Focal points of half–linear second order differential equations

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Abstract

In the paper the solutions of the half-linear differential equation

$$\Big(r(t)\Phi(x')\Big)' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1$$

with $x(a) \neq 0 = x(b)$ are studied. The equation is compared with another half–linear differential equation and with nonhomogenous equation having nonpositive right hand side. The presented results extend several theorems from the theory of linear differential equations to the half–linear differential equations.

1 Introduction

The aim of this paper is to study focal points of the half–linear differential equation

$$l[x] \equiv \Big(r(t)\Phi(x')\Big)' + c(t)\Phi(x) = 0 \tag{1.1}$$

where $\Phi(x) = x|x|^{p-2}$, $p > 1$, is a real constant and $r(t)$, $c(t)$ are real-valued continuous functions defined on the interval $I = [a, \infty)$, $r(t) > 0$ on I.

The domain of the operator l is defined to be the set of all continuous real-valued functions x defined on I such that x and $r\Phi(x')$ are continuously differentiable on I. The existence and uniqueness of solution and basic properties of the Eq. (1.1) were established in [4, 9].

Remark that if $p = 2$, then Eq. (1.1) is the well-known linear Sturm–Liouville second order differential equation.

New comparison theorems for the half–linear equations based on the existence of focal points are established in Section 2. These results extend the comparison theorems of Leighton and Reid known in the linear case. In the Section 3 the relationship between homogeneous Eq. (1.1) and the nonhomogenous equation

$$l[y] \equiv \Big(r(t)\Phi(y')\Big)' + c(t)\Phi(y) = f(t), \quad f(t) \leq 0 \text{ on } [a, \infty) \tag{1.2}$$

is studied. These investigations are motivated by the paper Navarro and Sarabia [10], where the linear equation is considered. The main tool in the proofs are Riccati technique and comparison theorems from Section 2, since the tool from [10] has no extension in the half–linear case.

First let us recall the main ideas of the variational technique for the Eq. (1.1), i.e. the relationship between Eq. (1.1) and the corresponding $p-$degree functional

$$\mathscr{J}(\eta; a, b) = \alpha|\eta(a)|^p + \beta|\eta(b)|^p + \int_a^b \Big(r(t)|\eta'(t)|^p - c(t)|\eta(t)|^p \Big)\, \mathrm{d}t\,, \tag{1.3}$$

which was studied in [7].

The equivalence between nonnegativity (positivity) of functional (1.3) and nonexistence of conjugate (focal, coupled) point is a powerful tool in the comparison and oscillation theory. See [5], where the Sturm-Picone and Leighton type comparison theorems for (1.1) are derived and [1, 2], where the variational technique is used to establish oscillation criteria for (1.1).

Definition 1.1. A point $d \in (a, b]$ is said to be the *focal point* to the point a *relative to the functional* (1.3) if there exists a nontrivial solution $y(t)$ of the Eq. (1.1) satisfying

$$\alpha\Phi(y(a)) - r(a)\Phi(y'(a)) = 0 \tag{1.4}$$

and $y(d) = 0$.

A point $b \in (a, \infty)$ is said to be the *focal point* to the point a if there exists a nontrivial solution of the Eq. (1.1) such that $x'(a) = 0 = x(b)$.

The focal point $b \in (a, \infty)$ is said to be *the first focal point* to the point a if the interval (a, b) does not contain a focal point to the point a.

For the focal point criteria for the Eq. (1.1) see [3]. The connection between functional (1.3) and Eq. (1.1) is expressed by the following lemma and presents the main tool used in Section 2.

Lemma 1.1 ([7])**.** *The functional* (1.3) *is nonnegative over the class of functions*

$$U = \{\eta \in AC[a, b] : \eta' \in L^p[a, b], \eta(b) = 0, \eta \not\equiv 0\}$$

if and only if the interval (a, b) *does not contain a focal point to the point* a *relative to the functional* (1.1)*, i.e. the solution which satisfies* (1.4) *has no zero on* (a, b)*.*

Remark 1.1. A closer examination of the proof of Lemma 1.1 shows that the lemma holds *verbatim et literatim* if the words nonnegative and (a, b) are replaced by positive and $(a, b]$, respectively.

2 Comparison theorems

Let us consider together with Eq. (1.1) the equation

$$L[y] \equiv \Big(R(t)\Phi(y')\Big)' + C(t)\Phi(y) = 0, \tag{2.1}$$

where $R(t)$ and $C(t)$ are continuous on $[a, b]$, $R(t)$ is a positive function and the inequality

$$R(t) \le r(t) \text{ for } t \in [a, b] \tag{2.2}$$

is satisfied. If, in addition,

$$c(t) \le C(t) \text{ for } t \in [a, b] \tag{2.3}$$

is satisfied, then (2.1) is called the *Sturmian majorant* of (1.1). Elbert [4] and Jaroš and Kusano [5] proved that under conditions (2.2) and (2.3) either between two zeros of the solution of Eq. (1.1) there exists zero of the solution of Eq. (2.1) or both solutions are constant multiples. This statement is a corollary of the following statement.

Theorem A ([5])**.** *Let x be a solution of* (1.1) *such that $x(a) = 0 = x(b)$, $x(t) \neq 0$ for every $t \in (a,b)$. If*

$$\int_a^b \Big[\Big(r(t) - R(t)\Big)|x'(t)|^p + \Big(C(t) - c(t)\Big)|x(t)|^p\Big]\,\mathrm{d}t \geq 0,$$

then every solution y of (2.1) *has zero in (a,b), or the Eq.* (1.1) *and* (2.1) *are identical and y is a constant multiple of x. If, in addition, strict inequality is satisfied, then y has a zero in (a,b). If $r, R \in C^1((a,b);(0,\infty))$ and*

$$\int_a^b \Big[\Big(C(t) - \frac{R(t)}{r(t)}c(t)\Big)|x(t)|^p + r(t)\Big(\frac{R(t)}{r(t)}\Big)' x(t)\Phi\big(x'(t)\big)\Big]\,\mathrm{d}t > 0,$$

then every solution of (2.1) *has a zero on (a,b).*

Remark that these results are usually referred as Leighton type, since in the linear case they are due to W. Leighton. Theorem A can be extended in a natural way to the case when $x(a) \neq 0 = x(b)$. This is the contents of the following theorem and its corollaries.

Theorem 2.1. *Let x be solution of* (1.1) *such that $x(b) = 0 \neq x(a)$, y be a solution of* (2.1) *such that $y(a) \neq 0$. Denote*

$$\alpha = r(a)\Phi\Big(\frac{x'(a)}{x(a)}\Big), \qquad A = R(a)\Phi\Big(\frac{y'(a)}{y(a)}\Big).$$

If

$$V[x] \equiv (\alpha - A)|x(a)|^p + \int_a^b \Big[\Big(r(t) - R(t)\Big)|x'(t)|^p + \Big(C(t) - c(t)\Big)|x(t)|^p\Big]\,\mathrm{d}t \geq 0,$$

then y has a zero on $(a,b]$. If, in addition, the sharp inequality is satisfied, then y has a zero on (a,b).

Proof. Denote

$$\mathscr{I}_\alpha(\eta; a, b) = \alpha|\eta(a)|^p + \int_a^b (r|\eta'|^p - c|\eta|^p)\,\mathrm{d}t$$
$$\mathscr{I}_A(\eta; a, b) = A|\eta(a)|^p + \int_a^b (R|\eta'|^p - C|\eta|^p)\,\mathrm{d}t\,.$$

The integration by parts yields that $\mathscr{I}_\alpha(x; a, b) = 0$ and hence

$$\mathscr{I}_A(x; a, b) = \mathscr{I}_A(x; a, b) - \mathscr{I}_\alpha(x; a, b) = -V[x] \leq 0 \quad (< 0).$$

Now the conclusion follows from Lemma 1.1 and the remark below this lemma. □

Corollary 2.1 ([8])**.** *Suppose* (2.2) *and* (2.3). *Let x, y be solutions of* (1.1), (2.1) *from Theorem 2.1. If*

$$A \leq \alpha, \tag{2.4}$$

then y has a zero on $(a,b]$. Moreover, y has a zero on (a,b) in the case when any of the following conditions is satisfied:

(i) *the sharp inequality in* (2.4) *is satisfied, or*
(ii) *$c(\cdot) \not\equiv C(\cdot)$ on (a,b), or*
(iii) *$R(t_0) < r(t_0)$ and $c(t_0) \neq 0$ in some $t_0 \in (a,b)$.*

Proof. Follows immediately from Theorem 2.1. □

Under additional condition on nonnegativity of the function $c(x)$ the inequalities (2.3) and (2.4) can be replaced by more general integral inequality, as shows the following corollary. In the linear case is this corollary due to Leighton [6, Theorem 1].

Corollary 2.2. *Suppose* (2.2), $\alpha \le 0$ *and* $c(t) \ge 0$ *for* $t \in [a,b]$. *Let* x, y *be solutions of* (1.1), (2.1) *from Theorem 2.1. If*

$$\alpha - A + \int_a^t \Big[C(s) - c(s)\Big]\,\mathrm{d}s \ge 0 \quad \textit{for every } t \in [a,b] \tag{2.5}$$

then the function $y(t)$ *has a zero for some* $t \in (a,b]$. *Moreover,* $y(t)$ *has a zero on* (a,b) *if the sharp inequality in* (2.5) *is satisfied, or if the condition* (iii) *of Corollary 2.1 holds.*

Proof. Without loos of generality we can consider $x(t) > 0$ for $t \in [a,b)$. The following integral can be written as a double integral

$$\int_a^b \Big[C(t) - c(t)\Big]|x(t)|^p\,\mathrm{d}t = \int_a^b \int_0^{|x(t)|^p} \Big[C(t) - c(t)\Big]\,\mathrm{d}s\,\mathrm{d}t\,.$$

From (1.1) it follows

$$r(t)\Phi(x'(t)) = \alpha\Phi(x(a)) - \int_a^t c(s)\Phi(x(s))\,\mathrm{d}s \qquad t \in [a,b)$$

and with regard to $\alpha \le 0$ and $c(t) \ge 0$ the function $x(t)$ is decreasing or it is constant in some interval $[a,d] \subseteq [a,b)$ and decreasing on $[d,b]$. The same is true also for $|x(t)|^p$. Then we can interchange the order of the integration and we obtain

$$\begin{aligned}\int_a^b \Big[C(t) - c(t)\Big]|x(t)|^p\,\mathrm{d}t &= \int_0^{|x(a)|^p} \int_0^{\varphi(s)} \Big[C(t) - c(t)\Big]\,\mathrm{d}t\,\mathrm{d}s \\ &\ge \int_0^{|x(a)|^p} [A - \alpha]\,\mathrm{d}s = [A - \alpha]|x(a)|^p.\end{aligned}$$

Note that $\varphi(s)$ is well-defined and at least continuous for $0 \le s \le |x(a)|^p$. By this computation we get

$$V[x] \ge \int_a^b \Big[r(t) - R(t)\Big]|x'(t)|^p\,\mathrm{d}t \ge 0,$$

and by Theorem 2.1 y has a zero on $(a,b]$. If the strict inequality is satisfied, then $V[x] > 0$ and y has a zero on (a,b). ☐

The next comparison theorem is an alternative to Theorem 2.1. It can be considered as a supplement to the second inequality in Theorem A.

Theorem 2.2. *Let* $r, R \in C^1([a,b];(0,\infty))$, $\alpha \le 0$, x, y *be solutions of* (1.1), (2.1) *from Theorem 2.1. If*

$$\begin{aligned}\mathcal{V}[x] \equiv \Big(\frac{R(a)}{r(a)}\alpha - A\Big)|x(a)|^p + \\ + \int_a^b \Big[\Big(C(t) - \frac{R(t)}{r(t)}c(t)\Big)|x(t)|^p + r(t)\Big(\frac{R(t)}{r(t)}\Big)' x(t)\Phi\big(x'(t)\big)\Big]\,\mathrm{d}t \ge 0,\end{aligned}$$

then y *has a zero on* $(a,b]$. *If in addition the sharp inequality holds, then* y *has zero on* (a,b).

Proof. Without loos of generality suppose that $x(t) > 0$ for $t \in [a,b)$. Then

$$\begin{aligned}L[x] = \Big(\frac{R}{r}r\Phi(x')\Big)' + C\Phi(x) &= \Big(\frac{R}{r}\Big)' r\Phi(x') + \frac{R}{r}\Big(r\Phi(x')\Big)' + C\Phi(x) \\ &= \Big[C - \frac{R}{r}c\Big]\Phi(x) + \Big(\frac{R}{r}\Big)' r\Phi(x')\end{aligned} \tag{2.6}$$

and since integration by parts shows that

$$\int_a^b \Big(\frac{R}{r}\Big)' r\Phi(x')x \,\mathrm{d}t \;=\; \big[R\Phi(x')x\big]_a^b - \int_a^b \Big[\frac{R}{r}x\Big(r\Phi(x')\Big)' + \frac{R}{r}r\Phi(x')x'\Big]\,\mathrm{d}t$$
$$= -R(a)\Phi\big(x'(a)\big)x(a) - \int_a^b \Big[R|x'|^p - \frac{R}{r}c|x|^p\Big]\,\mathrm{d}t\,,$$

it holds

$$\int_a^b xL[x]\,\mathrm{d}t \;=\; -\frac{R(a)}{r(a)}\alpha|x(a)|^p - \int_a^b \Big[R|x'|^p - C|x|^p\Big]\,\mathrm{d}t\,.$$

Hence

$$\mathscr{J}_A(x;a,b) = \Big(A - \frac{R(a)}{r(a)}\alpha\Big)|x(a)|^p + \int_a^b \Big[R|x'|^p - C|x|^p\Big]\,\mathrm{d}t \,+\, \frac{R(a)}{r(a)}\alpha|x(a)|^p$$
$$= \Big(A - \frac{R(a)}{r(a)}\alpha\Big)|x(a)|^p - \int_a^b xL[x]\,\mathrm{d}t \qquad (2.7)$$

and by (2.6)

$$\mathscr{J}_A(x;a,b) = \Big(A - \frac{R(a)}{r(a)}\alpha\Big)|x(a)|^p - \int_a^b \Big[\big(C - \frac{R}{r}c\big)\Phi(x) + \Big(\frac{R}{r}\Big)' r\Phi(x')\Big]\,\mathrm{d}t \;=\; -\mathcal{V}[x]$$

Now the statement follows from Lemma 1.1. □

The following corollary is of Reid's type, see [11, p. 29] and follows immediately from Theorem 2.2.

Corollary 2.3. *Let* $r, R \in C^1([a,b];(0,\infty))$, $c(t) \ge 0$ *for* $t \in [a,b]$. *Further let* x *and* y *be solutions of* (1.1) *and* (2.1), *respectively, such that* $x'(a) \le 0 = x(b)$, x *is positive on* $[a,b)$, $y(a) \ne 0$ *and* $\frac{x'(a)}{x(a)} \ge \frac{y'(a)}{y(a)}$. *If*

$$\Big(\frac{R(t)}{r(t)}\Big)' \le 0 \quad \text{and} \quad \frac{C(t)}{R(t)} \ge \frac{c(t)}{r(t)} \quad \text{for } t \in [a,b],$$

then y *has a zero on* $(a,b]$.

If we apply the method from the proof of Corollary 2.2, we get the following

Corollary 2.4. *Let* $r, R \in C^1([a,b];(0,\infty))$, $c(t) \ge 0$ *for* $t \in [a,b]$. *Further let* x *and* y *be solutions of* (1.1) *and* (2.1), *respectively, such that* $x'(a) \le 0 = x(b)$, x *is positive on* $[a,b)$, $y(a) \ne 0$. *If* $\Big(\frac{R(t)}{r(t)}\Big)' \le 0$ *and*

$$\frac{R(a)}{r(a)}\alpha - A + \int_a^t \Big[C(s) - \frac{R(s)}{r(s)}c(s)\Big]\,\mathrm{d}s \;\ge 0 \quad \text{for every } t \in [a,b],$$

then y *has a zero on* $(a,b]$.

The same method as in Theorem 2.1 can be applied to the case of the functional (1.3) defined on the class of functions with $x(a) = 0 \ne x(b)$. This is sketched in the following theorem.

Theorem 2.3. *Let* x *and* y *be solutions of* (1.1) *and* (2.1), *respectively, such that* $x(a) = 0 \ne x(b)$, $y(b) \ne 0$. *Denote*

$$\beta = -r(b)\Phi\Big(\frac{x'(b)}{x(b)}\Big), \qquad B = -R(b)\Phi\Big(\frac{y'(b)}{y(b)}\Big).$$

If

$$V_b[x] \equiv (\beta - B)|x(b)|^p + \int_a^b \Big[\Big(r(t) - R(t)\Big)|x'(t)|^p + \Big(C(t) - c(t)\Big)|x(t)|^p\Big]\,\mathrm{d}t \;\ge 0,$$

then y *has a zero on* $[a,b)$. *If in addition the sharp inequality is satisfied, then* y *has a zero on* (a,b).

Proof. Theorem follows from the fact that

$$\mathscr{J}_B(x;a,b) \equiv B|x(b)|^p + \int_a^b \Big[R(t)|x'(t)|^p - C(t)|x(t)|^p\Big]\,\mathrm{d}t \;= -V_b[x] \le 0$$

and from [7, Theorem 1] □

Corollary 2.5. *Suppose that* (2.2) *and* $c(t) \ge 0$ *for* $t \in (a,b)$ *holds. Let* x *and* y *be solutions of* (1.1) *and* (2.1)*, respectively,* $x'(b) \ge 0 = x(a)$*,* x *positive on* $[a,b)$*,* $y(b) \ne 0$*. Let* β*,* B *be the numbers from the preceding theorem. If*

$$\beta - B + \int_t^b \Big[C(s) - c(s)\Big]\,\mathrm{d}s \;\ge 0 \quad \textit{for every } t \in [a,b], \tag{2.8}$$

then the function $y(t)$ *has a zero for some* $t \in [a,b)$*. Moreover,* $y(t)$ *has a zero on* (a,b) *if the sharp inequality in* (2.8) *is satisfied, or if the condition* (iii) *of Theorem 2.1 holds.*

3 Nonhomogenous equation

In this section we will study the solution of the half-linear equation

$$l[x] \equiv \Big(r(t)\Phi(x')\Big)' + c(t)\Phi(x) = 0, \tag{1.1}$$

and nonhomogenous differential equation

$$l[y] \equiv \Big(r(t)\Phi(y')\Big)' + c(t)\Phi(y) = f(t), \quad f(t) \le 0 \text{ on } [a,b]. \tag{1.2}$$

In the following we present an useful tool in investigating of the operator $l[\cdot]$.

Riccati technique. If x is a function which has no zero on the interval (a,b), then the function $w(t) = r(t)\Phi\big(\frac{x'(t)}{x(t)}\big)$ is well defined on (a,b) and satisfies

$$R[w] := w' + c(t) + (p-1)r^{1-q}(t)|w|^q = \frac{l[x](t)}{\Phi(x(t))}, \tag{3.1}$$

where $q = \frac{p}{p-1}$ is the conjugate number to p. If x is a solution of the Eq. (1.1) then w solves the Riccati equation $R[w] = 0$. The point a (b) is a zero of the function x if and only if $w(a+) = \infty$ ($w(b-) = -\infty$). For every solution of Riccati equation the corresponding solution of Eq. (1.1) is unique up to the constant multiple.

Theorem 3.1. *Let* $\alpha, \alpha' \in \mathbb{R}$*,* $\alpha \ne 0$*,* x_1 *and* x_2 *be solutions of*

$$\begin{aligned} l[x_1] &= 0 \qquad x_1(a) = \alpha \qquad x_1'(a) = \alpha', \\ l[x_2] &\le 0 \qquad x_2(a) = \alpha \qquad x_2'(a) = \alpha' \end{aligned}$$

respectively. Suppose that the functions x_1*,* x_2 *have zero on* (a,∞) *and denote by* b_1 *and* b_2 *the first zero rigths of* a *of the function* x_1*,* x_2*, respectively. Define* $d = \min(b_1,b_2)$*. Then*

$$x_2(t) \le x_1(t) \quad \textit{for } t \in [a,d],$$

i.e. $x_2(t) \le x_1(t)$ *on the interval, where both functions are nonnegative (nonpositive).*

Proof. Let $\varepsilon > 0$ and denote $I_\varepsilon = [a, d-\varepsilon]$. Then $w_1(t) = r(t)\Phi\Big(\frac{x_1'(t)}{x_1(t)}\Big)$, $w_2(t) = r(t)\Phi\Big(\frac{x_2'(t)}{x_2(t)}\Big)$ are solutions of the Riccati equation $R[w_1] = 0$, $R[w_2] = \frac{l[x_2](t)}{\Phi\big(x_2(t)\big)}$ on I_ε, respectively and $w_1(a) = w_2(a)$.

Suppose $\alpha > 0$. Then $\frac{l[x_2](t)}{\Phi\left(x_2(t)\right)} \leq 0$ for $t \in I_\varepsilon$ and by standard argumentation $w_2(t) \leq w_1(t)$ for $t \in I_\varepsilon$. Hence,

$$x_2(t) = \alpha e^{\int_a^t \Phi^{-1}\left(\frac{w_2(s)}{r(s)}\right)\mathrm{d}s} \leq \alpha e^{\int_a^t \Phi^{-1}\left(\frac{w_1(s)}{r(s)}\right)\mathrm{d}s} = x_1(t) \tag{3.2}$$

for $t \in I_\varepsilon$.

If $\alpha < 0$, then $\frac{l[x_2](t)}{\Phi\left(x_2(t)\right)} \geq 0$ and $w_2(t) \geq w_1(t)$ for $t \in I_\varepsilon$, which implies (3.2). The limit process $\varepsilon \to 0+$ completes the proof. □

The following corollary is an extension of Theorem 1 and 3 from [10].

Corollary 3.1. *Let b be the first right focal point to the point a. Then every solution y of Eq. (1.2) which satisfies $y(a) > 0$ ($y(a) < 0$) and $y'(a) = 0$ has a zero (has no zero) on the interval $(a, b]$ ((a, b)).*

Proof. We choose $\alpha' = 0$ and $\alpha = y(a)$ in the previous theorem. Then b_1 is the first focal point to the point a and the conclusion follows from Theorem 3.1.

An easy alternative proof of the part concerning the positive solutions is the following: on the interval where y is positive is the function $y(t)$ solution of homogeneous equation

$$\Big(r(t)\Phi(x')\Big)' + \Big[c(t) - \frac{f(t)}{\Phi(y(t))}\Big]\Phi(x) = 0.$$

Now the conclusion follows from the fact that this equation is Sturmian majorant to (1.1) and from the Corollary 2.1. □

The following theorem compares solutions of two nonhomogenous equations.

Theorem 3.2. *Let x_i $(i = 1, 2)$ be the solutions of the equation*

$$l[x_i] = \Big(r(t)\Phi(x_i')\Big) + c(t)\Phi(x_i) = f_i(t),$$

where $f_2(t) \leq f_1(t) \leq 0$ on $[a, b]$. If

$$x_1(a) > x_2(a) > 0 \quad \textit{and} \quad \frac{x_1'(a)}{x_1(a)} \geq \frac{x_2'(a)}{x_2(a)},$$

then $x_1(t) > x_2(t)$ on the interval, where both $x_{1,2}$ are positive.

Proof. Suppose, by contradiction, that there exists a point d, $d \geq a$ such that $x_1(t) > x_2(t) > 0$ for $t \in [a, d)$ and $x_1(d) = x_2(d) > 0$. For $i = 1, 2$ the Riccati variable $w_i(t) = r(t)\Phi\left(\frac{x_i'(t)}{x_i(t)}\right)$ solves the Riccati equation

$$R[w_i] = w_i' + (p-1)r^{1-q}(t)|w_i|^q + c(t) - \frac{f_i(t)}{\Phi\big(x_i(t)\big)} = 0$$

on I and $w_1(a) \geq w_2(a)$. From the assumptions it follows

$$c(t) - \frac{f_2(t)}{\Phi\big(x_2(t)\big)} \geq c(t) - \frac{f_2(t)}{\Phi\big(x_1(t)\big)} \geq c(t) - \frac{f_1(t)}{\Phi\big(x_1(t)\big)} \quad \text{for } t \in [a, d].$$

Hence $w_2(t) \leq w_1(t)$ and

$$x_2(t) = x_2(a)e^{\int_a^t \Phi^{-1}\left(\frac{w_2(s)}{r(s)}\right)\mathrm{d}s} < x_1(a)e^{\int_a^t \Phi^{-1}\left(\frac{w_1(s)}{r(s)}\right)\mathrm{d}s} = x_1(t) \tag{3.3}$$

holds for $t \in [a, d]$. Particularly, for $t = d$ we have $x_2(d) < x_1(d)$, a contradiction. □

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