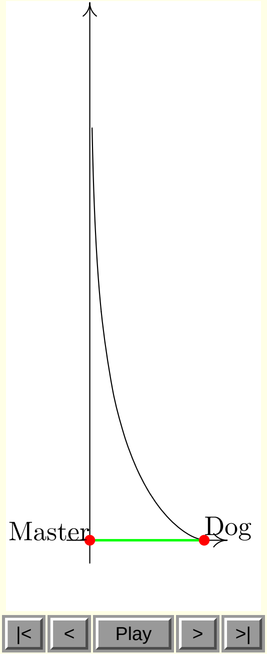
Tractrix

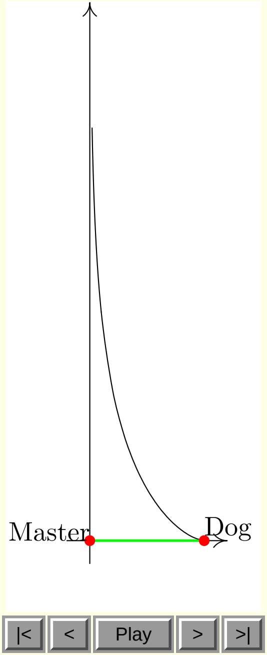


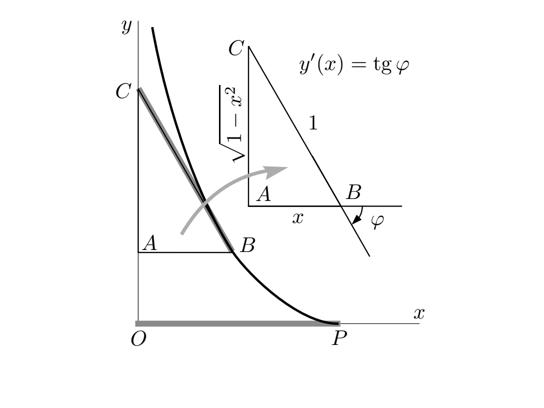
From
http://mathworld.wolfram.com/Tractrix.html:

- What is the *path* of an object starting off with a horizontal offset when it is dragged along by a string of *constant length* being pulled along a *straight vertical line*?

- By associating the object with a dog, the string with a leash, and the pull along a vertical line with the dog's master, the curve has the descriptive name hundkurve (**hound curve**) in German.

Tractrix





$$y' = -\frac{\sqrt{1-x^2}}{x}$$

$$y = -\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x$$

$$-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x$$

$$-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x$$

$$= \boxed{\begin{array}{c} 1-x^2=t^2 \\ -2x\,\mathrm{d}x = 2t\,\mathrm{d}t \\ -x\,\mathrm{d}x = t\,\mathrm{d}t \end{array}}$$

We use the substitution which removes the square root function.

$$-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x = -\int \frac{\sqrt{1-x^2}}{x^2}\,x\,\mathrm{d}x$$

$$= \boxed{\begin{array}{c} 1-x^2=t^2 \\ -2x\,\mathrm{d}x = 2t\,\mathrm{d}t \\ -x\,\mathrm{d}x = t\,\mathrm{d}t \end{array}}$$

We incorporate the terms from $\mathrm{d}x$ into the integral.

$$-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x = -\int \frac{\sqrt{1-x^2}}{x^2}\,x\,\mathrm{d}x$$

$$= \boxed{\begin{array}{c} 1-x^2=t^2 \\ -2x\,\mathrm{d}x = 2t\,\mathrm{d}t \\ -x\,\mathrm{d}x = t\,\mathrm{d}t \\ x^2 = 1-t^2 \end{array}} = \int \frac{t}{1-t^2}\,t\,\mathrm{d}t$$

We convert x^2 into the variable t and substitute.

$$-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x = -\int \frac{\sqrt{1-x^2}}{x^2}x\,\mathrm{d}x$$

$$= \boxed{\begin{gathered} 1-x^2=t^2 \\ -2x\,\mathrm{d}x = 2t\,\mathrm{d}t \\ -x\,\mathrm{d}x = t\,\mathrm{d}t \\ x^2 = 1-t^2 \end{gathered}} = \int \frac{t}{1-t^2}t\,\mathrm{d}t$$

$$= \int \frac{t^2}{1-t^2}\,\mathrm{d}t$$

We simplify.

$$-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x = -\int \frac{\sqrt{1-x^2}}{x^2}x\,\mathrm{d}x$$

$$= \boxed{\begin{aligned} 1-x^2 &= t^2 \\ -2x\,\mathrm{d}x &= 2t\,\mathrm{d}t \\ -x\,\mathrm{d}x &= t\,\mathrm{d}t \\ x^2 &= 1-t^2 \end{aligned}} = \int \frac{t}{1-t^2}t\,\mathrm{d}t$$

$$= \int \frac{t^2}{1-t^2}\,\mathrm{d}t = \int -1 + \frac{1}{1-t^2}\,\mathrm{d}t$$

We divide the numerator and convert into a sum of a polynomial and a rational function.

$$-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x = -\int \frac{\sqrt{1-x^2}}{x^2}x\,\mathrm{d}x$$

$$= \boxed{\begin{array}{c} 1-x^2=t^2 \\ -2x\,\mathrm{d}x = 2t\,\mathrm{d}t \\ -x\,\mathrm{d}x = t\,\mathrm{d}t \\ x^2 = 1-t^2 \end{array}} = \int \frac{t}{1-t^2}t\,\mathrm{d}t$$

$$= \int \frac{t^2}{1-t^2}\,\mathrm{d}t = \int -1+\frac{1}{1-t^2}\,\mathrm{d}t$$

$$= -t + \frac{1}{2}\ln\frac{1+t}{1-t}$$

We integrate.

$$-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x = -\int \frac{\sqrt{1-x^2}}{x^2} x\,\mathrm{d}x$$

$$= \boxed{\begin{array}{c} 1-x^2 = t^2 \\ -2x\,\mathrm{d}x = 2t\,\mathrm{d}t \\ -x\,\mathrm{d}x = t\,\mathrm{d}t \\ x^2 = 1-t^2 \end{array}} = \int \frac{t}{1-t^2} t\,\mathrm{d}t$$

$$= \int \frac{t^2}{1-t^2}\,\mathrm{d}t = \int -1 + \frac{1}{1-t^2}\,\mathrm{d}t$$

$$= -t + \frac{1}{2}\ln\frac{1+t}{1-t}$$

$$= -t + \frac{1}{2}\ln\frac{(1+t)^2}{(1-t)(1+t)} = -t + \frac{1}{2}\ln\frac{(1+t)^2}{1-t^2}$$

We simplify.

$$-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x = -\int \frac{\sqrt{1-x^2}}{x^2}x\,\mathrm{d}x$$

$$= \boxed{\begin{array}{c} 1-x^2 = t^2 \\ -2x\,\mathrm{d}x = 2t\,\mathrm{d}t \\ -x\,\mathrm{d}x = t\,\mathrm{d}t \\ x^2 = 1-t^2 \end{array}} = \int \frac{t}{1-t^2}t\,\mathrm{d}t$$

$$= \int \frac{t^2}{1-t^2}\,\mathrm{d}t = \int -1 + \frac{1}{1-t^2}\,\mathrm{d}t$$

$$= -t + \frac{1}{2}\ln\frac{1+t}{1-t}$$

$$= -t + \frac{1}{2}\ln\frac{(1+t)^2}{(1-t)(1+t)} = -t + \frac{1}{2}\ln\frac{(1+t)^2}{1-t^2}$$

$$= -t + \ln\frac{1+t}{\sqrt{1-t^2}}$$

We simplify.

$$
\begin{aligned}
-\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x &= -\int \frac{\sqrt{1-x^2}}{x^2} x\,\mathrm{d}x \\
&= \boxed{\begin{gathered} 1-x^2 = t^2 \\ -2x\,\mathrm{d}x = 2t\,\mathrm{d}t \\ -x\,\mathrm{d}x = t\,\mathrm{d}t \\ x^2 = 1-t^2 \end{gathered}} = \int \frac{t}{1-t^2} t\,\mathrm{d}t \\
&= \int \frac{t^2}{1-t^2}\,\mathrm{d}t = \int -1 + \frac{1}{1-t^2}\,\mathrm{d}t \\
&= -t + \frac{1}{2}\ln\frac{1+t}{1-t} \\
&= -t + \frac{1}{2}\ln\frac{(1+t)^2}{(1-t)(1+t)} = -t + \frac{1}{2}\ln\frac{(1+t)^2}{1-t^2} \\
&= -t + \ln\frac{1+t}{\sqrt{1-t^2}} \\
&= -\sqrt{1-x^2} + \ln\frac{1+\sqrt{1-x^2}}{x} + C
\end{aligned}
$$

Now we use the back substitution.

$$y = -\sqrt{1-x^2} + \ln \frac{1+\sqrt{1-x^2}}{x} + C$$

$$y = -\sqrt{1-x^2} + \ln \frac{1+\sqrt{1-x^2}}{x} + C$$

$$y(1) = 0$$

$$0 = -\sqrt{1-1} + \ln \frac{1+\sqrt{1-1}}{1} + C$$

$$C = 0$$

- We use the initial position of the dog to find the value of C.

$$y = -\sqrt{1-x^2} + \ln\frac{1+\sqrt{1-x^2}}{x} + C$$

$$y(1) = 0$$

$$0 = -\sqrt{1-1} + \ln\frac{1+\sqrt{1-1}}{1} + C$$

$$C = 0$$

$$y = -\sqrt{1-x^2} + \ln\frac{1+\sqrt{1-x^2}}{x}$$

The analytical form of the curve is

$$y = -\sqrt{1-x^2} + \ln\frac{1+\sqrt{1-x^2}}{x}.$$

Further reading

- http://mathworld.wolfram.com/Tractrix.html
- http://www.pballew.net/tractrix.html
- http://en.wikipedia.org/wiki/Tractrix