

**Conjugacy criteria for half-linear ODE  
in theory of PDE  
with generalized  $p$ -Laplacian  
and mixed powers**

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$$\begin{aligned} \operatorname{div} \left( A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\ + c(x) |y|^{p-2} y + \sum_{i=1}^m c_i(x) |y|^{p_i-2} y = e(x), \end{aligned} \quad (\text{E})$$

- $x = (x_1, \dots, x_n)_{i=1}^n \in \mathbb{R}^n$ ,  $p > 1$ ,  $p_i > 1$ ,
- $A(x)$  is elliptic  $n \times n$  matrix with differentiable components,  $c(x)$  and  $c_i(x)$  are Hölder continuous functions,  $\vec{b}(x) = (b_1(x), \dots, b_n(x))$  is continuous  $n$ -vector function,
- $\nabla = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)_{i=1}^n$  and  $\operatorname{div} = \frac{\partial}{\partial x_1} + \dots + \frac{\partial}{\partial x_n}$  is are the usual nabla and divergence operators,
- $q$  is a conjugate number to the number  $p$ , i.e.,  $q = \frac{p}{p-1}$ ,
- $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $\mathbb{R}^n$ ,  $\|\cdot\|$  is the usual norm in  $\mathbb{R}^n$ ,  $\|A\| = \sup \{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\} = \lambda_{\max}$  is the spectral norm
- **solution** of (E) in  $\Omega \subseteq \mathbb{R}^n$  is a differentiable function  $u(x)$  such that  $A(x) \|\nabla u(x)\|^{p-2} \nabla u(x)$  is also differentiable and  $u$  satisfies (E) in  $\Omega$
- $S(a) = \{x \in \mathbb{R}^n : \|x\| = a\}$ ,  
 $\Omega(a) = \{x \in \mathbb{R}^n : a \leq \|x\|\}$ ,  
 $\Omega(a, b) = \{x \in \mathbb{R}^n : a \leq \|x\| \leq b\}$

$$u'' + c(x)u = 0 \quad (1)$$

- Equation (1) is oscillatory if each solution has infinitely many zeros in  $[x_0, \infty)$ .
- Equation (1) is oscillatory if each solution has a zero  $[a, \infty)$  for each  $a$ .
- Equation (1) is oscillatory if each solution has conjugate points on the interval  $[a, \infty)$  for each  $a$ .
- All definition are equivalent (no accumulation of zeros and Sturm separation theorem).
- Equation is oscillatory if  $c(x)$  is large enough. Many oscillation criteria are expressed in terms of the integral  $\int^{\infty} c(x) dx$  (Hille and Nehari type)
- There are oscillation criteria which can detect oscillation even if  $\int^{\infty} c(x) dx$  is extremely small. These criteria are in fact series of conjugacy criteria.

$$(p(t)u')' + c(t)u + \sum_{i=1}^m c_i(t)|u|^{\alpha_i} \operatorname{sgn} u = e(t) \quad (2)$$

where  $\alpha_1 > \dots > \alpha_m > 1 > \alpha_{m+1} > \dots > \alpha_n > 0$ .

**Theorem A** (Sun, Wong (2007)). *If for any  $T \geq 0$  there exists  $a_1, b_1, a_2, b_2$  such that  $T \leq a_1 < b_1 \leq a_2 < b_2$  and*

$$\begin{cases} c_i(t) \geq 0 & t \in [a_1, b_1] \cup [a_2, b_2], \quad i = 1, 2, \dots, n \\ e(x) \leq 0 & t \in [a_1, b_1] \\ e(x) \geq 0 & t \in [a_2, b_2] \end{cases}$$

*and there exists a continuously differentiable function  $u(t)$  satisfying  $u(a_i) = u(b_i) = 0$ ,  $u(t) \neq 0$  on  $(a_i, b_i)$  and*

$$\int_{a_i}^{b_i} \{p(t)u'^2(t) - Q(t)u^2(t)\} dt \leq 0 \quad (3)$$

*for  $i = 1, 2$ , where*

$$Q(t) = k_0 |e(t)|^{\eta_0} \prod_{i=1}^m (c_i^{\eta_i}(t)) + c(t),$$

*$k_0 = \prod_{i=0}^m \eta_i^{-\eta_i}$  and  $\eta_i, i = 0, \dots, n$  are positive constants satisfying  $\sum_{i=1}^m \alpha_i \eta_i = 1$  and  $\sum_{i=0}^m \eta_i = 1$ ,*

*then all solutions of (2) are oscillatory.*

$$\Delta u + c(x)u = 0 \tag{4}$$

- Equation (4) is *oscillatory* if every solution has a zero on  $\{x \in \mathbb{R}^n : \|x\| \geq a\}$  for each  $a$ .
- Equation (4) is *nodally oscillatory* if every solution has a nodal domain on  $\{x \in \mathbb{R}^n : \|x\| \geq a\}$  for each  $a$ .
- Both definition are equivalent (Moss+Piepenbrink).

$$\operatorname{div}\left(\|\nabla u\|^{p-2} \nabla u\right) + c(x)|u|^{p-2}u = 0 \tag{5}$$

- Essentially the same approach to oscillation as in linear case
- The equivalence between two oscillations is open problem.

$$\begin{aligned} \operatorname{div} \left( A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\ + c(x)|y|^{p-2}y + \sum_{i=1}^m c_i(x)|y|^{p_i-2}y = e(x), \end{aligned} \quad (\text{E})$$

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## DETECTION OF OSCILLATION FROM ODE

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**Theorem B** (O. Došlý (2001)). *Equation*

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \quad (6)$$

*is oscillatory, if the ordinary differential equation*

$$\left( r^{n-1}|u'|^{p-2}u' \right)' + r^{n-1} \left( \frac{1}{\omega_n r^{n-1}} \int_{S(r)} c(x) \, dx \right) |u|^{p-2}u = 0 \quad (7)$$

*is oscillatory. The number  $\omega_n$  is the surface area of the unit sphere in  $\mathbb{R}^n$ .*

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for  $A(x) = a(\|x\|)I$ ,  $a(\cdot)$  differentiable).

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## OUR AIM

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- Extend method used in Theorem A to (E). Derive a general result, like Theorem B.
- Derive a result which does depend on more general expression, than the mean value of  $c(x)$  over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2009) excluded the possibility  $p_i > p$  for every  $i$ ).

$$\begin{aligned} \operatorname{div} \left( A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\ + c(x)|y|^{p-2}y + \sum_{i=1}^m c_i(x)|y|^{p_i-2}y = e(x), \end{aligned} \quad (\text{E})$$

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MODUS OPERANDI

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- Get rid of terms  $\sum_{i=1}^m c_i(x)|y|^{p_i-2}y$  and  $e(x)$  (join with  $c(x)|y|^{p-2}y$ ) and convert the problem into

$$\operatorname{div} \left( A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle + C(x)|y|^{p-2}y = 0.$$

- Derive Riccati type inequality in  $n$  variables.
- Derive Riccati type inequality in 1 variable.
- Use this inequality as a tool which transforms results from ODE to PDE.

Using generalized AG inequality  $\sum \alpha_i \geq \prod \left( \frac{\alpha_i}{\eta_i} \right)^{\eta_i}$ , if  $\alpha_i \geq 0$ ,  $\eta_i > 0$  and  $\sum \eta_i = 1$  we eliminate the right-hand side and terms with mixed powers.

**Lemma 1.** Let either  $y > 0$  and  $e(x) \leq 0$  or  $y < 0$  and  $e(x) \geq 0$ . Let  $\eta_i > 0$  be numbers satisfying  $\sum_{i=0}^m \eta_i = 1$  and  $\eta_0 + \sum_{i=1}^m p_i \eta_i = p$  and let  $c_i(x) \geq 0$  for every  $i$ . Then

$$\frac{1}{|y|^{p-2}y} \left( -e(x) + \sum_{i=1}^m c_i(x) |y|^{p_i-2}y \right) \geq C_1(x),$$

where

$$C_1(x) := \left| \frac{e(x)}{\eta_0} \right|^{\eta_0} \prod_{i=1}^m \left( \frac{c_i(x)}{\eta_i} \right)^{\eta_i}. \quad (8)$$

**Remark:** The numbers  $\eta_i$  from Lemma 1 exist, if  $p_i > p$  for some  $i$ .

**Lemma 2.** Suppose  $c_i(x) \geq 0$ . Let  $\eta_i > 0$  be numbers satisfying  $\sum_{i=1}^m \eta_i = 1$  and  $\sum_{i=1}^m p_i \eta_i = p$ .

Then

$$\frac{1}{|y|^{p-2}y} \sum_{i=1}^m c_i(x) |y|^{p_i-2}y \geq C_2(x),$$

where

$$C_2(x) := \prod_{i=1}^m \left( \frac{c_i(x)}{\eta_i} \right)^{\eta_i} \quad (9)$$

**Remark:** The numbers  $\eta_i$  from Lemma 2 exist iff  $p_i > p$  for some  $i$  and  $p_j < p$  for some  $j$ .



**Lemma 3.** Let  $y$  be a solution of (E) which does not have zero on  $\Omega$ . Suppose that there exists a function  $C(x)$  such that

$$C(x) \leq c(x) + \sum_{i=1}^m c_i(x) |y|^{p_i-p} - \frac{e(x)}{|y|^{p-2}y}$$

Denote  $\vec{w}(x) = A(x) \frac{\|\nabla y\|^{p-2} \nabla y}{|y|^{p-2}y}$ . The function  $\vec{w}(x)$  is well defined on  $\Omega$  and satisfies the inequality

$$\operatorname{div} \vec{w} + (p-1)\Lambda(x) \|\vec{w}\|^q + \langle \vec{w}, A^{-1}(x)\vec{b}(x) \rangle + C(x) \leq 0 \quad (10)$$

where

$$\Lambda(x) = \begin{cases} \lambda_{\max}^{1-q}(x) & 1 < p \leq 2, \\ \lambda_{\min} \lambda_{\max}^{-q}(x) & p > 2. \end{cases} \quad (11)$$

**Lemma 4.** Let (10) hold. Let  $l > 1$ ,  $l^* = \frac{l}{l-1}$  be two mutually conjugate numbers and  $\alpha \in C^1(\Omega, \mathbb{R}^+)$  be a smooth function positive on  $\Omega$ . Then

$$\operatorname{div}(\alpha(x)\vec{w}) + (p-1) \frac{\Lambda(x)\alpha^{1-q}(x)}{l^*} \|\alpha(x)\vec{w}\|^q - \frac{l^{p-1}\alpha(x)}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p + \alpha(x)C(x) \leq 0$$

holds on  $\Omega$ . If  $\left\| A^{-1}\vec{b} - \frac{\nabla \alpha}{\alpha} \right\| \equiv 0$  on  $\Omega$ , then this inequality holds with  $l^* = 1$ .

**Theorem 1.** Let the  $n$ -vector function  $\vec{w}$  satisfy inequality

$$\operatorname{div} \vec{w} + C_0(x) + (p-1)\Lambda_0(x) \|\vec{w}\|^q \leq 0$$

on  $\Omega(a, b)$ . Denote  $\tilde{C}(r) = \int_{S(r)} C_0(x) \, d\sigma$  and  $\tilde{R}(r) = \int_{S(r)} \Lambda_0^{1-p} \, d\sigma$ . Then the half-linear ordinary differential equation

$$\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0, \quad ' = \frac{d}{dr}$$

is disconjugate on  $[a, b]$  and it possesses solution which has no zero on  $[a, b]$ .

**Theorem 2.** Let  $l > 1$ . Let  $l^* = 1$  if  $\|\vec{b}\| \equiv 0$  and  $l^* = \frac{l}{l-1}$  otherwise. Further, let  $c_i(x) \geq 0$  for every  $i$ . Denote

$$\tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \Lambda^{1-p}(x) \, d\sigma$$

and

$$\tilde{C}(r) = \int_{S(r)} c(x) + C_1(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \vec{b}(x) \right\|^p \, d\sigma,$$

where  $\Lambda(x)$  is defined by (11) and  $C_1(x)$  is defined by (8).

Suppose that the equation

$$\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0$$

has conjugate points on  $[a, b]$ .

If  $e(x) \leq 0$  on  $\Omega(a, b)$ , then equation (E) has no positive solution on  $\Omega(a, b)$ .

If  $e(x) \geq 0$  on  $\Omega(a, b)$ , then equation (E) has no negative solution on  $\Omega(a, b)$ .

**Theorem 3** (non-radial variant of Theorem 2). Let  $l > 1$  and let  $\Omega \subset \Omega(a, b)$  be an open domain with piecewise smooth boundary such that  $\text{meas}(\Omega \cap S(r)) \neq 0$  for every  $r \in [a, b]$ . Let  $c_i(x) \geq 0$  on  $\Omega$  for every  $i$  and let  $\alpha(x)$  be a function which is positive and continuously differentiable on  $\Omega$  and vanishes on the boundary and outside  $\Omega$ . Let  $l^* = 1$  if  $\left\| A^{-1}\vec{b} - \frac{\nabla\alpha}{\alpha} \right\| \equiv 0$  on  $\Omega$  and

$l^* = \frac{l}{l-1}$  otherwise. In the former case suppose also that the integral

$$\int_{S(r)} \frac{\alpha(x)}{\Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p d\sigma$$

which may have singularity on  $\partial\Omega$  if  $\Omega \neq \Omega(a, b)$  is convergent for every  $r \in [a, b]$ . Denote

$$\tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \alpha(x) \Lambda^{1-p}(x) d\sigma$$

and

$$\tilde{C}(r) = \int_{S(r)} \alpha(x) \left( c(x) + C_1(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p \right) d\sigma,$$

where  $\Lambda(x)$  is defined by (11) and  $C_1(x)$  is defined by (8) and suppose that equation

$$\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0$$

has conjugate points on  $[a, b]$ .

If  $e(x) \leq 0$  on  $\Omega(a, b)$ , then equation (E) has no positive solution on  $\Omega(a, b)$ .

If  $e(x) \geq 0$  on  $\Omega(a, b)$ , then equation (E) has no negative solution on  $\Omega(a, b)$ .

**Theorem 4.** Let  $l$ ,  $\Omega$ ,  $\alpha(x)$ ,  $\Lambda(x)$  and  $\tilde{R}(r)$  be defined as in Theorem 3 and let  $c_i(x) \geq 0$  and  $\mathbf{e}(x) \equiv \mathbf{0}$  on  $\Omega(a, b)$ . Denote

$$\tilde{C}(r) = \int_{S(r)} \alpha(x) \left( c(x) + C_2(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \right) d\sigma,$$

where  $C_2(x)$  is defined by (9). If the equation

$$\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0$$

has conjugate points on  $[a, b]$ , then every solution of equation (E) has zero on  $\Omega(a, b)$ .

Similar theorems can be derived also for estimates of terms with mixed powers based on different methods than AG inequality (see R. M., Non-linear Analysis TMA 73 (2010)).

**Conjugacy criteria for half-linear ODE in theory of PDE with generalized  $p$ -Laplacian and mixed powers**

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$$\operatorname{div} \left( A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{B}(x) |\nabla u|^{q-2} \nabla u + c(x) |u|^{p-2} u + \sum_{i=1}^m d_i(x) |u|^{p_i-2} u = r(x) \quad (5)$$

- $\bullet$   $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $p > 1, p_i > 1$ .
- $\bullet$   $A(x)$  is elliptic  $n \times n$  matrix with differentiable components,  $c(x)$  and  $d_i(x)$  are Hölder continuous functions,  $\tilde{B}(x) = (\tilde{b}_1(x), \dots, \tilde{b}_n(x))$  is continuous  $n$ -vector function.
- $\bullet$   $\forall u \in \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$  and  $\operatorname{div} \frac{\partial}{\partial x_i} \dots \frac{\partial}{\partial x_i} u = 0$  are the usual scalar and divergence equations.
- $\bullet$   $q$  is a conjugate number to the number  $p$ , i.e.,  $\frac{1}{p} + \frac{1}{q} = 1$ .
- $\bullet$   $(\cdot, \cdot)$  is the usual scalar product in  $\mathbb{R}^n$ ,  $|\cdot|$  is the usual norm in  $\mathbb{R}^n$ ,  $|\cdot|_A = \sup \{ |Ax| : x \in \mathbb{R}^n, |x| = 1 \} = |A_{\max}|$  is the spectral norm.
- $\bullet$  solution of (5) in  $\Omega \subset \mathbb{R}^n$  is a differentiable function  $u(x)$  such that  $A(x) |\nabla u(x)|^{p-2} \nabla u(x)$  is also differentiable and  $u$  satisfies (5) in  $\Omega$ .
- $\bullet$   $B(x) = \{x \in \mathbb{R}^n : |x| = a\}$ ,
- $\bullet$   $D(x) = \{x \in \mathbb{R}^n : |x| \leq a\}$ ,
- $\bullet$   $D(x, b) = \{x \in \mathbb{R}^n : |x - b| \leq a\}$ .

CONCEPT OF OSCILLATION FOR LINEAR PDE

$\operatorname{div} \left( c(x) u \right) = 0 \quad (4)$

- $\bullet$  Equation (7) is oscillatory if every solution has a zero on  $\{x \in \mathbb{R}^n : |x| \geq a\}$  for each  $a$ .
- $\bullet$  Equation (7) is non-oscillatory if every solution has a nodal domain on  $\{x \in \mathbb{R}^n : |x| \geq a\}$  for each  $a$ .

CONCEPT OF OSCILLATION FOR HALF-LINEAR PDE

$\operatorname{div} \left( |\nabla u|^{p-2} \nabla u \right) + c(x) |u|^{p-2} u = 0 \quad (5)$

- $\bullet$  Equivalently the same approach to oscillation as in linear case.
- $\bullet$  The equivalence between two oscillations is a open problem.

**Lemma 3.** Let  $y$  be a solution of (7) which does not have zero in  $\Omega$ . Suppose that there exists a function  $C(x)$  such that

$$C(x) \leq c(x) + \sum_{i=1}^m d_i(x) |p_i - p| \frac{c(x)}{p_i - p}$$

Denote  $\theta(x) = A(x) |\nabla u|^{p-2} \nabla u$ . The function  $\theta(x)$  is well defined on  $\Omega$  and satisfies the inequality

$$\operatorname{div} \left( (p-1) A(x) |\theta|^{p-2} \theta + (A^{-1}(x) \theta) \right) + C(x) \leq 0 \quad (10)$$

where

$$A(x) = \left( \frac{A_{11}(x)}{A_{22}(x)}, \dots, \frac{A_{nn}(x)}{A_{nn}(x)} \right) \quad 1 < p \leq 2 \quad (11)$$

**Lemma 4.** Let (7) hold. Let  $t > 1$ ,  $t' = \frac{t}{t-1}$  be two mutually conjugate numbers and  $a \in C^1(\mathbb{R}^n, \mathbb{R}^n)$  be a smooth function positive at its zero.

$$\operatorname{div} \left( c(x) u \right) = (p-1) \frac{A(x) |\nabla u|^{p-2} \nabla u}{p-1} + \sum_{i=1}^m \frac{d_i(x) |u|^{p_i-2} u}{p_i - p} + c(x) |u|^{p-2} u \leq 0$$

hold on  $\Omega$ . If  $A^{-1}(x) \frac{A(x) |\nabla u|^{p-2} \nabla u}{p-1} + \sum_{i=1}^m \frac{d_i(x) |u|^{p_i-2} u}{p_i - p} + c(x) |u|^{p-2} u \leq 0$  on  $\Omega$ , then this inequality holds with  $t' = 1$ .

$$\operatorname{div} \left( A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{B}(x) |\nabla u|^{q-2} \nabla u + c(x) |u|^{p-2} u + \sum_{i=1}^m d_i(x) |u|^{p_i-2} u = r(x) \quad (5)$$

DETECTION OF OSCILLATION FROM ODE

**Theorem B** (D. Džur (2018)). Equation

$$\operatorname{div} \left( |\nabla u|^{p-2} \nabla u \right) + c(x) |u|^{p-2} u = 0 \quad (6)$$

is oscillatory if the ordinary differential equation

$$\left( r'' + (r' - u'' u') \frac{p-1}{p} \right) \left( \frac{1}{r} \right)' + c(x) |u|^{p-2} u = 0 \quad (7)$$

is oscillatory. The number  $u'$  is the surface area of the unit sphere in  $\mathbb{R}^n$ .

DIXON 2012

- $\bullet$  Extended method used in Theorem 7F to (7F). Derive a general result, like Theorem B.
- $\bullet$  Derive a result which does depend on more general expression, than the mean value of  $c(x)$  over spheres centered in the origin.
- $\bullet$  Remove restrictions used by previous authors (for example Xu (2008) excluded the possibility  $p_i \neq p$  for every  $i$ ).

$$\operatorname{div} \left( A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{B}(x) |\nabla u|^{q-2} \nabla u + c(x) |u|^{p-2} u + \sum_{i=1}^m d_i(x) |u|^{p_i-2} u = r(x) \quad (5)$$

DETECTION OF OSCILLATION FROM ODE

**Theorem B** (D. Džur (2018)). Equation

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DIXON 2012

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- $\bullet$  Remove restrictions used by previous authors (for example Xu (2008) excluded the possibility  $p_i \neq p$  for every  $i$ ).

**Theorem 1.** Let the  $n$ -vector function if study inequality

$$\operatorname{div} \left( C(x) u \right) = (p-1) A(x) |\theta|^{p-2} \theta \leq 0$$

on  $D(x, b)$ . Denote  $C(x) = \left( C_1(x), \dots, C_n(x) \right)$  and  $R(x) = \int_{x_0}^x A^{-1}(t) dt$ . Then the half-linear ordinary differential equation

$$\left( R(x) |r'|^{p-2} r' \right)' + \tilde{C}(x) |r'|^{q-2} r' = 0, \quad r' = \frac{dr}{dx}$$

disintegrates on  $[a, b]$  and it possesses solution which has no zero on  $[a, b]$ .

**Theorem 2.** Let  $t > 1$ . Let  $t' = 1 + \frac{1}{t}$  and  $t'' = \frac{t}{t-1}$  respectively. Further, let  $c(x) \geq 0$  for every  $x$ . Denote

$$R(x) = (t-1)^{t-1} \int_{x_0}^x A^{-1}(t) dt$$

and

$$C(x) = \int_{x_0}^x c(x) + C_1(x) - \frac{p-1}{p} \frac{r''}{R(x)^{t-1}} \left( A^{-1}(x) R(x) \right)' dx,$$

where  $A(x)$  is defined by (7F) and  $C_1(x)$  is defined by (7F). Suppose that the equation

$$\left( R(x) |r'|^{p-2} r' \right)' - C(x) |r'|^{q-2} r' = 0$$

has conjugate points on  $[a, b]$ .

If  $C(x) \geq 0$  on  $D(x, b)$ , then equation (7F) has no positive solution on  $D(x, b)$ .  
 If  $c(x) \geq 0$  on  $D(x, b)$ , then equation (7F) has no negative solution on  $D(x, b)$ .

CONCEPT OF OSCILLATION FOR ODE

$r'' + c(x)r = 0 \quad (1)$

$r'' + c(x)r = 0$

- $\bullet$  Equation (7F) is oscillatory if each solution has infinitely many zeros in  $[a_0, \infty)$ .
- $\bullet$  Equation (7F) is non-oscillatory if each solution has a zero  $[a_0, \infty)$  for each  $a_0$ .
- $\bullet$  Equation (7F) is oscillatory if each solution has conjugate points on the interval  $[a_0, \infty)$  for each  $a_0$ .
- $\bullet$  All definition are equivalent (see accumulation of zeros and Sturm separation theorem).
- $\bullet$  Equation is oscillatory if  $c(x)$  is large enough. Many oscillation criteria are expressed in terms of the integral  $\int_{a_0}^{\infty} c(x) dx$  (Hilb and Nehari type).
- $\bullet$  These are oscillation criteria which can detect oscillation even if  $\int_{a_0}^{\infty} c(x) dx$  is extremely small.
- $\bullet$  These criteria are in fact series of conjugacy criteria.

$$\operatorname{div} \left( A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{B}(x) |\nabla u|^{q-2} \nabla u + c(x) |u|^{p-2} u + \sum_{i=1}^m d_i(x) |u|^{p_i-2} u = r(x) \quad (5)$$

MIXED OPERATORS

- $\bullet$  Get rid of terms  $\sum_{i=1}^m d_i(x) |u|^{p_i-2} u$  and  $c(x)$  (gain with  $c(x) |u|^{p-2} u$ ) and convert the problem into
$$\operatorname{div} \left( A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{B}(x) |\nabla u|^{q-2} \nabla u + c(x) |u|^{p-2} u = 0,$$
- $\bullet$  Derive Riccati type inequality in  $x$  variables.
- $\bullet$  Derive Riccati type inequality in  $t$  variable.
- $\bullet$  Use this inequality as a tool which transforms results from ODE to PDE.

**Theorem 3** (non-radial variant of Theorem 7F). Let  $t > 1$  and let  $\Omega \subset D(x, b)$  be an open domain with piecewise smooth boundary such that  $\operatorname{meas}(\Omega \cap S(x)) \neq 0$  for every  $x \in [a, b]$ . Let  $c(x) \geq 0$  on  $\Omega$  for every  $x$  and let  $c(x)$  be a function which is positive and continuously differentiable on  $\Omega$  and vanishes on the boundary and outside  $\Omega$ . Let  $t' = 1 + \frac{1}{t}$ ,  $t'' = \frac{t}{t-1}$  on  $\Omega$  and  $t' = \frac{t}{t-1}$  otherwise. In the former case suppose also that the integral

$$\int_{x_0}^x \frac{a(t)}{A(t)} dt + \int_{x_0}^x \frac{1}{A(t)} \left( A^{-1}(t) R'(t) - \frac{R'(t) A'(t)}{A(t)^2} \right) dt$$

which may singularity on  $\partial \Omega \cap D(x, b)$  is convergent for every  $x \in [a, b]$ . Denote

$$R(x) = (t-1)^{t-1} \int_{x_0}^x A^{-1}(t) dt$$

and

$$C(x) = \int_{x_0}^x c(x) + C_1(x) - \frac{p-1}{p} \frac{r''}{R(x)^{t-1}} \left( A^{-1}(x) R'(x) - \frac{R'(x) A'(x)}{A(x)^2} \right) dx,$$

where  $A(x)$  is defined by (7F) and  $C_1(x)$  is defined by (7F) and suppose that equation

$$\left( R(x) |r'|^{p-2} r' \right)' - C(x) |r'|^{q-2} r' = 0$$

has conjugate points on  $[a, b]$ .

If  $C(x) \geq 0$  on  $D(x, b)$ , then equation (7F) has no positive solution on  $D(x, b)$ .  
 If  $c(x) \geq 0$  on  $D(x, b)$ , then equation (7F) has no negative solution on  $D(x, b)$ .

EQUATION WITH MIXED POWERS

$$(p(t) r')' + (q(t) r) = \sum_{i=1}^m d_i(t) |r|^{p_i-2} r + c(t) \quad (2)$$

where  $a_0 > \dots > a_m > 1 > a_{m+1} > \dots > a_n > 0$ .

**Theorem A** (Sun/Wong (2017)). If for any  $T \geq 0$  there exists  $a_1, b_1, a_2, b_2$  such that  $T \leq a_1 < b_1 \leq a_2 < b_2$  and

$$\int_{a_1}^{b_1} d_i(t) dt > 0, \int_{a_2}^{b_2} d_i(t) dt > 0, \int_{a_1}^{b_1} c(t) dt \leq 0, \int_{a_2}^{b_2} c(t) dt \leq 0, \int_{a_1}^{b_1} d_i(t) dt > 0, \int_{a_2}^{b_2} d_i(t) dt > 0, \int_{a_1}^{b_1} c(t) dt \leq 0, \int_{a_2}^{b_2} c(t) dt \leq 0$$

and there exists a continuously differentiable function  $\phi(t)$  satisfying  $\phi(a_i) = \phi(b_i) = 0$ ,  $\phi(t) > 0$  on  $(a_i, b_i)$  and

$$\int_{a_1}^{b_1} \left( p(t) \phi'(t) - q(t) \phi(t) \right) dt \leq 0$$

for  $i = 1, 2$ , where

$$Q(t) = q(t) \phi(t) - \sum_{i=1}^m d_i(t) \phi(t) + c(t),$$

$\lambda_1 = \int_{a_1}^{b_1} \phi(t) dt$  and  $\eta_1 = 0, \dots, \eta_m$  are positive constants satisfying  $\sum_{i=1}^m \eta_i = 1$  and  $\sum_{i=1}^m \eta_i = 1$ , then all solutions of (7F) are oscillatory.

Using generalized AG inequality  $\sum_{i=1}^m \left( \frac{a_i}{p_i} \right)^{p_i} + a_{n+1} \geq 0$ ,  $a_i \geq 0$ ,  $\eta_i > 0$  and  $\sum_{i=1}^m \eta_i = 1$  we elaborate the right-hand side and terms with mixed points.

**Lemma 1.** Let either  $\eta > 0$  and  $c(x) \leq 0$  or  $\eta < 0$  and  $c(x) \geq 0$ . Let  $\eta_i > 0$  for numbers satisfying  $\sum_{i=1}^m \eta_i = 1$  and  $\eta_i > 0$ ,  $\sum_{i=1}^m \eta_i p_i = p$  and let  $c(x) \geq 0$  for every  $x$ . Then

$$\frac{\operatorname{div} \theta}{|p-2|} - c(x) + \sum_{i=1}^m d_i(x) |u|^{p_i-2} u \geq C(x) u,$$

where

$$C(x) = \left( \frac{c(x)}{p-2} \right) + \sum_{i=1}^m \left( \frac{d_i(x)}{p_i} \right) \left( \frac{c(x)}{p-2} \right)^{\frac{p_i-1}{p_i}}$$

**Remark:** The numbers  $\eta_i$  from Lemma 7F exist, if  $p_i > p$  for some  $i$ .

**Lemma 2.** Suppose  $c(x) \geq 0$ . Let  $\eta_i > 0$  for numbers satisfying  $\sum_{i=1}^m \eta_i = 1$  and  $\sum_{i=1}^m \eta_i p_i = p$ . Then

$$\frac{\operatorname{div} \theta}{|p-2|} - c(x) + \sum_{i=1}^m d_i(x) |u|^{p_i-2} u \geq C(x) u,$$

where

$$C(x) = \left( \frac{c(x)}{p-2} \right) + \sum_{i=1}^m \left( \frac{d_i(x)}{p_i} \right) \left( \frac{c(x)}{p-2} \right)^{\frac{p_i-1}{p_i}}$$

**Remark:** The numbers  $\eta_i$  from Lemma 7F exist, if  $p_i > p$  for some  $i$  and  $p_i < p$  for some  $i$ .

**Theorem 4.** Let  $C_1(x), A(x)$  and  $R(x)$  be defined as in Theorem 7F and let  $c(x) \geq 0$  and  $r(x) = 0$  on  $D(x, b)$ . Denote

$$C(x) = \int_{x_0}^x c(x) + C_1(x) - \frac{p-1}{p} \frac{r''}{R(x)^{t-1}} \left( A^{-1}(x) R'(x) - \frac{R'(x) A'(x)}{A(x)^2} \right) dx,$$

where  $C_1(x)$  is defined by (7F) if the equation

$$\left( R(x) |r'|^{p-2} r' \right)' - C(x) |r'|^{q-2} r' = 0$$

has conjugate points on  $[a, b]$ , then every solution of equation (7F) has zero on  $D(x, b)$ .

Similar theorems can be derived also for estimations of terms with mixed powers using different methods than AG inequality (see R. Mařík, Non-linear Analysis, TMA 71 (2012)).