

**Conjugacy criteria for half-linear ODE  
in theory of PDE  
with generalized  $p$ -Laplacian  
and mixed powers**

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$$\begin{aligned} \operatorname{div} \left( A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\ + c(x) |y|^{p-2} y + \sum_{i=1}^m c_i(x) |y|^{p_i-2} y = e(x), \end{aligned} \quad (\text{E})$$

- $x = (x_1, \dots, x_n)_{i=1}^n \in \mathbb{R}^n$ ,  $p > 1$ ,  $p_i > 1$ ,
- $A(x)$  is elliptic  $n \times n$  matrix with differentiable components,  $c(x)$  and  $c_i(x)$  are Hölder continuous functions,  $\vec{b}(x) = (b_1(x), \dots, b_n(x))$  is continuous  $n$ -vector function,
- $\nabla = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)_{i=1}^n$  and  $\operatorname{div} = \frac{\partial}{\partial x_1} + \dots + \frac{\partial}{\partial x_n}$  are the usual nabla and divergence operators,
- $q$  is a conjugate number to the number  $p$ , i.e.,  $q = \frac{p}{p-1}$ ,
- $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $\mathbb{R}^n$ ,  $\|\cdot\|$  is the usual norm in  $\mathbb{R}^n$ ,  $\|A\| = \sup \{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\} = \lambda_{\max}$  is the spectral norm
- **solution** of (??) in  $\Omega \subseteq \mathbb{R}^n$  is a differentiable function  $u(x)$  such that  $A(x) \|\nabla u(x)\|^{p-2} \nabla u(x)$  is also differentiable and  $u$  satisfies (??) in  $\Omega$
- $S(a) = \{x \in \mathbb{R}^n : \|x\| = a\}$ ,  
 $\Omega(a) = \{x \in \mathbb{R}^n : a \leq \|x\|\}$ ,  
 $\Omega(a, b) = \{x \in \mathbb{R}^n : a \leq \|x\| \leq b\}$

$$u'' + c(x)u = 0 \quad (1)$$

- Equation (??) is oscillatory if each solution has infinitely many zeros in  $[x_0, \infty)$ .
- Equation (??) is oscillatory if each solution has a zero  $[a, \infty)$  for each  $a$ .
- Equation (??) is oscillatory if each solution has conjugate points on the interval  $[a, \infty)$  for each  $a$ .
- All definition are equivalent (no accumulation of zeros and Sturm separation theorem).
- Equation is oscillatory if  $c(x)$  is large enough. Many oscillation criteria are expressed in terms of the integral  $\int^{\infty} c(x) dx$  (Hille and Nehari type)
- There are oscillation criteria which can detect oscillation even if  $\int^{\infty} c(x) dx$  is extremely small. These criteria are in fact series of conjugacy criteria.

$$(p(t)u')' + c(t)u + \sum_{i=1}^m c_i(t)|u|^{\alpha_i} \operatorname{sgn} u = e(t) \quad (2)$$

where  $\alpha_1 > \dots > \alpha_m > 1 > \alpha_{m+1} > \dots > \alpha_n > 0$ .

**Theorem A** (Sun, Wong (2007)). *If for any  $T \geq 0$  there exists  $a_1, b_1, a_2, b_2$  such that  $T \leq a_1 < b_1 \leq a_2 < b_2$  and*

$$\begin{cases} c_i(t) \geq 0 & t \in [a_1, b_1] \cup [a_2, b_2], \quad i = 1, 2, \dots, n \\ e(x) \leq 0 & t \in [a_1, b_1] \\ e(x) \geq 0 & t \in [a_2, b_2] \end{cases}$$

*and there exists a continuously differentiable function  $u(t)$  satisfying  $u(a_i) = u(b_i) = 0$ ,  $u(t) \neq 0$  on  $(a_i, b_i)$  and*

$$\int_{a_i}^{b_i} \{p(t)u'^2(t) - Q(t)u^2(t)\} dt \leq 0 \quad (3)$$

*for  $i = 1, 2$ , where*

$$Q(t) = k_0 |e(t)|^{\eta_0} \prod_{i=1}^m (c_i^{\eta_i}(t)) + c(t),$$

*$k_0 = \prod_{i=0}^m \eta_i^{-\eta_i}$  and  $\eta_i, i = 0, \dots, n$  are positive constants satisfying  $\sum_{i=1}^m \alpha_i \eta_i = 1$  and  $\sum_{i=0}^m \eta_i = 1$ ,*

*then all solutions of (??) are oscillatory.*

$$\Delta u + c(x)u = 0 \tag{4}$$

- Equation (??) is *oscillatory* if every solution has a zero on  $\{x \in \mathbb{R}^n : \|x\| \geq a\}$  for each  $a$ .
- Equation (??) is *nodally oscillatory* if every solution has a nodal domain on  $\{x \in \mathbb{R}^n : \|x\| \geq a\}$  for each  $a$ .
- Both definition are equivalent (Moss+Piepenbrink).

$$\operatorname{div}\left(\|\nabla u\|^{p-2} \nabla u\right) + c(x)|u|^{p-2}u = 0 \tag{5}$$

- Essentially the same approach to oscillation as in linear case
- The equivalence between two oscillations is open problem.

$$\begin{aligned} \operatorname{div} \left( A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\ + c(x)|y|^{p-2}y + \sum_{i=1}^m c_i(x)|y|^{p_i-2}y = e(x), \end{aligned} \quad (\text{E})$$

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## DETECTION OF OSCILLATION FROM ODE

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**Theorem B** (O. Došlý (2001)). *Equation*

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \quad (6)$$

*is oscillatory, if the ordinary differential equation*

$$\left( r^{n-1}|u'|^{p-2}u' \right)' + r^{n-1} \left( \frac{1}{\omega_n r^{n-1}} \int_{S(r)} c(x) \, dx \right) |u|^{p-2}u = 0 \quad (7)$$

*is oscillatory. The number  $\omega_n$  is the surface area of the unit sphere in  $\mathbb{R}^n$ .*

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for  $A(x) = a(\|x\|)I$ ,  $a(\cdot)$  differentiable).

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## OUR AIM

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- Extend method used in Theorem ?? to (??). Derive a general result, like Theorem B.
- Derive a result which does depend on more general expression, than the mean value of  $c(x)$  over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2009) excluded the possibility  $p_i > p$  for every  $i$ ).

$$\begin{aligned} \operatorname{div} \left( A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\ + c(x)|y|^{p-2}y + \sum_{i=1}^m c_i(x)|y|^{p_i-2}y = e(x), \end{aligned} \quad (\text{E})$$

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MODUS OPERANDI

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- Get rid of terms  $\sum_{i=1}^m c_i(x)|y|^{p_i-2}y$  and  $e(x)$  (join with  $c(x)|y|^{p-2}y$ ) and convert the problem into

$$\operatorname{div} \left( A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle + C(x)|y|^{p-2}y = 0.$$

- Derive Riccati type inequality in  $n$  variables.
- Derive Riccati type inequality in 1 variable.
- Use this inequality as a tool which transforms results from ODE to PDE.

Using generalized AG inequality  $\sum \alpha_i \geq \prod \left( \frac{\alpha_i}{\eta_i} \right)^{\eta_i}$ , if  $\alpha_i \geq 0$ ,  $\eta_i > 0$  and  $\sum \eta_i = 1$  we eliminate the right-hand side and terms with mixed powers.

**Lemma 1.** Let either  $y > 0$  and  $e(x) \leq 0$  or  $y < 0$  and  $e(x) \geq 0$ . Let  $\eta_i > 0$  be numbers satisfying  $\sum_{i=0}^m \eta_i = 1$  and  $\eta_0 + \sum_{i=1}^m p_i \eta_i = p$  and let  $c_i(x) \geq 0$  for every  $i$ . Then

$$\frac{1}{|y|^{p-2}y} \left( -e(x) + \sum_{i=1}^m c_i(x) |y|^{p_i-2}y \right) \geq C_1(x),$$

where

$$C_1(x) := \left| \frac{e(x)}{\eta_0} \right|^{\eta_0} \prod_{i=1}^m \left( \frac{c_i(x)}{\eta_i} \right)^{\eta_i}. \quad (8)$$

**Remark:** The numbers  $\eta_i$  from Lemma ?? exist, if  $p_i > p$  for some  $i$ .

**Lemma 2.** Suppose  $c_i(x) \geq 0$ . Let  $\eta_i > 0$  be numbers satisfying  $\sum_{i=1}^m \eta_i = 1$  and  $\sum_{i=1}^m p_i \eta_i = p$ .

Then

$$\frac{1}{|y|^{p-2}y} \sum_{i=1}^m c_i(x) |y|^{p_i-2}y \geq C_2(x),$$

where

$$C_2(x) := \prod_{i=1}^m \left( \frac{c_i(x)}{\eta_i} \right)^{\eta_i} \quad (9)$$

**Remark:** The numbers  $\eta_i$  from Lemma ?? exist iff  $p_i > p$  for some  $i$  and  $p_j < p$  for some  $j$ .



**Lemma 3.** Let  $y$  be a solution of (??) which does not have zero on  $\Omega$ . Suppose that there exists a function  $C(x)$  such that

$$C(x) \leq c(x) + \sum_{i=1}^m c_i(x) |y|^{p_i-p} - \frac{e(x)}{|y|^{p-2}y}$$

Denote  $\vec{w}(x) = A(x) \frac{\|\nabla y\|^{p-2} \nabla y}{|y|^{p-2}y}$ . The function  $\vec{w}(x)$  is well defined on  $\Omega$  and satisfies the inequality

$$\operatorname{div} \vec{w} + (p-1)\Lambda(x) \|\vec{w}\|^q + \langle \vec{w}, A^{-1}(x)\vec{b}(x) \rangle + C(x) \leq 0 \quad (10)$$

where

$$\Lambda(x) = \begin{cases} \lambda_{\max}^{1-q}(x) & 1 < p \leq 2, \\ \lambda_{\min} \lambda_{\max}^{-q}(x) & p > 2. \end{cases} \quad (11)$$

**Lemma 4.** Let (??) hold. Let  $l > 1$ ,  $l^* = \frac{l}{l-1}$  be two mutually conjugate numbers and  $\alpha \in C^1(\Omega, \mathbb{R}^+)$  be a smooth function positive on  $\Omega$ . Then

$$\operatorname{div}(\alpha(x)\vec{w}) + (p-1) \frac{\Lambda(x)\alpha^{1-q}(x)}{l^*} \|\alpha(x)\vec{w}\|^q - \frac{l^{p-1}\alpha(x)}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p + \alpha(x)C(x) \leq 0$$

holds on  $\Omega$ . If  $\left\| A^{-1}\vec{b} - \frac{\nabla \alpha}{\alpha} \right\| \equiv 0$  on  $\Omega$ , then this inequality holds with  $l^* = 1$ .

**Theorem 1.** Let the  $n$ -vector function  $\vec{w}$  satisfy inequality

$$\operatorname{div} \vec{w} + C_0(x) + (p-1)\Lambda_0(x) \|\vec{w}\|^q \leq 0$$

on  $\Omega(a, b)$ . Denote  $\tilde{C}(r) = \int_{S(r)} C_0(x) \, d\sigma$  and  $\tilde{R}(r) = \int_{S(r)} \Lambda_0^{1-p} \, d\sigma$ . Then the half-linear ordinary differential equation

$$\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0, \quad ' = \frac{d}{dr}$$

is disconjugate on  $[a, b]$  and it possesses solution which has no zero on  $[a, b]$ .

**Theorem 2.** Let  $l > 1$ . Let  $l^* = 1$  if  $\|\vec{b}\| \equiv 0$  and  $l^* = \frac{l}{l-1}$  otherwise. Further, let  $c_i(x) \geq 0$  for every  $i$ . Denote

$$\tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \Lambda^{1-p}(x) \, d\sigma$$

and

$$\tilde{C}(r) = \int_{S(r)} c(x) + C_1(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \vec{b}(x) \right\|^p \, d\sigma,$$

where  $\Lambda(x)$  is defined by (??) and  $C_1(x)$  is defined by (??).

Suppose that the equation

$$\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0$$

has conjugate points on  $[a, b]$ .

If  $e(x) \leq 0$  on  $\Omega(a, b)$ , then equation (??) has no positive solution on  $\Omega(a, b)$ .

If  $e(x) \geq 0$  on  $\Omega(a, b)$ , then equation (??) has no negative solution on  $\Omega(a, b)$ .

**Theorem 3** (non-radial variant of Theorem ??). Let  $l > 1$  and let  $\Omega \subset \Omega(a, b)$  be an open domain with piecewise smooth boundary such that  $\text{meas}(\Omega \cap S(r)) \neq 0$  for every  $r \in [a, b]$ . Let  $c_i(x) \geq 0$  on  $\Omega$  for every  $i$  and let  $\alpha(x)$  be a function which is positive and continuously differentiable on  $\Omega$  and vanishes on the boundary and outside  $\Omega$ . Let  $l^* = 1$  if  $\left\| A^{-1}\vec{b} - \frac{\nabla\alpha}{\alpha} \right\| \equiv 0$  on  $\Omega$  and

$l^* = \frac{l}{l-1}$  otherwise. In the former case suppose also that the integral

$$\int_{S(r)} \frac{\alpha(x)}{\Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p d\sigma$$

which may have singularity on  $\partial\Omega$  if  $\Omega \neq \Omega(a, b)$  is convergent for every  $r \in [a, b]$ . Denote

$$\tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \alpha(x) \Lambda^{1-p}(x) d\sigma$$

and

$$\tilde{C}(r) = \int_{S(r)} \alpha(x) \left( c(x) + C_1(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p \right) d\sigma,$$

where  $\Lambda(x)$  is defined by (??) and  $C_1(x)$  is defined by (??) and suppose that equation

$$\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0$$

has conjugate points on  $[a, b]$ .

If  $e(x) \leq 0$  on  $\Omega(a, b)$ , then equation (??) has no positive solution on  $\Omega(a, b)$ .

If  $e(x) \geq 0$  on  $\Omega(a, b)$ , then equation (??) has no negative solution on  $\Omega(a, b)$ .

**Theorem 4.** Let  $l$ ,  $\Omega$ ,  $\alpha(x)$ ,  $\Lambda(x)$  and  $\tilde{R}(r)$  be defined as in Theorem ?? and let  $c_i(x) \geq 0$  and  $\mathbf{e}(x) \equiv \mathbf{0}$  on  $\Omega(a, b)$ . Denote

$$\tilde{C}(r) = \int_{S(r)} \alpha(x) \left( c(x) + C_2(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \right) d\sigma,$$

where  $C_2(x)$  is defined by (??). If the equation

$$\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0$$

has conjugate points on  $[a, b]$ , then every solution of equation (??) has zero on  $\Omega(a, b)$ .

Similar theorems can be derived also for estimates of terms with mixed powers based on different methods than AG inequality (see R. M., Non-linear Analysis TMA 73 (2010)).