4. 4. 2019

Derivace

1) $y = x^2 e^x$ $\qquad \dfrac{dy}{dx} = 2x e^x + x^2 e^x$

2) $y = \dfrac{x^2 + a}{x^3}$ $\qquad \dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^{-1} + a x^{-3} \right) = (-1).x^{-2} - 3a\, x^{-4} =$

$$= -\frac{1}{x^2} - \frac{3a}{x^4} = -\frac{3a + x^2}{x^4}$$

3) $y = e^{-kx}$ $\qquad \dfrac{dy}{dx} = -k e^{-kx}$

4) $y = \pi x^3 + 2\pi x^2$ $\qquad \dfrac{dy}{dx} = 3\pi x^2 + 4\pi x$

5) $y = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$ $\qquad \dfrac{dy}{dx} = \frac{1}{2}(x+1)^{-1/2} = \dfrac{1}{2\sqrt{x+1}}$

Lineární aproximace

1) $y = x \cdot e^x \Rightarrow y(0) = 0$ $\qquad \dfrac{dy}{dx} = e^x + x e^x \Big|_{x=0} = 1$

$$y \simeq x$$

2) $y = r x \left(1 - \dfrac{x}{K}\right) \Rightarrow y(0) = 0$ $\qquad \dfrac{dy}{dx} = r - \dfrac{2r}{K} x \Big|_{x=0} = r$

$y = r x - \dfrac{r}{K} x^2$

$$y \simeq r x$$

r … množní parametr

3) $y(K) = 0$ $\qquad \dfrac{dy}{dx}(K) = r - \dfrac{2r}{K} \cdot K = r - 2r = -r$ (viz minulý příklad)

$$y \simeq -r(x - K)$$

4) $y = \sqrt{x} \Rightarrow y(1) = 1$, $\quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}} \Big|_{x=1} = \dfrac{1}{2} \Rightarrow y \simeq 1 + \dfrac{1}{2}(x-1)$

5) $y = \dfrac{1}{\sqrt{x}} \Rightarrow y(1) = 1$, $\quad \dfrac{dy}{dx} = -\dfrac{1}{2}(x)^{-3/2} \Big|_{x=1} = -\dfrac{1}{2} \Rightarrow y \simeq 1 - \dfrac{1}{2}(x-1)$

Integrály:

1) $\int x^2 \sin(x^3)\,dx \quad \left| \begin{array}{l} x^3 = t \\ 3x^2 dx = dt \\ x^2 dx = \frac{1}{3} dt \end{array} \right| = \frac{1}{3} \int \sin t \, dt =$

$= -\frac{1}{3} \cos t + C = -\frac{1}{3} \cos(x^3) + C$

2) $\int x^2 + 3\sqrt{x}\,dx = \int x^2 + 3x^{\frac{1}{2}} dx = \frac{1}{3} x^3 + 3 \cdot \frac{2}{3} x^{3/2} + C =$

$= \frac{1}{3} x^3 + 2x^{\frac{3}{2}} + C$

3) $\int_0^1 \frac{1}{x+1}\,dx = \left[\ln|x+1| \right]_0^1 = \ln 2 - \ln 1 =$

$= \ln 2$

4) $\int_{-1}^{1} \frac{1}{x^2+1}\,dx = \left[\operatorname{arctg} x \right]_{-1}^{1} = \operatorname{arctg} 1 - (\operatorname{arctg} -1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

5) $\frac{1}{\pi} \int_0^{\pi} \sin(kx)\,dx = \frac{1}{\pi} \left[-\frac{1}{k} \cos(kx) \right]_0^{\pi} = \frac{1}{\pi} \left[-\frac{1}{k} \cos(k\pi) + \frac{1}{k} \cos 0 \right]$

$= \frac{1}{k\pi} \left[1 - \cos(k\pi) \right]$

$k = 1 \qquad \frac{1}{\pi} \int_0^{\pi} \sin(kx)\,dx = \frac{1}{1 \cdot \pi} \left[1 - (-1) \right] = \frac{2}{\pi}$

$k = 2 \qquad$ –"– $\qquad = \frac{1}{2\pi} \left[1 - 1 \right] = 0$

$k = 3 \qquad$ –"– $\qquad = \frac{1}{3\pi} \left[1 - (-1) \right] = \frac{2}{3\pi}$

$k = 4 \qquad$ –"– $\qquad = \frac{1}{4\pi} \left[1 - 1 \right] = 0$

$k = 5 \qquad$ –"– $\qquad = \frac{1}{5\pi} \left[1 - (-1) \right] = \frac{2}{5\pi}$

$k = 6 \qquad = 0$

$\vdots \qquad \vdots$

Řešení

1) $\frac{dr}{dt} = k \cdot r^2 t$

α) $r = 0$

β) $\frac{dr}{r^2} = k \cdot t \, dt$

$$\int \frac{dr}{r^2} = \int k t \, dt$$

$$-\frac{1}{r} = \frac{1}{2} k t^2 + c$$

$$\boxed{\begin{gathered} r = 0 \\ -\frac{1}{r} = \frac{1}{2} k t^2 + c \end{gathered}}$$

2) $\frac{dy}{dx} = k \frac{y}{y^2+1}$

α) $y = 0$

β) $\frac{y^2+1}{y} \, dy = k \cdot dx$

$$\int \frac{y^2+1}{y} \, dy = \int k \, dx$$

$$\int y + \frac{1}{y} \, dy = \int k \, dx$$

$$\frac{1}{2} y^2 + \ln|y| = kx + c$$

$$\boxed{\begin{gathered} y = 0 \\ \frac{1}{2} y^2 + \ln|y| = kx + c \end{gathered}}$$

3) $\frac{dy}{dx} = \frac{x}{y}$

α) konstantní řešení neexistuje

β)

$$y \, dy = x \, dx$$

$$\int y \, dy = \int x \, dx$$

$$\frac{1}{2} y^2 = \frac{1}{2} x^2 + c$$

$$\boxed{\frac{1}{2} y^2 = \frac{1}{2} x^2 + c}$$

Gravitace:

1) $\frac{\alpha}{R^2} = \beta \cdot R \Rightarrow \beta = \frac{\alpha}{R^3}$ $\qquad F = \begin{cases} \frac{\alpha}{r^2} & r > R \\ \frac{\alpha}{R^3} r & r \geq R \end{cases}$

2) $\frac{1}{12}\int_{R-12}^{R} \frac{\alpha}{R^3} r \, dr =$ (střední hodnota lineární funkce) $= \frac{\alpha}{R^3}(R-6)$

(funkční hodnota v polovině)

3) $\frac{1}{36000}\int_{R}^{R+36000} \frac{\alpha}{r^2} dr = \frac{1}{36000}\left[-\alpha r^{-1}\right]_{R}^{R+36000} =$

$= \frac{1}{36000}\left[-\frac{\alpha}{R+36000} + \frac{\alpha}{R}\right] = \frac{\alpha}{R(R+36000)}$

4) a) nad Zemí: $F = \frac{\alpha}{r^2}$, $\frac{dF}{dr} = -2\alpha r^{-3}$, $\frac{dF}{dr}(R) = -2\alpha R^{-3}$

$\Delta F \approx \frac{dF}{dr}(R)\,\Delta h = -\frac{2\alpha}{R^3}\Delta h$

b) pod Zemí: $F = \frac{\alpha}{R^3} \cdot r$, $\frac{dF}{dr} = \frac{\alpha}{R^3}$
($\Delta h < 0$)

$\Delta F \approx \frac{dF}{dr}(R)\Delta h = \frac{\alpha}{R^3}\Delta h$ .. poloviční ve srovnání s případem a)

Rozhledna a) $\frac{dH}{dh} = 3{,}57 \cdot \frac{1}{2} h^{-1/2}$ $\qquad \frac{dH}{dh}(5) = 0{,}8 \, \frac{km}{m}$

každý metr výšky pozorovatele se projeví tak, že horizont je o 800 m dál.

b) Náklady: kx, kde x je výška,
Tržby: $a \cdot \sqrt{x}$
Zisk: $f(x) = a\sqrt{x} - kx$, $\frac{df}{dx} = a \cdot \frac{1}{2}\frac{1}{\sqrt{x}} - k$

$\frac{df}{dx} = 0$ pro $x = \left(\frac{a}{2k}\right)^2$

f'	$+$		$-$
f	$\nearrow$	$\left(\frac{a}{2k}\right)^2$	$\searrow$

Zisk má maximum pro $x = \left(\frac{a}{2k}\right)^2$