligned}$$

Hence, by Lemma 3.3, Eq. (3.2) cannot possess a solution positive on the domain $\Omega(a,b)\cap\Omega$. Since a can be arbitrary large, the equation is oscillatory in Ω. □

Remark 3.1. Theorem 3.1 with $\Omega = \mathbb{R}^n$, $\alpha(x) \equiv 1$ and $K = 1$ is known, see [Došlý, Mařík, 2001, Theorem 3.5]. The test function from (3.5) is in this case in the form $y(x) = z(\|x\|)$. See also [Jaroš, Kusano, Yoshida, 2000] for slightly more general equation than (3.2).

Remark 3.2. Oscillation criteria for Eq. (3.6) can be found in monograph [Došlý, Řehák, 2005] and the references therein.

Remark 3.3. For $n = 2$ on the plane with Cartesian coordinates (x_1, x_2) let $r(x_1, x_2)$, $\varphi(x_1, x_2)$ be the polar coordinates of the point $x = (x_1, x_2)$. Let us denote the half-plane $x_2 \ge 0$ by Ω and put $\alpha(x_1, x_2) = \sin^2 \varphi(x_1, x_2)$ for $(x_1, x_2) \in \Omega$ (i.e. $\varphi \in [0, \pi]$) and $\alpha(x_1, x_2) \equiv 0$ otherwise. The function $C(t)$ from Eq. (3.6) is affected by the values of the function $c(x_1, x_2)$ over the half-plane Ω only. Application of any of the known oscillation criteria for Eq. (3.6) produces a criterion for oscillation of (3.2) over the domain Ω. This criterion will not be disturbed by the values of the function $c(x_1, x_2)$ on the half-plane $x_2 \le 0$ which may be relatively “bad”. Among others, the criterion may be applicable also in the cases when the values on the lower half-plane $x_2 \le 0$ causes that $\int_{S(t)} c(x)\,\mathrm{d}\sigma = 0$. In these cases radial oscillation criteria (in the sense of Remark 2.3 on page 4) fail to detect the oscillation.

Summary

In this book we study the partial differential equation with $p-$Laplacian and the nonlinearity of Emden-Fowler type

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) + c(x)\Phi(u) = 0 \tag{1}$$

and its generalizations. Here $p > 1$, Φ is signed power function $\Phi(u) = |u|^{p-2}u = |u|^{p-1}\operatorname{sgn} u$, $x = (x_1, x_2, \ldots, x_n)$, the vector norm $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^n$ and $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ is the usual nabla operator. The sets $\Omega(a)$ and $S(a)$ are sets in $\mathbb{R}^n$ defined as follows:

$$\Omega(a) = \left\{x \in \mathbb{R}^n : a \le \|x\|\right\},$$
$$S(a) = \left\{x \in \mathbb{R}^n : \|x\| = a\right\}.$$

The function $c(x)$ called potential is assumed to be integrable on every compact subset of $\Omega(1)$. It is worth to mention that we do not assume anything concerning either the fixed sign or the radial symmetry of the potential $c(x)$. The solution of Eq. (1) is every differentiable function $u : \Omega(1) \to \mathbb{R}$ such that $\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_i}$ is differentiable with respect to x_i and u satisfies Eq. (1) almost everywhere on $\Omega(1)$.

The number q is the conjugate number to p, i.e., $q = \frac{p}{p-1}$. The number ω_n is the surface area of the unit sphere in $\mathbb{R}^n$ and the vector $\vec{\nu}(x)$ is the normal unit vector to the sphere $S(\|x\|)$ oriented outwards, i.e. $\vec{\nu}(x) = (x_1, \ldots, x_n)\|x\|^{-1}$.

A well-known linear oscillation theory is established for the equation

$$\Delta u + c(x)u = 0. \tag{2}$$

According to this theory, there are two different concepts of oscillation – *weak oscillation* and *strong (nodal) oscillation*. Equation (2) is said to be *weakly oscillatory* if every solution has a zero outside every ball in $\mathbb{R}^n$ and *strongly oscillatory* if every solution has a nodal domain outside every ball in $\mathbb{R}^n$. In this book the weak oscillation is used.

The classical Sturm theory states that

$$u'' + c(x)u = 0 \tag{3}$$

is oscillatory if the function $c(x)$ is sufficiently large and it is known that these classical Sturmian comparison theorems can be extended to Eq. (1) and consequently, Eq. (1) is oscillatory if the function $c(x)$ is sufficiently large. Most oscillation criteria arise essentially from oscillation criterion for Eq. (3) by replacing the onedimensional potential with integral mean value of the n-variable potential $c(x)$ from Eq. (1) where the mean value is evaluated over spheres centered in the origin. Thus the function $c(x)$ is usually embedded in the integrals

over spheres in absolute majority of oscillation criteria and as an unwanted side-effect, the information about distribution of the potential over the sphere is lost. To remove this disadvantage we derive several oscillation results in which the distribution of the potential $c(x)$ over spheres is also allowed to play a role. These criteria are called *nonradial* oscillation criteria.

Let us emphasize that following the nonradial approach we obtain oscillation criteria which are applicable also to the cases when the equation is strongly asymmetric with respect to origin and the mean value of the potential $c(x)$ is small. The possible applications include for example criteria which depend on the function $g(r) = \int_{S(r)} \rho(x)c(x)\,\mathrm{d}\sigma$, where $\rho(x)$ is n-variable function (which does not depend on $\|x\|$ only). The oscillation criteria of this type are applicable also in such extreme cases when $\int_{S(r)} c(x)\,\mathrm{d}S = 0$ and these criteria can be used also to detect oscillation over more general exterior domains, than the exterior of a ball. The author believes that nonradial criteria are more natural for partial differential equations and provide deeper insight into the oscillation properties specific for partial differential equations.

Typical result is the following.

Theorem 1. *Denote*

$$\begin{aligned} D &= \big\{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t \ge \|x\| \ge t_0\big\},\\ D_0 &= \big\{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t > \|x\| \ge t_0\big\}. \end{aligned}$$

Let $H(t,x) \in C(D,[0,\infty))$, and $\rho(x) \in C^1(\Omega(t_0),(0,\infty))$ be such that the function $H(t,x)$ has continuous partial derivative with respect to x_i ($i = 1..n$) on D_0. Denote

$$\begin{aligned} \Omega_{0,t}(a,b) &= \big\{x \in \mathbb{R}^n : a \le \|x\| \le b, H(t,x) \ne 0\big\},\\ S_{0,t}(a) &= \big\{x \in \mathbb{R}^n : \|x\| = a, H(t,x) \ne 0\big\}. \end{aligned}$$

and suppose that the following conditions hold

(i) *If $\|x\| = t \ge t_0$, then $H(t,x) = 0$.*

(ii) *If $H(t,x) = 0$ for some $(t,x) \in D_0$, then $\|\nabla H(t,x)\| = 0$.*

(iii) *There exists function $k(s) \in C([t_0,\infty),(0,\infty))$ such that the function $f(t,s) := k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma = k(s)\int_{S_{0,t}(s)} H(t,x)\,\mathrm{d}\sigma$ is positive and nonincreasing with respect to s for every $t > s \ge t_0$.*

(iv) *The vector–valued function $\vec{h}(t,x)$ defined on D_0 by*

$$\vec{h}(t,x) = \nabla H(t,x) + \frac{H(t,x)}{\rho(x)}\nabla\rho(x) \tag{4}$$

satisfies

$$\int_{\Omega_{0,t}(t_0,t)} H^{1-p}(t,x)\|\vec{h}(t,x)\|^p \rho(x)\,\mathrm{d}x < \infty \tag{5}$$

for $t > t_0$.

If

$$\limsup_{t\to\infty}\left(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\right)^{-1} \times \int_{\Omega_{0,t}(t_0,t)}\left[H(t,x)\rho(x)c(x)-\frac{\|\vec{h}(t,x)\|^p\rho(x)}{p^pH^{p-1}(t,x)}\right]\mathrm{d}x = \infty \tag{6}$$

then Eq. (1) *is oscillatory.*

To explain the difference between this theorem and the usual radially symmetric criteria consider an unbounded domain $\Omega\subset\Omega(t_0)$ with smooth boundary $\partial\Omega$. If in addition to the conditions of Theorem 1 the function $H(t,x)$ vanishes outside Ω and both $H(t,x)$ and $\|\nabla H(t,x)\|$ vanish on $\partial\Omega$ for every $t\geq t_0$, then it follows that Eq. (1) is oscillatory in Ω. Hence Theorem 1 can be used to formulate explicit oscillation criteria on general types of domains. Examples of the oscillation criteria on half–plane are given on page 35.

Many of the results are proved also for more general equations

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right)+\left\langle\vec{b}(x),\|\nabla u\|^{p-2}\nabla u\right\rangle+c(x)\Phi(u)=0. \tag{7}$$

and

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+\Big\langle\vec{b}(x),\|\nabla u\|^{p-2}\nabla u\Big\rangle+c(x)\Phi(u)=0 \tag{8}$$

As a particular example of general oscillation criterion we present the following two Theorems 2 and 3.

Theorem 2. *For a real number $l>1$ define the functions*

$$\begin{aligned} a(r)&=(l^*)^{p-1}\int_{S(r)}\|A(x)\|^p\lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma\,,\\ b(r)&=\int_{S(r)}\left[c(x)-\frac{l^{p-1}}{\lambda_{\min}^{p-1}(x)}\frac{\|\vec{b}(x)\|^p}{p^p}\right]\mathrm{d}\sigma\,. \end{aligned} \tag{9}$$

where $l^*=\frac{l}{l-1}$ is the conjugate number to the number l if $\|\vec{b}(x)\|\neq 0$ and $l^*=1$ if $\|\vec{b}(x)\|=0$. If the equation

$$\Big(a(r)\Phi(u')\Big)'+b(r)\Phi(u)=0. \tag{10}$$

is oscillatory, then Eq. (8) *is also oscillatory.*

It turns out that the same result can be proved also for more general equations, which are majorants to (8). These equations cover for example

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+\Big\langle\vec{b}(x),\|\nabla u\|^{p-2}\nabla u\Big\rangle+c(x)f(u)=0, \tag{11}$$

where $f(u)$ is a differentiable function which satisfies $f(0)=0$, $uf(u)>0$ for $u\neq 0$ and

$$\frac{f'(u)}{f^{2-q}(u)}\geq p-1. \tag{12}$$

Equation (11) is sometimes called super-half-linear equation.

If the function $f(u)$ satisfies (12) with $p-1$ replaced by $\varepsilon > 0$, it is sufficient to replace $f(u)$ and $c(s)$ by $f^*(u) = \epsilon^* f(u)$ and $c^*(x) = \frac{1}{\epsilon^*} c(x)$, respectively, where $\epsilon^* = \left(\frac{p-1}{\epsilon}\right)^{p-1}$. The function $f^*(u)$ satisfies (12) and $f(u)c(x) = f^*(u)c^*(x)$ holds.

Finally, it is possible to use this method also to prove nonexistence of positive solution of the equation

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u\Big\rangle + B(x,u) = 0,$$

where

$$B(x,u) \geq c(x) f(u) \quad \text{for } u \geq 0$$

and the function $f(u)$ satisfies hypotheses stated above.

The following theorem is a variant of Theorem 2 and presents sharper result, but covers the case $1 < p \leq 2$ only.

Theorem 3. *Let* $1 < p \leq 2$. *For a real number* $l > 1$ *define the functions*

$$\begin{aligned} \widehat{a}(r) &= (l^*)^{p-1} \int_{S(r)} \lambda_{\max}(x)\, \mathrm{d}\sigma\,, \\ \widehat{b}(r) &= \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{p^p} \lambda_{\max}(x) \|\vec{b}(x) A^{-1}(x)\|^p \right] \mathrm{d}\sigma\,, \end{aligned} \tag{13}$$

where $l^* = \frac{l}{l-1}$ *is the conjugate number to the number* l *if* $\|\vec{b}(x)\| \neq 0$ *and* $l^* = 1$ *if* $\|\vec{b}(x)\| = 0$. *Here* $\vec{b}(x)A^{-1}(x)$ *denotes the matrix product of row matrix* $\big(b_1(x), \dots, b_n(x)\big)$ *and the inverse* $A^{-1}(x)$. *If the equation*

$$\Big(\widehat{a}(r)\Phi(u')\Big)' + \widehat{b}(r)\Phi(u) = 0 \tag{14}$$

is oscillatory, then Eq. (8) *is also oscillatory.*

An application of these theorems and known oscillation criteria for ordinary differential equations yields effective oscillation criteria for equation (8) which are sharper and more general than the results published in the literature.

The last part of this book concerns equations and inequalities related to Eq. (1).

Souhrn

Tato publikace je věnována studiu parciální diferenciální rovnice s p-laplaciánem a nelinearitou Emden-Fowlerova typu

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) + c(x)\Phi(u) = 0 \tag{1}$$

a některým jejím zobecněním. Zde $p > 1$, Φ je obecná mocninná funkce opatřená znaménkem argumentu $\Phi(u) = |u|^{p-2}u = |u|^{p-1}\operatorname{sgn} u$, $x = (x_1, x_2, \dots, x_n)$, vektorová norma $\|\cdot\|$ je běžná Euklidovská norma v prostoru $\mathbb{R}^n$ a $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ je obvyklý nabla operátor. Symboly $\Omega(a)$ a $S(a)$ označují následující množiny v $\mathbb{R}^n$:

$$\Omega(a) = \left\{x \in \mathbb{R}^n : a \le \|x\|\right\},$$
$$S(a) = \left\{x \in \mathbb{R}^n : \|x\| = a\right\}.$$

Funkce $c(x)$ se nazývá potenciál a předpokládáme, že tato funkce je integrovatelná na každé kompaktní podmnožině množiny $\Omega(1)$. Zdůrazněme na tomto místě, že nečiníme žádné předpoklady o znaménku této funkce nebo o její radiální symetrii. Pod pojmem řešení rovnice (1) rozumíme každou absolutně spojitou funkci $u : \Omega(1) \to \mathbb{R}$, pro kterou je $\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_i}$ absolutně spojitá vzhledem k x_i a u splňuje rovnici (1) skoro všude na $\Omega(1)$.

Symbolem q označujeme konjugované číslo k číslu p, tj. $q = \frac{p}{p-1}$. Symbolem ω_n označujeme povrch jednotkové koule v $\mathbb{R}^n$ a vektor $\vec{\nu}(x)$ je normálový vektor ke kulové ploše $S(\|x\|)$ orientovaný vně, tj. $\vec{\nu}(x) = (x_1, \dots, x_n)\|x\|^{-1}$.

Pro rovnici

$$\Delta u + c(x)u = 0, \tag{2}$$

která je speciálním případem rovnice (1) pro $p = 2$, je vybudována rozsáhlá oscilační teorie. V této teorii jsou rozlišovány dva druhy oscilace – *slabá oscilace* a *silná (nodální) oscilace*. Rovnice (2) je oscilatorická ve *slabém* smyslu, pokud každé její řešení má nulový bod vně libovolně velké koule v $\mathbb{R}^n$ a v *silném* smyslu, pokud pro každé její řešení u existuje vně libovolně velké koule množina Ω taková, že funkce u je rovna nule na hranici této množiny. V této práci se budeme zabývat slabou oscilací. (Poznamenejme, že oba druhy oscilací jsou ekvivalentní v lineárním případě $p = 2$, problém ekvivalence obou přístupů v případě obecného p je však dosud otevřen.)

Podle klasické Sturmovy srovnávací teorie je rovnice

$$u'' + c(x)u = 0 \tag{3}$$

oscilatorická, pokud je funkce $c(x)$ dostatečně velká pro velká x. Je známo, že mnoho částí této teorie je možno zobecnit i na rovnici (1). V důsledku této skutečnosti je rovnice (1)

oscilatorická, pokud je funkce $c(x)$ dostatečně velká. Ve většině oscilačních kritérií je funkce jedné proměnné $c(x)$ z rovnice (3) nahrazena integrální střední hodnotou funkce n proměnných $c(x)$ z rovnice (1), přičemž střední hodnota je počítána na sférách se středem v počátku. V důsledku toho funkce $c(x)$ ve velké většině oscilačních kritérií vystupuje prostřednictvím své střední hodnoty na sférách se středem v počátku a nežádoucím doprovodným jevem je fakt, že po výpočtu této integrální střední hodnoty ztrácíme informaci o rozložení potenciálu v jednotlivých směrech. Abychom odstranili tento nežádoucí jev, jsou v práci odvozeny výsledky, ve kterých hraje roli nejenom střední hodnota funkce $c(x)$, ale i její rozložení v prostoru. Kritéria tohoto typu nazýváme díky jejich podstatě *neradiální* oscilační kritéria.

Zdůrazněme, že díky neradiálnímu přístupu jsme schopni odvodit oscilační kritéria, která jsou aplikovatelná i na případy, kdy funkce c je silně radiálně nesymetrická vzhledem k počátku a její střední hodnota na sférách se středem v počátku je malá. Možné aplikace zahrnují i tak extrémní případy, kdy střední hodnota funkce $c(x)$ je nulová, tj. platí $\int_{S(r)} c(x)\,\mathrm{d}S = 0$. Mezi další výhody tohoto přístupu patří i možnost detekce oscilace na obecnějších množinách, než je pouze vnější část n-rozměrné koule. Autor věří, že tento přístup je přirozený pro parciální diferenciální rovnice a poskytuje mnohem lepší náhled na oscilační teorii parciálních diferenciálních rovnic a případné rozdíly mezi parciálními a obyčejnými diferenciálními rovnicemi.

Typickým výsledkem je následující věta.

Věta 1. *Označme*

$$\begin{aligned} D &= \big\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t \geq \|x\| \geq t_0\big\}, \\ D_0 &= \big\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t > \|x\| \geq t_0\big\}. \end{aligned}$$

Buď $H(t,x) \in C(D,[0,\infty))$*, a funkce* $\rho(x) \in C^1(\Omega(t_0),(0,\infty))$ *buď taková, že funkce* $H(t,x)$ *má spojitou parciální derivaci podle* x_i *(*$i = 1..n$*) na množině* D_0*. Označme*

$$\begin{aligned} \Omega_{0,t}(a,b) &= \big\{x \in \mathbb{R}^n : a \leq \|x\| \leq b, H(t,x) \neq 0\big\}, \\ S_{0,t}(a) &= \big\{x \in \mathbb{R}^n : \|x\| = a, H(t,x) \neq 0\big\}. \end{aligned}$$

a předpokládejme, že jsou splněny následující podmínky

(i) *Jestliže* $\|x\| = t \geq t_0$*, pak* $H(t,x) = 0$*.*

(ii) *Jestliže* $H(t,x) = 0$ *pro některá* $(t,x) \in D_0$*, potom* $\|\nabla H(t,x)\| = 0$*.*

(iii) *Existuje funkce* $k(s) \in C([t_0,\infty),(0,\infty))$ *taková, že funkce* $f(t,s) := k(s)\int_{S(s)} H(t,x)\,\mathrm{d}\sigma = k(s)\int_{S_{0,t}(s)} H(t,x)\,\mathrm{d}\sigma$ *je kladná a neklesající vzhledem k proměnné* s *pro každé* $t > s \geq t_0$*.*

(iv) *Vektorová funkce* $\vec{h}(t,x)$ *definovaná na* D_0 *vztahem*

$$\vec{h}(t,x) = \nabla H(t,x) + \frac{H(t,x)}{\rho(x)}\nabla\rho(x) \tag{4}$$

splňuje

$$\int_{\Omega_{0,t}(t_0,t)} H^{1-p}(t,x)\|\vec{h}(t,x)\|^p\rho(x)\,\mathrm{d}x < \infty \tag{5}$$

pro $t > t_0$*.*

Jestliže

$$\limsup_{t\to\infty}\left(\int_{S(t_0)} H(t,x)\,\mathrm{d}\sigma\right)^{-1} \times \int_{\Omega_{0,t}(t_0,t)}\left[H(t,x)\rho(x)c(x)-\frac{\|\vec{h}(t,x)\|^p\rho(x)}{p^pH^{p-1}(t,x)}\right]\mathrm{d}x = \infty, \tag{6}$$

potom rovnice (1) *je oscilatorická.*

Pro objasnění rozdílu mezi touto větou a obvyklými postačujícími podmínkami pro oscilaci uvažujme ohraničenou oblast $\Omega \subset \Omega(t_0)$ s hladkou hranicí $\partial\Omega$. Pokud vedle podmínky Věty 1 je funkce $H(t,x)$ navíc rovna nule mimo oblast Ω a obě funkce $H(t,x)$ a $\|\nabla H(t,x)\|$ jsou nulové na hranici $\partial\Omega$ pro všechna $t \geq t_0$, pak je rovnice (1) oscilatorická na množině Ω. Větu 1 je tedy možno použít pro formulaci explicitních oscilačních kritérií na obecnějších množinách, než vnější část koule. Jako příklad aplikace tohoto typu odvozujeme v práci například oscilační kritéria zaručující oscilaci na polorovině.

Velká část výsledků týkajících se oscilace je odvozena pro obecnější rovnice

$$\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right)+\left\langle\vec{b}(x),\|\nabla u\|^{p-2}\nabla u\right\rangle+c(x)\Phi(u)=0 \tag{7}$$

a

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+\left\langle\vec{b}(x),\|\nabla u\|^{p-2}\nabla u\right\rangle+c(x)\Phi(u)=0. \tag{8}$$

Jako příklad obecných oscilačních kritérií formulujme následující dvě věty, Větu 2 a 3.

Věta 2. *Pro reálné číslo $l > 1$ definujme funkce*

$$\begin{aligned} a(r) &= (l^*)^{p-1}\int_{S(r)}\|A(x)\|^p\lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma\,,\\ b(r) &= \int_{S(r)}\left[c(x)-\frac{l^{p-1}}{\lambda_{\min}^{p-1}(x)}\frac{\|\vec{b}(x)\|^p}{p^p}\right]\mathrm{d}\sigma\,. \end{aligned} \tag{9}$$

kde $l^* = \frac{l}{l-1}$ je konjugované číslo k číslu l pokud $\|\vec{b}(x)\| \neq 0$ a $l^* = 1$ pokud $\|\vec{b}(x)\| = 0$. Jestliže je rovnice

$$\Big(a(r)\Phi(u')\Big)' + b(r)\Phi(u) = 0 \tag{10}$$

oscilatorická, potom je oscilatorická i rovnice (8).

Je možno ukázat, že analogické výsledky mohou být dokázány i pro obecnější rovnice, které v jistém smyslu rovnici (8) majorizují. Tyto obecnější rovnice zahrnují například

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big)+\left\langle\vec{b}(x),\|\nabla u\|^{p-2}\nabla u\right\rangle+c(x)f(u)=0, \tag{11}$$

kde $f(u)$ je diferencovatelná funkce splňující $f(0) = 0$, $uf(u) > 0$ pro $u \neq 0$ a

$$\frac{f'(u)}{f^{2-q}(u)} \geq p-1. \tag{12}$$

Pokud navíc funkce $f(u)$ splňuje nerovnici (12) s nějakou jinou kladnou konstantou než $p-1$, například s konstantou $\varepsilon > 0$, stačí zaměnit funkce $f(u)$ a $c(x)$ za funkce $f^*(u) = \epsilon^* f(u)$ a $c^*(x) = \frac{1}{\epsilon^*} c(x)$, kde $\epsilon^* = \left(\frac{p-1}{\epsilon}\right)^{p-1}$. Tímto se rovnice nezmění a nové funkce již mají požadované vlastnosti.

Dále je možné analogickou cestou dokázat podmínky zaručující neexistenci kladného řešení rovnice

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \Big\rangle + B(x,u) = 0,$$

kde

$$B(x,u) \geq c(x)f(u) \quad \text{pro } u \geq 0$$

a funkce $f(u)$ splňuje podmínky z předešlých odstavců.

Následující věta je silnější variantou Věty 2, pokrývá ovšem pouze případ $1 < p \leq 2$.

Věta 3. *Buď* $1 < p \leq 2$ *reálné číslo. Pro reálné číslo* $l > 1$ *definujme funkce*

$$\begin{aligned} \widehat{a}(r) &= (l^*)^{p-1} \int_{S(r)} \lambda_{\max}(x)\, \mathrm{d}\sigma\,, \\ \widehat{b}(r) &= \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{p^p} \lambda_{\max}(x) \|\vec{b}(x)A^{-1}(x)\|^p \right] \mathrm{d}\sigma\,, \end{aligned} \tag{13}$$

kde $l^* = \frac{l}{l-1}$ *je konjugované číslo k číslo* l*, pokud* $\|\vec{b}(x)\| \neq 0$*, a* $l^* = 1$ *pokud* $\|\vec{b}(x)\| = 0$*. Zde symbolem* $\vec{b}(x)A^{-1}(x)$ *označujeme maticový součin řádkové matice* $\big(b_1(x), \dots, b_n(x)\big)$ *a inverzní matice* $A^{-1}(x)$*. Jestliže je rovnice*

$$\Big(\widehat{a}(r)\Phi(u')\Big)' + \widehat{b}(r)\Phi(u) = 0 \tag{14}$$

oscilatorická, je oscilatorická i rovnice (8).

Aplikací těchto vět a známých oscilačních kritérií pro obyčejné diferenciální rovnice získáváme oscilační kritéria pro parciální diferenciální rovnici (8) a taková kritéria v mnoha případech zlepšují a zpřesňují dosavadní známé a publikované výsledky.

Závěrečná část publikace je věnována rovnicím a nerovnicím příbuzným s rovnicí (1).

Klíčová slova: parciální diferenciální rovnice, diferenciální rovnice druhého řádu, p-laplacián, oscilační teorie

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