Local maxima and minima
1-st derivative test
Interactive tests

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Look at three or four or twenty my quizzes and then fill in my questionnaire please!

1. Theory

Definition 1 (local extrema) *Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in Dom(f)$. The function f is said to take on its local maximum at the point x_0 if there exists a neighborhood $\mathrm{N}(x_0)$ of the point x_0 such that $f(x_0) \geq f(x)$ for all $x \in \mathrm{N}(x_0)$.*
The function f is said to take on its sharp local maximum at the point x_0 if there exists a neighborhood $\mathrm{N}(x_0)$ of the point x_0 such that $f(x_0) > f(x)$ for $x \in N(x_0) \setminus \{x_0\}$.
If the opposite inequalities hold, then the function f is said to take on its local minimum or sharp local minimum at the point x_0.
A common word for local minimum and maximum is a local extremum (pl. extrema). A common word for the sharp local maximum and the sharp local minimum is a sharp local extremum.

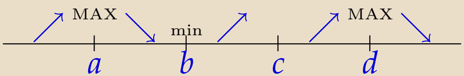
In plain words, the function f takes on its sharp local maximum at the point $x = a$ if there is no greater value than $f(a)$ in a neighborhood of the point a (i.e. close to the point $x = a$).

There exists a close relationship between the existence of a local extremum and monotonicity of the function, as the following theorem shows.

Theorem 1 (sufficient conditions for (non-)existence of local extrema) *Let f be a function defined and continuous in some neighborhood of x_0.*

- *If the function f is increasing in some left-hand side neighborhood of the point x_0 and decreasing in some right-hand side neighborhood of the point x_0, then the function f takes on its sharp local maximum at the point x_0.*
- *If the function f is decreasing in some left-hand side neighborhood of the point x_0 and increasing in some right-hand side neighborhood of the point x_0, then the function f takes on its sharp local minimum at the point x_0.*
- *If the function f is either increasing or decreasing in some (two-sided) neighborhood of the point x_0, then there is no local extremum of the function f at x_0.*

Graphically the situation can be represented in the following scheme.



The following definition concerns points with a very close relationship to local extrema.

Definition 2 (stationary point) *The point x_0 is said to be a stationary point of the function f if $f'(x_0) = 0$.*

Theorem 2 (relationship between stationary point and local extremum) *Let f be a function defined in x_0. If the function f takes on a local extremum at $x = x_0$, then the derivative of the function f at the point x_0 either does not exist or equals zero and hence $x = x_0$ is a stationary point of the function f.*

Theorem 3 (relationship between derivative and monotonicity) *Let f be a function. Suppose that f is differentiable on the open interval I.*

- *If $f'(x) > 0$ on I, then the function f is increasing on I.*
- *If $f'(x) < 0$ on I, then the function f is decreasing on I.*

The first derivative test.

1. We find a natural domain of the function f.
2. We find the derivative $f'(x)$ of the function $f(x)$.
3. We solve the equation $f'(x) = 0$. This yields the stationary points of the function.
4. We specify the domain of the derivative $f'(x)$.
5. Suppose that we have found all stationary points and all points of discontinuity of the derivative. We mark these points on the real line.
6. The real line is divided into several subintervals now. On each subinterval the derivative preserves its sign (in harmony with Theorem of Bolzano.
7. We choose an arbitrary test number in each subinterval, say ξ in the i-th subinterval. We find the value of the derivative at this number, $f'(\xi)$, and from the sign of the derivative we specify whether the function f is increasing or decreasing at ξ and, consequently, on the whole subinterval.
8. We find the points where the function is continuous and the type of monotonicity changes. At these points the function $f(x)$ takes on a local extremum.
9. To recognize the type of an extremum, we consider the type of monotonicity on the right and on the left.

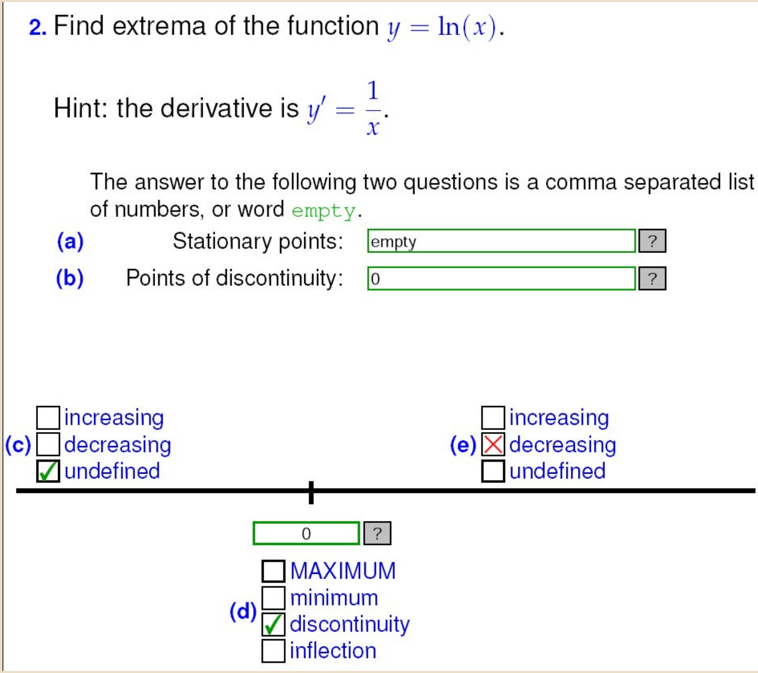
2. Test

Quiz In the following problems you are given a function and its derivative.

- Find stationary points of the function.
- Find discontinuities of the derivative.
- Mark these points on the real axis (well ordered).
- Find the type of monotonicity on subintervals. If there is a subinterval which does not belong to the domain of the function, mark `undefined`.
- Find local maxima, minima and stationary points where there is inflection.

The first two or three problems are very very simple, just to learn how to manage the test (see also the next page)

There are no stationary points, one point of discontinuity, the function is not defined on $(-\infty, 0)$ and increasing on $(0, \infty)$ (user entered wrong answer to the last question).



1. Find extrema of the function $y = x^2$.

Hint: the derivative is $y' = 2x$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

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2. Find extrema of the function $y = \ln(x)$.

Hint: the derivative is $y' = \dfrac{1}{x}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

3. Find extrema of the function $y = -\frac{1}{9}x^4 + \frac{2}{3}x^2$.

Hint: the derivative is $y' = -\frac{4}{9}x(x^2 - 3)$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

(i) ☐ increasing ☐ decreasing ☐ undefined

[ ] ? [ ] ? [ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(h) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

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4. Find extrema of the function $y = 4x^3 - 3x^4$.

Hint: the derivative is $y' = 12x^2(1 - x)$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

[ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

5. Find extrema of the function $y = \left(\frac{1+x}{1-x}\right)^2$.

Hint: the derivative is $y' = -4\frac{x+1}{(x-1)^3}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: ?

(b) Points of discontinuity: ?

(c) increasing / decreasing / undefined

(e) increasing / decreasing / undefined

(g) increasing / decreasing / undefined

?

(d) MAXIMUM / minimum / discontinuity / inflection

?

(f) MAXIMUM / minimum / discontinuity / inflection

6. Find extrema of the function $y = x + \dfrac{4}{x}$.

Hint: the derivative is $y' = \dfrac{(x-2)(x+2)}{x^2}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: ☐ ?

(b) Points of discontinuity: ☐ ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

(i) ☐ increasing ☐ decreasing ☐ undefined

☐ ? ☐ ? ☐ ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(h) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

7. Find extrema of the function $y = \dfrac{x}{(x+1)^2}$.

Hint: the derivative is $y' = \dfrac{1-x}{(x+1)^3}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

[ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

8. Find extrema of the function $y = x^2 - 2\ln x$.

Hint: the derivative is $y' = 2\frac{(x-1)(x+1)}{x}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

[ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

9. Find extrema of the function $y = \dfrac{x^2}{1-x}$.

Hint: the derivative is $y' = \dfrac{x(2-x)}{(1-x)^2}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

(i) ☐ increasing ☐ decreasing ☐ undefined

[ ] ? [ ] ? [ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(h) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

10. Find extrema of the function $y = 1 + x^2 - \frac{x^4}{2}$.

Hint: the derivative is $y' = -2x(x-1)(x+1)$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(d) [ ] ? ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(e) ☐ increasing ☐ decreasing ☐ undefined

(f) [ ] ? ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(g) ☐ increasing ☐ decreasing ☐ undefined

(h) [ ] ? ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(i) ☐ increasing ☐ decreasing ☐ undefined

11. Find extrema of the function $y = \dfrac{x-2}{\sqrt{x^2+1}}$.

Hint: the derivative is $y' = \dfrac{2x+1}{(x^2+1)^{\frac{3}{2}}}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c)
- ☐ increasing
- ☐ decreasing
- ☐ undefined

(e)
- ☐ increasing
- ☐ decreasing
- ☐ undefined

[ ] ?

(d)
- ☐ MAXIMUM
- ☐ minimum
- ☐ discontinuity
- ☐ inflection

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12. Find extrema of the function $y = \dfrac{x^2}{x^2+1}$.

Hint: the derivative is $y' = \dfrac{2x}{(1+x^2)^2}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c)
- [ ] increasing
- [ ] decreasing
- [ ] undefined

(e)
- [ ] increasing
- [ ] decreasing
- [ ] undefined

[ ] ?

(d)
- [ ] MAXIMUM
- [ ] minimum
- [ ] discontinuity
- [ ] inflection

13. Find extrema of the function $y = \left(\dfrac{1+x}{1-x}\right)^2$.

Hint: the derivative is $y' = -4\dfrac{x+1}{(x-1)^3}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: ☐ ?

(b) Points of discontinuity: ☐ ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

☐ ? ☐ ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

14. Find extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

Hint: the derivative is $y' = -8\frac{(x+1)^3}{(x-1)^5}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ? [ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

15. Find extrema of the function $y = \dfrac{x}{1+x^2}$.

Hint: the derivative is $y' = \dfrac{1-x^2}{(1+x^2)^2}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ? [ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

16. Find extrema of the function $y = \dfrac{1+x^2}{1-x^2} = -1 + \dfrac{2}{1-x^2}$.

Hint: the derivative is $y' = \dfrac{4x}{(1-x^2)^2}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(d) [ ] ? ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(e) ☐ increasing ☐ decreasing ☐ undefined

(f) [ ] ? ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(g) ☐ increasing ☐ decreasing ☐ undefined

(h) [ ] ? ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(i) ☐ increasing ☐ decreasing ☐ undefined

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17. Find extrema of the function $y = \dfrac{\ln^2 x}{x}$.

Hint: the derivative is $y' = \dfrac{\ln x(2 - \ln x)}{x^2}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: ?

(b) Points of discontinuity: ?

(c) increasing / decreasing / undefined

(e) increasing / decreasing / undefined

(g) increasing / decreasing / undefined

(i) increasing / decreasing / undefined

? ? ?

(d) MAXIMUM / minimum / discontinuity / inflection

(f) MAXIMUM / minimum / discontinuity / inflection

(h) MAXIMUM / minimum / discontinuity / inflection

18. Find extrema of the function $y = \dfrac{\ln x}{\sqrt{x}}$.

Hint: the derivative is $y' = \dfrac{2 - \ln x}{2x^{\frac{3}{2}}}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ? [ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

19. Find extrema of the function $y = \dfrac{e^x}{1+x}$.

Hint: the derivative is $y' = \dfrac{xe^x}{(x+1)^2}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ? [ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

20. Find extrema of the function $y = x^{\frac{2}{3}} e^{-x}$.

Hint: the derivative is $y' = \dfrac{2-3x}{3\sqrt[3]{x}} e^{-x}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ? [ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

21. Find extrema of the function $y = x^2 e^{-x}$.

Hint: the derivative is $y' = x(2 - x)e^{-x}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ? [ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

22. Find extrema of the function $y = xe^{\frac{1}{x}}$.

Hint: the derivative is $y' = \dfrac{x-1}{x}e^{\frac{1}{x}}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ? [ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

23. Find extrema of the function $y = \frac{x^2}{2} - \ln(1+x)$.

Hint: the derivative is $y' = \frac{x^2+x-1}{x+1}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(e) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(g) ☐ increasing ☐ decreasing ☐ undefined

24. Find extrema of the function $y = x - \ln(1 + x^2)$.

Hint: the derivative is $y' = \dfrac{(x-1)^2}{x^2+1}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c)
- ☐ increasing
- ☐ decreasing
- ☐ undefined

(e)
- ☐ increasing
- ☐ decreasing
- ☐ undefined

[ ] ?

(d)
- ☐ MAXIMUM
- ☐ minimum
- ☐ discontinuity
- ☐ inflection

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25. Find extrema of the function $y = \dfrac{x^2}{2} + \dfrac{8}{x^3}$.

Hint: the derivative is $y' = \dfrac{x^5 - 24}{x^4}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(e) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

(g) ☐ increasing ☐ decreasing ☐ undefined

26. Find extrema of the function $y = (x+1)^{10} e^{-x}$.

Hint: the derivative is $y' = e^{-x}(x+1)^9(9-x)$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c) ☐ increasing ☐ decreasing ☐ undefined

(e) ☐ increasing ☐ decreasing ☐ undefined

(g) ☐ increasing ☐ decreasing ☐ undefined

[ ] ?

(d) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

[ ] ?

(f) ☐ MAXIMUM ☐ minimum ☐ discontinuity ☐ inflection

27. Find extrema of the function $y = \frac{x^2}{2^x}$.

Hint: the derivative is $y' = \frac{x(2 - x\ln 2)}{2^x}$.

The answer to the following two questions is a comma separated list of numbers, or word `empty`.

(a) Stationary points: [ ] ?

(b) Points of discontinuity: [ ] ?

(c)
- ☐ increasing
- ☐ decreasing
- ☐ undefined

(e)
- ☐ increasing
- ☐ decreasing
- ☐ undefined

(g)
- ☐ increasing
- ☐ decreasing
- ☐ undefined

[ ] ?

(d)
- ☐ MAXIMUM
- ☐ minimum
- ☐ discontinuity
- ☐ inflection

[ ] ?

(f)
- ☐ MAXIMUM
- ☐ minimum
- ☐ discontinuity
- ☐ inflection