OSCILLATION CRITERIA FOR THE SCHRÖDINGER PDE

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Abstract. In this paper oscillatory properties of the Schrödinger partial differential equation are investigated. Using Riccati technique a Hartman–Wintner type criterion is proved. Further, some oscillation criteria which appeared only recently for the ordinary differential equations, are extended to the partial differential equation.

1 Introduction

Let us consider the Schrödinger partial differential equation

$$\Delta u + c(x)u = 0 \qquad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \tag{1.1}$$

The function $c(x)$ is assumed to be integrable on each compact subset of the set $\Omega = \{x \in \mathbb{R}^n : ||x|| \geq 1\}$. By a solution of equation (1.1) is understood the function $u : \Omega \to \mathbb{R}$ which is absolutely continuous with first partial derivatives in each compact subset of Ω and satisfy the equation (1.1) almost everywhere on Ω.

Oscillation properties of the Schrödinger, or more generally, of elliptic partial differential equations are studied for example in [3, 8, 11, 12].

In Müller–Pfeiffer [8] and Toraev [12] the variational method is used to derive oscillation criteria for elliptic partial differential equations. Here the equation (1.1) is said to be oscillatory if there exists a *nodal domain* outside of arbitrary ball in $\mathbb{R}^n$. In Allegretto [1] is such a oscillation called *strong oscillation*; an alternative term is *nodal oscillation.* Schminke in [11] introduced Riccati technique for studying oscillation properties of (1.1). Using this method it is natural to define oscillation as follows.

Definition 1.1. The function $f(x)$ is said to be *oscillatory* if $f(x)$ has zero outside of arbitrary ball in $\mathbb{R}^n$ centered in the origin and it is said *nonoscillatory* otherwise. The equation (1.1) is said to be *nonoscillatory* if it has nonoscillatory solution and *oscillatory* otherwise.

By the n-dimensional version of the Sturm separation theorem the equation (1.1) is oscillatory if it is nodally oscillatory. Equivalence between these two types of oscillation, which is obvious in the case $n = 1$, has been proved in Allegretto [1] for the function $c(x)$ sufficiently smooth. In Moss–Piepenbrik [7] an alternative proof of this equivalence was given and the condition on smoothness of the function $c(x)$ was weakened into the condition the function $c(x)$ to be locally Hölder continuous.

The aim of this paper is to give some new oscillation criteria, i.e. criteria which guarantees that if there exists a solution u of the equation (1.1) on Ω, then the function u is oscillatory. These results extends some known criteria for oscillation of the equation (1.1)

Denote

$$C(t) = \frac{1}{t} \int_1^t \int_{1 \le ||x|| \le T} ||x||^{1-n} c(x) \, \mathrm{d}x \; \mathrm{d}T \, . \tag{1.2}$$

We distinguish two cases.

1. There exists a finite limit
$$\lim_{t \to \infty} C(t) = C_0. \tag{1.3}$$

2. The condition (1.3) fails to hold and $\liminf_{t\to\infty} C(t) > -\infty$.

In the first case the method introduced for $n = 1$ in Chantladze, Kandelaki and Lomtatidze [2] is used in proving our main results. Denote

$$\begin{aligned} Q(t) &= t\Big(C_0 - \int_{1 \le ||x|| \le t} ||x||^{1-n} c(x) \, \mathrm{d}x \Big), \\ H(t) &= \frac{1}{t} \int_{1 \le ||x|| \le t} ||x||^{3-n} c(x) \, \mathrm{d}x \\ Q_* = \liminf_{t\to\infty} Q(t), &\quad Q^* = \limsup_{t\to\infty} Q(t), \\ H_* = \liminf_{t\to\infty} H(t), &\quad H^* = \limsup_{t\to\infty} H(t). \end{aligned}$$

The oscillation criteria are formulated in terms of the functions $C(t)$, $Q(t)$, $H(t)$ and numbers Q_*, Q^*, H_*, H^*.

In the second case the equation (1.1) is oscillatory by Theorem 3.1 below.

The paper is organized as follows. In the next section we present auxiliary results concerning the nonoscillatory Schrödinger equation. Main results are formulated in Section 3. The Section 4 is devoted to some remarks and comments.

2 Preliminary results

We use the Riccati technique introduced in [11]. Denote $\Omega_r = \{x \in \mathbb{R}^n : ||x|| \geq r\}$ If u is a solution of (1.1) which is positive in Ω_r for some $r > 0$, then the vector variable $w = \frac{\nabla u}{u}$ is defined on Ω_r and solves here the *Riccati type equation*

$$\operatorname{div} w + c(x) + ||w||^2 = 0, \tag{2.1}$$

where $||\cdot||$ is the usual Euclidean norm in $\mathbb{R}^n$. Conversely, if there exists a solution of the equation (2.1) defined in Ω_r, then there exists a positive solution of the equation (1.1). Clearly the equation (1.1) is nonoscillatory if and only if there exists a number $a \in \mathbb{R}$ and a solution u of equation (1.1) positive on Ω_a, i.e. there exists a solution w of (2.1) defined on the set Ω_a. In this section we will suppose that the equation (1.1) is nonoscillatory and the Riccati equation has a solution on Ω_a.

The following Lemma is an extension of Lemma 7.1 in Chapter XI from Hartman [4] and plays a crucial role in our later considerations.

Lemma 2.1. *Let the equation* (1.1) *be nonoscillatory, i.e.* (1.1) *has a positive solution on* Ω_a *for some* $a \geq 1$. *The following statements are equivalent:*

(i) It holds

$$\int_{a \leq ||x|| \leq \infty} ||x||^{1-n} ||w||^2 \,\mathrm{d}x < \infty; \tag{2.2}$$

(ii) There exists a finite limit

$$\lim_{t \to \infty} C(t) = C_0; \tag{2.3}$$

(iii) It holds

$$\liminf_{t \to \infty} C(t) > -\infty. \tag{2.4}$$

Proof. Let w be the solution of Riccati equation defined on the set Ω_a. The Riccati equation, Gauss divergence theorem and the identity

$$||x||^{1-n} \operatorname{div} w = \operatorname{div}(||x||^{1-n} w) - (1-n)||x||^{-n} \langle w, j \rangle,$$

where j is the unit outside normal to the sphere in $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$, implies that w satisfies the equality

$$\begin{aligned} \int_{a \leq ||x|| \leq t} ||x||^{1-n} c(x) \,\mathrm{d}x &+ \int_{a \leq ||x|| \leq t} ||x||^{1-n} ||w||^2 \,\mathrm{d}x \\ &- \int_{a \leq ||x|| \leq t} (1-n)||x||^{-n} \langle w, j \rangle \,\mathrm{d}x + \int_{||x||=t} ||x||^{1-n} \langle w, j \rangle \,\mathrm{d}S \\ &- \int_{||x||=a} ||x||^{1-n} \langle w, j \rangle \,\mathrm{d}S = 0 \end{aligned} \tag{2.5}$$

for $t \geq a$.

"(i)=>(ii)" Suppose that (2.2) holds. Then the Cauchy inequality implies

$$\begin{aligned}
&\int_{a\le||x||\le t} ||x||^{-n}||w||\,\mathrm{d}x \\
&\qquad \le \Big(\int_{a\le||x||\le t} ||x||^{1-n}||w||^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_{a\le||x||\le t} ||x||^{-n-1}\,\mathrm{d}x\Big)^{1/2} \\
&\qquad = \Big(\int_{a\le||x||\le t} ||x||^{1-n}||w||^2\,\mathrm{d}x\Big)^{1/2}\Big(\omega_n\int_a^t \frac{1}{s^2}\,\mathrm{d}s\Big)^{1/2},
\end{aligned}$$

where ω_n is the measure of the unit sphere in $\mathbb{R}^n$ and

$$\int_{a\le||x||\le\infty} ||x||^{-n}\langle w,j\rangle\,\mathrm{d}x\;<\infty \tag{2.6}$$

converges. From (2.5) and (2.6) we get

$$\begin{aligned}
\widehat{C}-\int_{1\le||x||\le t} ||x||^{1-n}c(x)\,\mathrm{d}x\;&=\int_{||x||=t} ||x||^{1-n}\langle w,j\rangle\,\mathrm{d}S \\
&-\int_{t\le||x||\le\infty} ||x||^{1-n}||w||^2\,\mathrm{d}x\;+(1-n)\int_{t\le||x||\le\infty} ||x||^{-n}\langle w,j\rangle\,\mathrm{d}x\,,
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
\widehat{C}=-&\int_{a\le||x||\le\infty} ||x||^{1-n}||w||^2\,\mathrm{d}x\;+\int_{1\le||x||\le a} ||x||^{1-n}c(x)\,\mathrm{d}x \\
+&\int_{||x||=a} ||x||^{1-n}\langle w,j\rangle\,\mathrm{d}S\;+(1-n)\int_{a\le||x||\le\infty} ||x||^{-n}\langle w,j\rangle\,\mathrm{d}x
\end{aligned}$$

is a finite number. We will show that

$$\widehat{C}=C_0 \tag{2.8}$$

Then it will follow that $\widehat{C}$ in fact does not depend on the choice of the number a.

From (2.7) and from the inequality

$$|\alpha+\beta+\gamma|^2\le 4|\alpha|^2+4|\beta|^2+4|\gamma|^2$$

it follows

$$\frac{1}{T}\int_a^T\Big|\widehat{C}-\int_{1\le||x||\le t} ||x||^{1-n}c(x)\,\mathrm{d}x\Big|^2\,\mathrm{d}t$$

$$\le\frac{4}{T}\int_a^T\Big|\int_{||x||=t} ||x||^{1-n}\langle w,j\rangle\,\mathrm{d}S\Big|^2\,\mathrm{d}t \tag{2.9}$$

$$+\frac{4}{T}\int_a^T\Big|\int_{t\le||x||\le\infty} ||x||^{1-n}||w||^2\,\mathrm{d}x\Big|^2\,\mathrm{d}t \tag{2.10}$$

$$+\frac{4(1-n)^2}{T}\int_a^T\Big|\int_{t\le||x||\le\infty} ||x||^{-n}\langle w,j\rangle\,\mathrm{d}x\Big|^2\,\mathrm{d}t\,. \tag{2.11}$$

By l'Hospital rule, (2.2) and (2.6) the terms (2.10) and (2.11) tend to zero for $T \to \infty$. The Cauchy inequality implies

$$
\begin{aligned}
&\frac{1}{T}\int_a^T \left| \int_{||x||=t} ||x||^{1-n} \langle w, j\rangle \, \mathrm{d}S \right|^2 \mathrm{d}t \\
&\qquad \le \frac{1}{T}\int_a^T \left(\int_{||x||=t} ||x||^{1-n} ||w||^2 \, \mathrm{d}S \right) \left(\int_{||x||=t} ||x||^{1-n} \, \mathrm{d}S \right) \mathrm{d}t \\
&\qquad = \frac{1}{T}\omega_n \int_{a\le ||x||\le T} ||x||^{1-n} ||w||^2 \, \mathrm{d}x
\end{aligned}
$$

and using (2.2) the term (2.9) tends to zero. Hence

$$
\frac{1}{T}\int_a^T \left| \widehat{C} - \int_{1\le ||x||\le t} ||x||^{1-n} c(x) \, \mathrm{d}x \right|^2 \mathrm{d}t \ \to 0 \qquad \text{for } T \to \infty. \tag{2.12}
$$

Cauchy inequality implies

$$
\left| \frac{1}{T}\int_a^T f(t) \, \mathrm{d}t \right| \le \frac{1}{T}\int_a^T |f(t)| \, \mathrm{d}t \ \le \left(\frac{1}{T}\int_a^T |f(t)|^2 \, \mathrm{d}t \right)^{1/2}.
$$

Hence

$$
\begin{aligned}
&\left(\frac{1}{T}\int_a^T \left| \widehat{C} - \int_{1\le ||x||\le t} ||x||^{1-n} c(x) \, \mathrm{d}x \right|^2 \mathrm{d}t \right)^{1/2} \\
&\qquad \ge \left| \frac{1}{T}\int_a^T \left(\widehat{C} - \int_{1\le ||x||\le t} ||x||^{1-n} c(x) \, \mathrm{d}x \right) \mathrm{d}t \right| \\
&\qquad = \left| \widehat{C} - \widehat{C}\frac{a}{T} - \frac{1}{T}\int_a^T \int_{1\le ||x||\le t} ||x||^{1-n} c(x) \, \mathrm{d}x \ \mathrm{d}t \right|
\end{aligned}
$$

and from (2.12) it follows that (2.3) with $\widehat{C} = C_0$ holds.

The implication "(ii)=>(iii)" is trivial.

"(iii)=>(i)" Suppose that (2.4) holds and (2.2) does not hold. Define the function

$$
Z(t) := \int_a^t \int_{a\le ||x||\le s} ||x||^{1-n} ||w||^2 \, \mathrm{d}x \ \mathrm{d}s \, .
$$

This function satisfies

$$
\frac{Z(t)}{t} \to \infty \text{ for } t \to \infty. \tag{2.13}
$$

From (2.5) we get

$$
\begin{aligned}
&\frac{1}{T}Z(T) - \frac{1}{T}\int_a^T \int_{a\le ||x||\le t} (1-n) ||x||^{-n} \langle w, j\rangle \, \mathrm{d}x \ \mathrm{d}t \\
&\quad + \frac{1}{T}\int_a^T \int_{||x||=t} ||x||^{1-n} \langle w, j\rangle \, \mathrm{d}S \ \mathrm{d}t \\
&= -\frac{1}{T}\int_a^T \int_{a\le ||x||\le t} ||x||^{1-n} c(x) \, \mathrm{d}x \ \mathrm{d}t \, + \frac{T-a}{T} \int_{||x||=a} ||x||^{1-n} \langle w, j\rangle \, \mathrm{d}S \, .
\end{aligned}
$$

If (2.4) holds, then the right-hand side is bounded from above and less than $\frac{1}{4T}Z(T)$ for T large enough. Hence

$$\begin{aligned}\frac{3}{4}Z(T) \leq & \left| \int_a^T \int_{a\leq ||x||\leq t} (1-n)||x||^{-n}\langle w, j\rangle \,\mathrm{d}x \;\mathrm{d}t \right| \\ & + \left| \int_a^T \int_{||x||=t} ||x||^{1-n}\langle w, j\rangle \,\mathrm{d}S \;\mathrm{d}t \right|\end{aligned}$$

for T large enough. The Cauchy inequality gives

$$\begin{aligned}&\left| \int_a^T \int_{||x||=t} ||x||^{1-n}\langle w, j\rangle \,\mathrm{d}S \;\mathrm{d}t \right| \\ &\quad \leq \left(\int_{a\leq ||x||\leq T} ||x||^{1-n}||w||^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{a\leq ||x||\leq T} ||x||^{1-n} \,\mathrm{d}x \right)^{1/2} \\ &\quad = \left(\int_{a\leq ||x||\leq T} ||x||^{1-n}||w||^2 \,\mathrm{d}x \right)^{1/2} \left(\int_a^T \omega_n \,\mathrm{d}t \right)^{1/2} \\ &\quad \leq \big(\omega_n T Z'(T)\big)^{1/2}.\end{aligned}$$

A similar computation gives

$$\begin{aligned}&\left| \int_a^T \int_{a\leq ||x||\leq t} (1-n)||x||^{-n}\langle w, j\rangle \,\mathrm{d}x \;\mathrm{d}t \right| \\ &\quad \leq (n-1)\left(\omega_n \int_a^T \int_a^t s^{-2} \,\mathrm{d}s \;\mathrm{d}t \right)^{1/2} \left(\int_a^T \int_{a\leq ||x||\leq t} ||x||^{1-n}||w||^2 \,\mathrm{d}x \;\mathrm{d}t \right)^{1/2} \\ &\quad \leq (n-1)\left(\omega_n \frac{T}{a} \right)^{1/2} \left(\int_a^T \int_{a\leq ||x||\leq t} ||x||^{1-n}||w||^2 \,\mathrm{d}x \;\mathrm{d}t \right)^{1/2}.\end{aligned}$$

From (2.13) it follows that

$$\frac{\omega_n}{a} \leq \frac{1}{4^2(n-1)^2}\frac{Z(T)}{T}$$

for T large enough, hence

$$\begin{aligned}&\left| \int_a^T \int_{a\leq ||x||\leq t} (1-n)||x||^{1-n}\langle w, j\rangle \,\mathrm{d}x \;\mathrm{d}t \right| \\ &\quad \leq (n-1)\frac{1}{4(n-1)}\Big(Z(T)\Big)^{1/2}\Big(Z(T)\Big)^{1/2} = \frac{1}{4}Z(T)\end{aligned}$$

for T large enough. Combining the above computations we get

$$\frac{1}{2}Z(T) \leq \big(\omega_n T Z'(T)\big)^{1/2}$$

and from here

$$4\omega_n \frac{Z'(T)}{Z^2(T)} \geq \frac{1}{T} \qquad \text{for } T \text{ large enough.}$$

Integration of this inequality from T_0 to ∞ gives a convergent integral on the left–hand side and divergent integral on the right–hand side. This contradiction ends the proof. □

In what follows the function $\rho(t)$ is defined

$$\rho(t) = \int_{||x||=t} ||x||^{1-n} \langle w, j \rangle \, \mathrm{d}x \tag{2.14}$$

for every $t \geq a$. In Lemmas 2.2 and 2.3 we give an a priori bound for the function $t\rho(t)$ in terms of Q_* and H_*.

Lemma 2.2. *Let* (1.3) *holds. If* (1.1) *is nonoscillatory and* $\frac{(n-2)^2-1}{4}\omega_n \leq Q_* \leq \frac{(n-2)^2}{4}\omega_n$ *holds, then*

$$\liminf_{t\to\infty} t\rho(t) \geq \frac{\omega_n}{2}\left[2 - n - \sqrt{(n-2)^2 - \frac{4Q_*}{\omega_n}}\right]. \tag{2.15}$$

Proof. Let w be the solution of the Riccati equation (2.1) defined on Ω_a for some $a \in \mathbb{R}$. From Cauchy inequality we have

$$\rho^2(t) \leq \omega_n \int_{||x||=t} ||x||^{1-n} ||w||^2 \, \mathrm{d}S \qquad \text{for } t \geq a. \tag{2.16}$$

The equalities (2.7), (2.8) and the inequality

$$\alpha x - \beta x^2 \leq \frac{\alpha^2}{4\beta} \qquad \text{for } \beta > 0 \tag{2.17}$$

imply for $t \geq a$

$$\begin{aligned} t\rho(t) &\geq Q(t) + t\int_t^\infty \left[(n-1)s\rho(s) + \frac{1}{\omega_n}s^2\rho^2(s)\right]\frac{1}{s^2}\,\mathrm{d}s \\ &\geq Q(t) - \frac{(n-1)^2}{4}\omega_n. \end{aligned} \tag{2.18}$$

Denote $r = \liminf_{t\to\infty} t\rho(t)$. If $r = \infty$, there is nothing to prove. Suppose that $r < \infty$. If $Q_* = \frac{(n-2)^2-1}{4}\omega_n$, the statement follows from (2.18). Suppose $Q_* > \frac{(n-2)^2-1}{4}\omega_n$ and choose $0 < \varepsilon < Q_* - \frac{(n-2)^2-1}{4}\omega_n$. Then there exists $t_\varepsilon \geq a$ such that $t\rho(t) \geq r - \varepsilon$ and $Q(t) \geq Q_* - \varepsilon$ for $t \geq t_\varepsilon$. Using (2.18) we get

$$\begin{aligned} t\rho(t) &\geq r - \varepsilon > r - Q_* + \frac{(n-2)^2-1}{4}\omega_n \\ &\geq -\frac{(n-1)^2}{4}\omega_n + \frac{(n-2)^2-1}{4}\omega_n = -\frac{n-1}{2}\omega_n \end{aligned}$$

for $t \geq t_\varepsilon$. From here and from the fact that the function $(n-1)x + \frac{1}{\omega_n}x^2$ is increasing for $x \geq -\frac{n-1}{2}\omega_n$ we get

$$(n-1)t\rho(t) + \frac{1}{\omega_n}t^2\rho^2(t) \geq (n-1)(r-\varepsilon) + \frac{1}{\omega_n}(r-\varepsilon)^2$$

for $t \geq t_\varepsilon$ and the first inequality in (2.18) implies

$$t\rho(t) \geq Q_* - \varepsilon + (n-1)(r-\varepsilon) + \frac{1}{\omega_n}(r-\varepsilon)^2 \tag{2.19}$$

for $t \geq t_\varepsilon$. From here using the limit process $\lim_{\varepsilon\to 0} \liminf_{t\to\infty}$ we get

$$r \geq \frac{1}{\omega_n} r^2 + (n-1)r + Q_*$$

which immediately implies (2.15). $\square$

Lemma 2.3. *If* (1.1) *is nonoscillatory and* $\frac{(n-2)^2-1}{4}\omega_n \leq H_* \leq \frac{(n-2)^2}{4}\omega_n$ *holds, then*

$$\limsup_{t\to\infty} t\rho(t) \leq \frac{\omega_n}{2}\left[2-n+\sqrt{(n-2)^2-\frac{4H_*}{\omega_n}}\,\right]. \tag{2.20}$$

Proof. Let w be the solution of the Riccati equation (2.1) defined on Ω_a for some a, $1 \leq a \leq \tau \leq T$. Derivation of (2.5) with respect to t, multiplication by t^2 and integration between τ and T we get

$$\begin{aligned}
&\int_\tau^T t^2 \int_{||x||=t} ||x||^{1-n} c(x)\,\mathrm{d}S\ \mathrm{d}t + \int_\tau^T t^2 \int_{||x||=t} ||x||^{1-n}||w||^2\,\mathrm{d}S\ \mathrm{d}t \\
&- \int_\tau^T (1-n)t^2 \int_{||x||=t} ||x||^{-n}\langle w,j\rangle\,\mathrm{d}S\ \mathrm{d}t + T^2 \int_{||x||=T} ||x||^{1-n}\langle w,j\rangle\,\mathrm{d}S \\
&-\tau^2 \int_{||x||=\tau} ||x||^{1-n}\langle w,j\rangle\,\mathrm{d}S - \int_\tau^T 2t \int_{||x||=t} ||x||^{1-n}\langle w,j\rangle\,\mathrm{d}S\ \mathrm{d}t = 0,
\end{aligned}$$

where the last three terms arises from integration by parts. From here using (2.14) and (2.16) we get

$$\begin{aligned}
T\rho(T) + H(T) &= \frac{\tau^2}{T} \int_{||x||=\tau} ||x||^{1-n}\langle w,j\rangle\,\mathrm{d}S \\
&\quad + \frac{1}{T}\int_\tau^T \Bigg[(3-n)t \int_{||x||=t} ||x||^{1-n}\langle w,j\rangle\,\mathrm{d}S \\
&\qquad - t^2 \int_{||x||=t} ||x||^{1-n}||w||^2\,\mathrm{d}S\Bigg]\,\mathrm{d}t \\
&\quad + \frac{1}{T}\int_1^\tau t^2 \int_{||x||=t} ||x||^{1-n}c(x)\,\mathrm{d}S\ \mathrm{d}t \\
&\leq \frac{\tau^2}{T}\rho(\tau) + \frac{1}{T}\int_\tau^T \left[(3-n)t\rho(t) - \frac{1}{\omega_n}t^2\rho^2(t)\right]\mathrm{d}t + \frac{\tau}{T}H(\tau)
\end{aligned} \tag{2.21}$$

Using (2.17) we get

$$T\rho(T) + H(T) \leq \frac{(3-n)^2}{4}\omega_n + \frac{\tau}{T}\big(\tau\rho(\tau) + H(\tau)\big). \tag{2.22}$$

Denote $R = \limsup_{t\to\infty} t\rho(t)$. Now if $R = -\infty$, there is nothing to prove. Suppose $R > -\infty$. From (2.22) it follows that

$$R \leq -H_* + \frac{(3-n)^2}{4}\omega_n.$$

From here we conclude that lemma is true for $H_* = \frac{(n-2)^2-1}{4}\omega_n$. Suppose that $H_* > \frac{(n-2)^2-1}{4}\omega_n$. Then for each $0 < \varepsilon < H_* - \frac{(n-2)^2-1}{4}\omega_n$ there exists $t_\varepsilon \geq a$ such that

$$t\rho(t) \leq R + \varepsilon \leq -H_* + \frac{(3-n)^2}{4}\omega_n + \varepsilon < (3-n)\frac{\omega_n}{2}$$

and

$$H(t) \geq H_* - \varepsilon$$

for $t \geq t_\varepsilon$. The inequality (2.21) and the fact that the function $(3-n)x - \frac{1}{\omega_n}x^2$ is increasing for $x \leq (3-n)\frac{\omega_n}{2}$ imply

$$\begin{aligned} t\rho(t) \leq &\varepsilon - H_* + \frac{t_\varepsilon}{T}\big(t_\varepsilon\rho(t_\varepsilon) + H(t_\varepsilon)\big) + \frac{1}{t}\int_{t_\varepsilon}^t \left[(3-n)s\rho(s) - \frac{1}{\omega_n}s^2\rho^2(s)\right] \mathrm{d}s \\ \leq &\varepsilon - H_* + \frac{t_\varepsilon}{T}\big(t_\varepsilon\rho(t_\varepsilon) + H(t_\varepsilon)\big) \\ &+ \frac{1}{t}\int_{t_\varepsilon}^t \left[(3-n)(R+\varepsilon) - \frac{1}{\omega_n}(R+\varepsilon)^2\right] \mathrm{d}t \end{aligned}$$

for every $t \geq t_\varepsilon$. Hence

$$R \leq -H_* + (3-n)R - \frac{1}{\omega_n}R^2$$

which implies (2.20). □

Remark 2.1. From the proofs of Lemmas 2.2 and 2.3 it follows that the inequalities (2.18) (2.21) and (2.22) are valid for every Q_* and H_*.

3 Main results

The following theorem is a Hartman–Wintner type oscillation criterion (see [5] and [13] for the case $n = 1$).

Theorem 3.1. *If*

$$-\infty < \liminf_{t\to\infty} C(t) < \limsup_{t\to\infty} C(t) \leq \infty \tag{3.1}$$

or if

$$\lim_{t\to\infty} C(t) = \infty, \tag{3.2}$$

then the equation (1.1) *is oscillatory.*

Proof. Suppose, by contradiction, that (3.1) holds and there exists number r such that positive solution of (1.1) on Ω_r exists. Hence the corresponding solution of Riccati equation is defined on Ω_r. The first inequality in (3.1) and the (iii)=>(ii) part of Lemma 2.1 shows that there exists a finite limit $\lim_{t\to\infty} C(t)$ which contradicts (3.1). The proof for (3.2) is the same. □

In view of this theorem the following text deals with the case when there exists a finite limit (1.3).

Theorem 3.2. *Let* (1.3) *hold and*

$$\limsup_{t\to\infty} \frac{t}{\ln t}\Big(C_0 - C(t)\Big) > \frac{(n-2)^2}{4}\omega_n. \tag{3.3}$$

Then the equation (1.1) *is oscillatory.*

Proof. Suppose, by contradiction, that there exists number $a \geq 1$ and a solution $w(x)$ of the Riccati equation defined on Ω_a. We make use of Lemma 2.1. From (2.7) with respect to (2.8) using integration by parts and from inequalities (2.16), (2.17) it follows that

$$\begin{aligned}
t\Big(C_0 - C(t)\Big) &= \int_{a\leq||x||\leq t} ||x||^{1-n}\langle w, j\rangle \,\mathrm{d}x \\
&\quad - \int_a^t \int_{T\leq||x||\leq\infty} ||x||^{1-n}||w||^2 \,\mathrm{d}x \;\mathrm{d}T \\
&\quad + \int_a^t (1-n) \int_{T\leq||x||\leq\infty} ||x||^{-n}\langle w, j\rangle \,\mathrm{d}x \;\mathrm{d}T \;+ a\big(C_0 - C(a)\big) \\
&= -t \int_{t\leq||x||\leq\infty} ||x||^{1-n}||w||^2 \,\mathrm{d}x \;+ a \int_{a\leq||x||\leq\infty} ||x||^{1-n}||w||^2 \,\mathrm{d}x \\
&\quad - \int_a^t T \int_{||x||=T} ||x||^{1-n}||w||^2 \,\mathrm{d}S \;\mathrm{d}T \\
&\quad + (1-n)t \int_{t\leq||x||\leq\infty} ||x||^{-n}\langle w, j\rangle \,\mathrm{d}x \\
&\quad - (1-n)a \int_{a\leq||x||\leq\infty} ||x||^{-n}\langle w, j\rangle \,\mathrm{d}x \\
&\quad + (2-n) \int_{a\leq||x||\leq t} ||x||^{1-n}\langle w, j\rangle \,\mathrm{d}x \;+ a\big(C_0 - C(a)\big) \\
&\leq t \int_t^\infty \Big[-\omega_n^{-1}T^2\rho^2(T) + (1-n)T\rho(T)\Big] \frac{\mathrm{d}T}{T^2} \\
&\quad + \int_a^t \Big[-\omega_n^{-1}T^2\rho^2(T) + (2-n)T\rho(T)\Big] \frac{\mathrm{d}T}{T} \\
&\quad + a\big(C_0 - C(a)\big) + (n-1)a \int_{a\leq||x||\leq\infty} ||x||^{-n}\langle w, j\rangle \,\mathrm{d}x \\
&\quad + a \int_{a\leq||x||\leq\infty} ||x||^{1-n}||w||^2 \,\mathrm{d}x \\
&\leq \frac{(n-2)^2\omega_n}{4} \ln t + \mathrm{const}
\end{aligned}$$

Hence

$$\frac{t}{\ln t}\Big(C_0 - C(t)\Big) \leq \frac{(n-2)^2}{4}\omega_n + \frac{\mathrm{const}}{\ln t}$$

which contradicts (3.3). □

Corollary 3.1. *Assume* (1.3), $Q_* > -\infty$ *and*

$$\limsup_{t\to\infty} \frac{1}{\ln t} \int_{1\le ||x||\le t} ||x||^{2-n} c(x)\,\mathrm{d}x > \frac{(n-2)^2}{4}\omega_n. \tag{3.4}$$

Then the equation (1.1) *is oscillatory.*

Proof. The definition of the function $Q(t)$ and integration by parts gives

$$\begin{aligned} t\Big(C_0 - C(t)\Big) &= Q(t) + t\int_{1\le ||x||\le t} ||x||^{1-n} c(x)\,\mathrm{d}x \\ &\quad - \int_1^t \int_{1\le ||x||\le T} ||x||^{1-n} c(x)\,\mathrm{d}x\ \mathrm{d}T \\ &= Q(t) + \int_1^t T \int_{||x||=T} ||x||^{1-n} c(x)\,\mathrm{d}S\ \mathrm{d}T \\ &= Q(t) + \int_{1\le ||x||\le t} ||x||^{2-n} c(x)\,\mathrm{d}x\,. \end{aligned}$$

Now the statement follows from Theorem 3.2. □

For $c(x) \ge 0$ Corollary 3.1 was proved in Müller–Pfeiffer [8].

Corollary 3.2. *Let* (1.3) *holds and*

$$\liminf_{t\to\infty} [Q(t) + H(t)] > \frac{(n-2)^2}{2}\omega_n. \tag{3.5}$$

Then the equation (1.1) *is oscillatory.*

Proof. Integration by parts gives

$$\begin{aligned} \int_1^t Q(s)\,\mathrm{d}s &= tQ(t) - Q(1) - \int_1^t sQ'(s)\,\mathrm{d}s \\ &= tQ(t) - C_0 - \int_1^t sC_0\,\mathrm{d}s + \int_1^t s\int_{1\le ||x||\le s} ||x||^{1-n} c(x)\,\mathrm{d}x\ \mathrm{d}s \\ &\quad + \int_1^t s^2 \int_{||x||=s} ||x||^{1-n} c(x)\,\mathrm{d}S\ \mathrm{d}s = \frac{1}{2}[tQ(t) + tH(t)] - \frac{1}{2}C_0. \end{aligned}$$

Hence

$$Q(t) + H(t) = \frac{2}{t}\int_1^t Q(s)\,\mathrm{d}s + \frac{C_0}{t} \tag{3.6}$$

holds. Now the equality

$$\begin{aligned} t\Big(C_0 - C(t)\Big) &= \int_1^t \frac{Q(s)}{s}\,\mathrm{d}s + C_0 \\ &= C_0 + \frac{1}{t}\int_1^t Q(s)\,\mathrm{d}s + \int_1^t \frac{1}{s^2}\int_1^s Q(u)\,\mathrm{d}u\ \mathrm{d}s \end{aligned} \tag{3.7}$$

and Theorem 3.2 implies oscillation of the equation (1.1). □

Corollary 3.3. *Let* (1.3) *holds. Each of the conditions*

$$Q_* > \frac{(n-2)^2}{4}\omega_n, \tag{3.8}$$

$$H_* > \frac{(n-2)^2}{4}\omega_n \tag{3.9}$$

guarantees the oscillation of the equation (1.1).

Proof. Follows immediately from Theorem 3.2 using equality (3.7) for the statement (3.8) and equalities

$$C(t) = C(\tau) + \int_\tau^t \frac{\ln s}{s^2}\frac{1}{\ln s}\int_{1\le||x||\le s} ||x||^{2-n}c(x)\,\mathrm{d}x\ \mathrm{d}s$$

$$\frac{1}{\ln t}\int_{1\le||x||\le t} ||x||^{2-n}c(x)\,\mathrm{d}x\ = \frac{H(t)}{\ln t} + \frac{1}{\ln t}\int_1^t \frac{1}{s}H(s)\,\mathrm{d}s\,,$$

which can be checked directly using integration by parts, for the statement (3.9). □

In the case $n=2$ the conditions (3.3) and (3.4) can be weakened as follows.

Theorem 3.3. *Let* $n=2$ *and* (1.3) *holds. Each of the conditions*

$$\limsup_{t\to\infty} t\Big(C_0 - C(t)\Big) = \infty \tag{3.10}$$

$$Q_* > -\infty \quad \textit{and} \quad \limsup_{t\to\infty}\int_{1\le||x||\le t} c(x)\,\mathrm{d}x\ = \infty \tag{3.11}$$

guarantees the oscillation of the equation (1.1).

Proof. Follows from the proofs of Theorem 3.2 and Corollary 3.1. □

The following theorem completes Corollary 3.2.

Theorem 3.4. *Let* (1.3) *and*

$$\limsup_{t\to\infty}\,[Q(t)+H(t)] > \frac{(n-2)^2+1}{2}\omega_n \tag{3.12}$$

holds. Then the equation (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has positive solution on Ω_a for some $a>1$. Inequalities (2.18) and (2.22) (see also Remark 2.1) implied

$$\begin{aligned} Q(T)+H(T) &\le \frac{(n-1)^2}{4}\omega_n + \frac{(3-n)^2}{4}\omega_n + \frac{a}{T}\big(a\rho(a)+H(a)\big) \\ &= \frac{(n-2)^2+1}{2}\omega_n + \frac{a}{T}\big(a\rho(a)+H(a)\big) \end{aligned}$$

which contradicts (3.12). □

The following theorems treat with the case when the numbers Q_*, H_* does not satisfy the bound in (3.8), (3.9). We discus the cases with

$$\frac{(n-2)^2-1}{4}\omega_n \le Q_* \le \frac{(n-2)^2}{4}\omega_n \tag{3.13}$$

and (or)

$$\frac{(n-2)^2-1}{4}\omega_n \le H_* \le \frac{(n-2)^2}{4}\omega_n. \tag{3.14}$$

Theorem 3.5. *If* (1.3), (3.13) *and*

$$H^* > \frac{\omega_n}{2}\left[\frac{(n-2)^2+1}{2} + \sqrt{(n-2)^2 - \frac{4Q_*}{\omega_n}}\right] \tag{3.15}$$

holds, then (1.1) *is oscillatory.*

Proof. Suppose, by contradiction, that there exists a solution of (1.1) positive on Ω_a for some $a > 1$, (1.3) and (3.13) holds. From the inequality (2.22) it follows

$$H^* \le -\liminf_{t\to\infty} t\rho(t) + \frac{(n-3)^2}{4}\omega_n$$

and using (2.15) we get a contradiction with (3.15). □

Theorem 3.6. *If* (1.3), (3.14) *holds and if*

$$Q^* > \frac{\omega_n}{2}\left[\frac{(n-2)^2+1}{2} + \sqrt{(n-2)^2 - \frac{4H_*}{\omega_n}}\right], \tag{3.16}$$

then the equation (1.1) *is oscillatory.*

Proof. Suppose, by contradiction, that there exists a solution of (1.1) positive on Ω_a for some $a > 1$, (1.3) and (3.14) holds. From (2.18) we get the inequality

$$Q^* \le \limsup_{t\to\infty} t\rho(t) + \frac{(n-1)^2}{4}\omega_n,$$

and (2.20) leads to the contradiction with (3.16). □

If both (3.13) and (3.14) holds, the constants in Theorems 3.4–3.6 can be decreased, as shows the following theorem.

Theorem 3.7. *Let* (1.3), (3.13) *and* (3.14) *holds. Denote*

$$k_n = \frac{\omega_n}{2}\left[\sqrt{(n-2)^2 - \frac{4Q_*}{\omega_n}} + \sqrt{(n-2)^2 - \frac{4H_*}{\omega_n}}\right].$$

Then each of the conditions

$$Q^* > Q_* + k_n \tag{3.17}$$

$$H^* > H_* + k_n \tag{3.18}$$

$$\limsup_{t\to\infty}\,[Q(t) + H(t)] > H_* + Q_* + k_n \tag{3.19}$$

guarantees the oscillation of the equation (1.1).

Proof. Suppose, by contradiction, that there exists a solution of (2.1) defined on Ω_a for some $a \geq 1$, (1.3), (3.13), (3.14) hold and denote

$$m = \frac{\omega_n}{2}\left[2 - n - \sqrt{(n-2)^2 - \frac{4Q_*}{\omega_n}}\right] \tag{3.20}$$

$$M = \frac{\omega_n}{2}\left[2 - n + \sqrt{(n-2)^2 - \frac{4H_*}{\omega_n}}\right]. \tag{3.21}$$

Assume that $Q_* > \frac{(n-2)^2-1}{4}\omega_n$ and $H_* > \frac{(n-2)^2-1}{4}\omega_n$ hold. From (3.20), (3.21) and from Lemmas 2.2 and 2.3 it follows that for every $0 < \varepsilon < 1 - \min\left\{\sqrt{(n-2)^2 - \frac{4Q*}{\omega_n}}, \sqrt{(n-2)^2 - \frac{4H*}{\omega_n}}\right\}$ there exists $t_\varepsilon \geq a$ such that

$$(1-n)\frac{\omega_n}{2} < m - \varepsilon \leq t\rho(t) \leq M + \varepsilon < (3-n)\frac{\omega_n}{2} \tag{3.22}$$

and from here

$$(1-n)t\rho(t) - \frac{1}{\omega_n}t^2\rho^2(t) \leq (1-n)(m-\varepsilon) - \frac{1}{\omega_n}(m-\varepsilon)^2 \tag{3.23}$$

$$(3-n)t\rho(t) - \frac{1}{\omega_n}t^2\rho^2(t) \leq (3-n)(M+\varepsilon) - \frac{1}{\omega_n}(M+\varepsilon)^2 \tag{3.24}$$

for every $t \geq t_\varepsilon$. Now (2.18) and (2.21) imply for $t \geq t_\varepsilon$

$$\begin{aligned}
Q(t) &\leq t\rho(t) + t\int_t^\infty \left[(1-n)s\rho(s) - \frac{1}{\omega_n}s^2\rho^2(s)\right]\frac{\mathrm{d}s}{s^2}\\
&\leq t\rho(t) + (1-n)(m-\varepsilon) - \frac{1}{\omega_n}(m-\varepsilon)^2,\\
H(t) &\leq -t\rho(t) + \frac{\tau}{t}\big(\tau\rho(\tau) + H(\tau)\big)\\
&\quad + \frac{1}{t}\int_{t_\varepsilon}^t \left[(3-n)s\rho(s) - \frac{1}{\omega_n}s^2\rho^2(s)\right]\mathrm{d}s \leq\\
&\leq -t\rho(t) + (3-n)(M+\varepsilon) - \frac{1}{\omega_n}(M+\varepsilon)^2 + \frac{\tau}{t}\big(\tau\rho(\tau) + H(\tau)\big),\\
Q(t) + H(t) &\leq (3-n)(M+\varepsilon) - \frac{1}{\omega_n}(M+\varepsilon)^2\\
&\quad + (1-n)(m-\varepsilon) - \frac{1}{\omega_n}(m-\varepsilon)^2 + \frac{\tau}{T}\big(\tau\rho(\tau) + H(\tau)\big).
\end{aligned}$$

These inequalities together with Lemmas 2.2 and 2.3 imply

$$\begin{aligned}
Q^* &\leq M + (1-n)m - \frac{1}{\omega_n}m^2,\\
H^* &\leq -m + (3-n)M - \frac{1}{\omega_n}M^2\\
\limsup_{t\to\infty}\,[Q(t) + H(t)] &\leq (3-n)M - \frac{1}{\omega_n}M^2 + (1-n)m - \frac{1}{\omega_n}m^2
\end{aligned} \tag{3.25}$$

respectively. Now direct computation leads to the contradiction with (3.17), (3.18), (3.19) respectively. If $Q_* = \frac{(n-2)^2-1}{4}\omega_n$ or $H_* = \frac{(n-2)^2-1}{4}\omega_n$ the statement (3.17), (3.18) follows from Theorems 3.6, 3.5, respectively. In this case $m = \frac{1-n}{2}\omega_n$, $M = \frac{3-n}{2}\omega_n$, respectively and using estimation

$$(1-n)t\rho(t) - \frac{1}{\omega_n}t^2\rho^2(t) \le \frac{(1-n)^2}{4}\omega_n = (1-n)m - \frac{1}{\omega_n}m^2,$$
$$(3-n)t\rho(t) - \frac{1}{\omega_n}t^2\rho^2(t) \le \frac{(3-n)^2}{4}\omega_n = (3-n)M - \frac{1}{\omega_n}M^2$$

instead of (3.23) and (3.24) respectively, we again conclude (3.25) which leads to the contradiction with (3.19). □

4 Remarks and comments

Remark 4.1. If the limit

$$\lim_{t\to\infty}\int_{1\le||x||\le t} ||x||^{1-n}c(x)\,\mathrm{d}x \tag{4.1}$$

exists, then the limit (1.3) exists too and both limits are equal. If the limit (4.1) is finite then $Q(t)$ takes the form

$$Q(t) = t\int_{t\le||x||\le\infty} ||x||^{1-n}c(x)\,\mathrm{d}x\,.$$

In this case the criteria in Corollary 3.3 are for $n = 1$ and $c \ge 0$ due to Hille [6] and Nehari [9]. If $\lim_{t\to\infty}\int_{1\le||x||\le t}||x||^{1-n}c(x)\,\mathrm{d}x = \infty$ then (1.1) is oscillatory by Theorem 3.1. Remark that the existence of (4.1) is not necessary for existence of the limit (1.3).

Remark 4.2. An example of the equation

$$\Delta u + \frac{(n-2)^2}{4||x||^2}u = 0$$

which possesses positive solution $||x||^{\frac{2-n}{2}}$ shows that the right–hand sides in (3.3)–(3.5), (3.8),(3.9) and (3.17)–(3.19) cannot be decreased. This example also shows that the condition

$$\lim_{t\to\infty}\int_{1\le||x||\le t} c(x)\,\mathrm{d}x = \infty$$

is not sufficient for oscillation of the equation (1.1) as in the one-dimensional case, as it is mentioned in [11]. In fact the above condition guarantees oscillation of the equation (1.1) for $n \le 2$, see [10, 11].

Remark 4.3. As it can be seen from the Definition 1.1, the above oscillation criteria give the sufficient condition for nonexistence of nonoscillatory solution, but does not guarantee existence of oscillatory solution. Nevertheless in the case when the function $c(x)$ is Hölder continuous, which is often studied in the literature, our criteria give the sufficient conditions for existence of oscillatory solution and also the sufficient condition for nodal oscillation of the equation (1.1).

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