Linear algebra and linear systems

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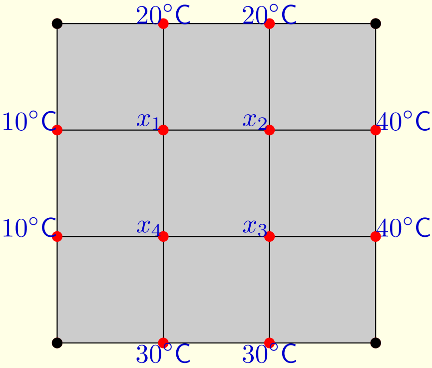
An important problem in the heat conduction is to find the stationary state for the temperature on the plate.
The Fourier's law of heat conduction:

$$\frac{\partial T}{\partial t} = k \cdot \left(\frac{\partial^2 T}{(\partial x)^2} + \frac{\partial^2 T}{(\partial y)^2} \right).$$

The left hand side vanishes in the stationary state. However, the equation is still very complicated even for zero left hand side and it is very hard to find alnalytical solution. Thus we will approximate the solution using more elementary techniques.

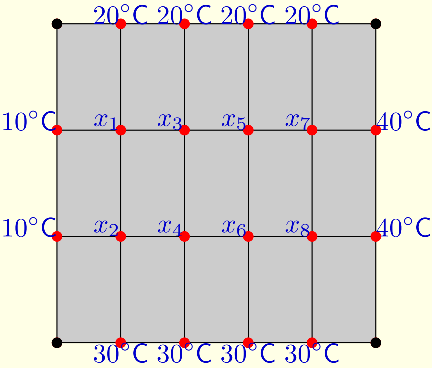


- Consider metal plate divided by the grid into 12 nodes (corners are not considered) as on the picture.
- The temperature on boudary is given and we have to find the temperature in inner nodes $x_1, \dots, x_4$.

Assumption: The temperature at every node is an average of the temepratures at its neighbors. Hence

$$\begin{cases} x_1 = \dfrac{1}{4}(30 + x_2 + x_4) \\ x_2 = \dfrac{1}{4}(60 + x_1 + x_3) \\ x_3 = \dfrac{1}{4}(70 + x_2 + x_4) \\ x_4 = \dfrac{1}{4}(40 + x_1 + x_3) \end{cases} \Rightarrow \begin{cases} 4x_1 - x_2 - x_4 = 30 \\ -x_1 + 4x_2 - x_3 = 60 \\ -x_2 + 4x_3 - x_4 = 70 \\ -x_1 - x_3 + 4x_4 = 40 \end{cases}$$

We get a linear system in four unknowns.



For more detailed computation we divide the plate into more nodes and get linear system

$$A\vec{x} = \vec{b},$$

where the colmun matrix is given by the temperatures on the boundary and the matrix A is band matrix (nonzero numbers are in a bend around the main diagonal)

$$A = \begin{pmatrix} 4 & -1 & -1 & & & & & \\ -1 & 4 & 0 & -1 & & & & \\ -1 & 0 & 4 & -1 & -1 & & & \\ & -1 & -1 & 4 & 0 & -1 & & \\ & & -1 & 0 & 4 & -1 & -1 & \\ & & & -1 & -1 & 4 & 0 & -1 \\ & & & & -1 & 0 & 4 & -1 \\ & & & & & -1 & -1 & 4 \end{pmatrix}$$

(zeros outside the bend around the main diagonal are omitted).

- The same approach is used in the practical problems. The only difference lies in the fact that we use a fine mesh with many nodes which yields system involving many (several thousands) equations. To solve such a system (or to ensure solvability of such a system) we use various concepts, ideas and tricks from linear algebra.
- In the practical problems the average used in this simple problem is replaced by a sililar, but slightly more refined expression, inspired by the second derivatives in the equation of the heat conduction.
- In a similar way we can approximate solution of some other important equations in phzsics, like the Navier–Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = g - \frac{\nabla p}{\rho} + \mu \nabla^2 \vec{v}$$

governing fluid dynamics and many others.

