On $p-$Degree Functionals With One Free End Point

ROBERT MAŘÍK

Department of Mathematics, Faculty of Science, Masaryk University
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
E-mail: marik@math.muni.cz

Abstract. We study the scalar functionals corresponding to the second order half-linear differential equation as an extension of the quadratic functionals. We follow the work [7] concerning the functionals with zero boundary conditions and give an analogy of the Reid's Roundabout theorem for the functional with one free end point. For brevity we focus ourselves only on the regular functionals.

AMS Subject Classification. 34C10, 49K15.

Keywords. Half–linear differential equation, associated functional, Picone identity, focal points, coupled points.

1 Introduction

Let us consider the second order half-linear differential equation

$$\text{(E)} \qquad \Bigl(r(t)\Phi(y')\Bigr)' + q(t)\Phi(y) = 0,$$

where $\Phi(y) = y|y|^{p-2}$, $p > 1$ is a constant and $r(t)$, $q(t)$ are real-valued continuous functions defined on a given non-degenerate compact interval I, $r(t) > 0$ on I. The domain of the operator on the left hand side is defined to be the set of all continuous real-valued functions y defined on I such that y and $r\Phi(y')$ are continuously differentiable on I.

In [7] was studied the functional corresponding to the equation (E) and was introduced the necessary and sufficient condition for nonnegativity of this functional, as a natural generalization of quadratic functionals.

Supported by the grant No. 201/96/0410 of the Czech Grant Agency and grant 758/98 of Development of Czech Universities.

The aim of this paper is to continue this study and give a necessary and sufficient condition for positivity (nonnegativity) of the functional with one free end point

$$\mathcal{L}(\eta; a, b) = \alpha|\eta(a)|^p + \int_a^b \Big[r(t)|\eta'(t)|^p - q(t)|\eta(t)|^p \Big] \,\mathrm{d}t$$

on the class of functions $\eta \in U_{*0}$ such that $\eta \in AC[a,b]$, $\eta'(t) \in L^p[a,b]$ with boundary condition $\eta(b) = 0$. Any function of class U_{*0} is said to be an *admissible function.*

We will show that method used for quadratic functional case in [2] can be applied to the $p-$degree functional $\mathcal{L}(\eta; a, b)$.

We use the concept of Riccati equation and focal points. If $y(t)$ is solution of (E) which has no zero on I, then the function $w(t) = r(t)\Phi(y'(t)/y(t))$ is defined on I and satisfies the *Riccati-type equation*

$$w'(t) + q(t) + (p-1)r(t)^{1-k}|w(t)|^k = 0 \tag{1}$$

where $k > 0$ is such that $1/p + 1/k = 1$ holds. This fact can be verified by direct computation. The function $w(t)$ is said to be solution of (1) associated with solution $y(t)$ of equation (E). Conversely, if $w(t)$ is defined on I, then the associated solution $y(t)$ of equation (E) is given up to constant multiple and has no zero on I.

A point $c \in (a, b]$ is said to be *focal point* to a (relative to functional $\mathcal{L}(\eta; a, b)$) if the nontrivial solution $y(t)$ of (E) satisfying

$$\alpha\Phi(y(a)) - r(a)\Phi(y'(a)) = 0 \tag{2}$$

has zero c. This solution is given uniquely up to the constant multiple.

As in [7], our main tool is modified Picone identity from Jaroš–Kusano [6]:

Lemma A. *Let η be AC function on I, $y(t)$ be solution of* (E) *which has no zero on I and $w(t)$ be corresponding solution of* (1). *It holds*

$$r(t)|\eta'(t)|^p - q|\eta(t)|^p = \Big(w(t)|\eta(t)|^p\Big)' + P_p(\eta, w, r, t) \tag{3}$$

for all $t \in I$ for which $\eta'(t)$ exists, where

$$P_p(\eta, w, r, t) = r(t)|\eta'(t)|^p + (p-1)r^{1-k}(t)|w(t)|^k|\eta(t)|^p - pw(t)\Phi(\eta(t))\eta'(t).$$

The function $P_p(\eta, w, r, t)$ satisfies $P_p(\eta, w, r, t) \ge 0$ and $P_p(\eta, w, r, t) = 0$ if and only if η is constant multiple of y.

If we integrate the Picone identity (3) from a to b and the term $w(a+)|\eta(a+)|^p$, resp. $w(b-)|\eta(b-)|^p$ is of the type "$\infty \cdot 0$" we use the following lemma, which is due to Elbert [5].

Lemma B [5]. *Let $\eta \in AC[a,b]$, $\eta' \in L^p[a,b]$ and let w be solution of* (1) *defined on (a,b). If $\eta(a) = 0$, resp. $\eta(b) = 0$, then it holds $\lim_{t\to a^+} w(t)|\eta(t)|^p = 0$ resp. $\lim_{t\to b^-} w(t)|\eta(t)|^p = 0$.*

2 The functional with one free end point

First we give a condition for nonnegativity of $\mathcal{L}(\eta; a, b)$ in terms of Riccati–equation solutions. Denote $y_b(t)$ the solution of (E) which satisfies initial conditions $y_b(b) = 0$, $y_b'(b) = 1$ and $w_b(t)$ the corresponding solution of Riccati equation.

Lemma 1. *Functional $\mathcal{L}(\eta; a, b)$ is nonnegative on the class of functions U_{*0} if and only if $w_b(t)$ is defined on $[a, b)$ and satisfies $\alpha - w_b(a) \geq 0$.*

Proof. " $\Rightarrow$ ": Suppose that $w_b(t)$ is not defined in a point $c \in [a, b)$ and it is defined on (c, b), this means that $y_b(c) = 0$ and $y(t) \neq 0$ for every $t \in (c, b)$. Let $\lambda \in (c, b)$ be a real number and define the admissible function

$$\eta_\lambda(t) = \begin{cases} y_b(\lambda) & t \in [a, \lambda] \\ y_b(t) & t \in [\lambda, b], \end{cases}$$

From definition of the function $\eta_\lambda(t)$ it follows

$$\begin{aligned} \mathcal{L}(\eta_\lambda; a, b) &= \alpha|\eta_\lambda(a)|^p + \int_a^\lambda \Big[r(t)|\eta_\lambda'(t)|^p - q(t)|\eta_\lambda(t)|^p\Big]\,\mathrm{d}t + \\ &+ \int_\lambda^b \Big[r(t)|y_b'(t)|^p - q(t)|y_b(t)|^p\Big]\,\mathrm{d}t = |y_b(\lambda)|^p\Big(\alpha - \int_a^\lambda q(t)\,\mathrm{d}t\Big) + r(t)\Phi\big(y_b'(t)\big)y_b(t)\Big|_\lambda^b - \\ &- \int_\lambda^b y_b(t)\Big[\Big(r(t)\Phi\big(y_b'(t)\big)\Big)' + q(t)\Phi\big(y_b(t)\big)\Big]\,\mathrm{d}t = \\ &= |y_b(\lambda)|^p\Big(\alpha - r(\lambda)\Phi(y_b'(\lambda)/y_b(\lambda)) - \int_a^\lambda q(t)\,\mathrm{d}t\Big), \end{aligned}$$

where the second integral was computed using integration by parts. If $\lambda \to c^+$ then the expression in parenthesis tends to $-\infty$ and clearly there exists $\lambda_0 \in (c, b)$ such that $\mathcal{L}(\eta_{\lambda_0}; a, b) < 0$, a contradiction. This means that $w_b(t)$ is defined on $[a, b)$. The function y_b is admissible and a similar computation as above gives $0 \leq \mathcal{L}(y_b; a, b) = |y_b(a)|^p\Big(\alpha - w_b(a)\Big)$. It holds $y_b(a) \neq 0$, hence $\alpha - w_b(a) \geq 0$.

" $\Leftarrow$ ": Let $w_b(t)$ be defined on $[a, b)$, $\alpha - w_b(a) \geq 0$ and η be an admissible function. Integrating Picone's identity from a to $b - \varepsilon$, letting $\varepsilon \to 0+$ and using Lemma B we get $\mathcal{L}(\eta; a, b) = |\eta(a)|^p\Big(\alpha - w_b(a)\Big) + \int_a^b P_p(\eta, w_b, r, t)\,\mathrm{d}t \geq 0$ and hence the functional is nonnegative. $\square$

Next theorem is an analogue of the well known Reid's Roundabout theorem.

Theorem 1. *The following statements are equivalent.*

(i) $\mathcal{L}(\eta; a, b)$ *is nonnegative on* U_{*0}.

(ii) *There exists no focal point to* a *on* (a, b).

(iii) *The solution of the Riccati equation satisfying* $w(a) = \alpha$ *is defined on* $[a, b)$.

(iv) *The solution of the Riccati equation* $w_b(t)$ *is defined on* $[a, b)$ *and satisfies* $\alpha - w_b(a) \geq 0$.

Proof. "(i) $\Rightarrow$ (ii)" Let $c \in (a, b)$ be the first focal point to a. Then there exists a nontrivial solution of (E) such that $\alpha\Phi(y(a)) - r(a)\Phi(y'(a)) = 0$, $y(c) = 0$, and $y(t) \neq 0$ for $t \in (a, c)$. Let $\lambda \in (a, c)$ be a real number and $\eta_\lambda(t)$ be an admissible function defined by

$$\eta_\lambda(t) = \begin{cases} y(t) & t \in [a, \lambda] \\ \frac{y(\lambda)}{b-\lambda}(b - t) & t \in [\lambda, b]. \end{cases}$$

Then

$$\begin{aligned}\mathcal{L}(\eta_\lambda; a, b) = \alpha|\eta_\lambda(a)|^p + \int_a^\lambda \big(r(t)|y'(t)|^p - q(t)|y(t)|^p\big)\,\mathrm{d}t + \\ + \Big|\frac{y(\lambda)}{b-\lambda}\Big|^p \int_\lambda^b \big(r(t) - q(t)|b-t|^p\big)\,\mathrm{d}t = \\ = |y(\lambda)|^p \Big[r(\lambda)\Phi\big(y'(\lambda)/y(\lambda)\big) + \frac{1}{|b-\lambda|^p}\int_\lambda^b \big(r(t) - q(t)|b-t|^p\big)\,\mathrm{d}t\Big].\end{aligned}$$

The first term in the parenthesis tends to $-\infty$ if λ tends to c from left and the second one is bounded. Hence there exists λ_0 such that $\mathcal{L}(\eta_{\lambda_0}; a, b) < 0$, a contradiction.

"(ii) $\Rightarrow$ (iii)" Follows immediately from the definition of focal point.

"(iii) $\Rightarrow$ (i)" If a solution of (1) satisfying $w(a) = \alpha$ is defined on $[a, b)$, then integrating Picone's identity from a to $b - \varepsilon$, letting $\varepsilon \to 0$ and using Lemma B we have $\mathcal{L}(\eta; a, b) = \int_a^b P_p(\eta, w, r, t)\,\mathrm{d}t \geq 0$ and the functional is nonnegative.

"(i) $\Leftrightarrow$ (iv)" See Lemma 1. $\square$

Theorem 1'. *The following statements are equivalent.*

(i) $\mathcal{L}(\eta; a, b)$ *is positive definite on* U_{*0}.

(ii) *There exists no focal point to* a *on* $(a, b]$.

(iii) *The solution of the Riccati equation satisfying* $w(a) = \alpha$ *is defined on* $[a, b]$.

(iv) *The solution of the Riccati equation* $w_b(t)$ *is defined on* $[a, b)$ *and satisfies* $\alpha - w_b(a) > 0$.

Sketch of the proof. "(i) $\Rightarrow$ (ii)" Let $c \in (a, b]$ be a focal point to a, and $y(t)$ be the solution of (E) which realizes this focal point. Let η be an admissible function, which is equal to $y(t)$ on $[a, c]$ and zero on $[c, b]$. Then integrating by parts we have $\mathcal{L}(\eta; a, b) = 0$, but $\eta \not\equiv 0$, a contradiction.

"(ii) $\Rightarrow$ (iii)" Follows immediately from the definition of focal point.

"(iii) $\Rightarrow$ (i)" Follows from the Picone identity and from properties of the function P_p.

"(i) $\Leftrightarrow$ (iv)" Analogously to Lemma 1. $\square$

3 Comparison theorem

As an application of the previous results we derive a comparison theorem for the half-linear differential equation. Let us consider two differential equations

$$\text{(A)} \qquad \Big(r(t)\Phi(x')\Big)' + q(t)\Phi(x) = 0$$

$$\text{(B)} \qquad \Big(R(t)\Phi(y')\Big)' + Q(t)\Phi(y) = 0,$$

where $r(t)$, $q(t)$, $R(t)$ and $Q(t)$ are continuous on $[a, b]$, $R(t)$ and $r(t)$ are positive and the inequalities

$$R(t) \le r(t) \text{ and } q(t) \le Q(t) \text{ for } t \in [a, b]$$

are satisfied. Jaroš and Kusano proved in [6] the following comparison theorem.

Theorem C [6]. *If there exists solution $x(t)$ of* (A) *such that $x(a) = 0 = x(b)$ and $x(t) \ne 0$ for all $t \in (a, b)$, then any solution $y(t)$ of equation* (B) *either has a zero on (a, b), or it is a constant multiple of x.*

This theorem describes behavior of the solution of the equation (B) in the interval between two zeros of the solution of the equation (A). For the case $x(a) \ne 0$ this statement can be extend using Theorems 1 and 1' and we get the following comparison theorem.

Theorem 2. *Let $x(t)$, $y(t)$ be solutions of* (A), (B), *$x(b) = 0$, $x(t) \ne 0$ for $t \in [a, b)$. If*

$$\text{(4)} \qquad R(a)\Phi\Big(\frac{y'(a)}{y(a)}\Big) \le r(a)\Phi\Big(\frac{x'(a)}{x(a)}\Big),$$

then the function $y(t)$ has a zero for some $t \in (a, b]$. Moreover, $y(t)$ has a zero on (a, b) in the case when any of the following conditions is satisfied:

(i) *the sharp inequality in* (4) *is satisfied, or*

(ii) *$q(\cdot) \not\equiv Q(\cdot)$ on (a, b), or*

(iii) *$R(t_0) < r(t_0)$ and $q(t_0) \ne 0$ in some $t_0 \in (a, b)$.*

Proof. From (4) it follows $y(a) \ne 0$. Denote

$$\alpha = r(a)\Phi\Big(\frac{x'(a)}{x(a)}\Big), \quad \mathcal{L}_\alpha(\eta; a, b) = \alpha|\eta(a)|^p + \int_a^b (r|\eta'|^p - q|\eta|^p)\,\mathrm{d}t$$

$$\beta = R(a)\Phi\Big(\frac{y'(a)}{y(a)}\Big), \quad \mathcal{L}_\beta(\eta; a, b) = \beta|\eta(a)|^p + \int_a^b (R|\eta'|^p - Q|\eta|^p)\,\mathrm{d}t.$$

Integration by parts shows $\mathcal{L}_\alpha(x;a,b)=0$. Hence

$$\mathcal{L}_\beta(x;a,b)=\mathcal{L}_\beta(x;a,b)-\mathcal{L}_\alpha(x;a,b)=$$
$$=(\beta-\alpha)|x(a)|^p+\int_a^b\Big(q(t)-Q(t)\Big)|x(t)|^p\,\mathrm{d}t+\int_a^b\Big(R(t)-r(t)\Big)|x'(t)|^p\,\mathrm{d}t\le 0.$$

Each of the last three terms is nonnpositive, hence the functional $\mathcal{L}_\beta(\eta;a,b)$ is not positive definite and by Theorem 1' there exists a focal point to a relative to the functional $\mathcal{L}_\beta(\eta;a,b)$ on $(a,b]$, or equivalently, $y(t)$ has a zero on $(a,b]$. Under the conditions (i), (ii) or (iii), the first, second or third term is negative, respectively. Hence the functional is not nonnegative and by Theorem 1 there exists a focal point to a on (a,b), i.e. $y(t)$ has a zero on (a,b). $\square$

Remark. In the literature there are several results devoted to the comparison theorems for nonlinear differential equations, which include half-linear differential equation (E). We refer to the paper Agarwal et al. [1] and references therein.

Remark that under certain special assumptions on functions r and q the first part of the conclusion of Theorem 2 can be obtained from [1, Theorem 2.1].

Acknowledgement. I would like to thank to Professor Zuzana Došlá for valuable help and many remarks, which contributed to the final form of this paper.

References

1. R. P. Agarwal, W.-C. Lian and C.-C. Yeh, *Levin's comparison theorem for nonlinear second order differential equations*, Appl. Math. Lett. **9** (1996), 29-35.
2. Z. Došlá, O. Došlý, *Singular quadratic functionals of one dependent variable*, Comment. Math. Univ. Carolinae **36** (1995), 219–237.
3. Z. Došlá, O. Došlý, *Quadratic functionals with general boundary conditions*, Appl. Math. Opt. **36** (1997), 243–262.
4. Z. Došlá, P. Zezza, *Conjugate points in the theory of variations and optimal control theory via the quadratic form theory.*, Diff. Eq. and Dyn. Syst. **2** (1994), 137–152.
5. Á. Elbert, *A half-linear second order differential equation*, Coll. Math. Soc. János Bolyai 30. Qual. theory of diff. eq. Szeged (Hungary) (1979), 153–179.
6. J. Jaroš, T. Kusano, *A Picone type identity for second order half-linear differential equations* (to appear).
7. R. Mařík, *Nonnegativity of functionals corresponding to the second order half-linear differential equation* (to appear).