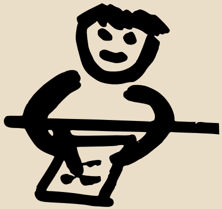
Second order linear differential equation
Variation of constants
Interactive tests

Robert Mařík

May 28, 2007

Look at three or four or twenty my quizzes and then fill in my questionnaire please!



1. Theory

Definition 1 (second order linear differential equation) *Let p, q be real numbers, f be functions continuous on the interval I. The equation*

$$y'' + py' + qy = f(x) \tag{1}$$

is said to be a second order linear differential equation with constant coefficients.

Definition 2 (special types of 2nd order LDE) *Equation (1) is said to be homogeneous if $f(x) = 0$ for all $x \in I$ and nonhomogeneous otherwise.*

Definition 3 (associated homogeneous equation) *Consider nonhomogeneous equation (1). Homogeneous equation*

$$y'' + py' + qy = 0. \tag{2}$$

with the left-hand side identical with equation (1) is called a homogeneous equation associated to the nonhomogeneous equation (1).

Theorem 1 (general solution of nonhomogeneous second order LDE) *Let $y_1(x)$ and $y_2(x)$ be fundamental system of solutions of the homogeneous LDE (2) and $y_p(x)$ be an arbitrary particular solution of the nonhomogeneous LDE (1). Then the function*

$$y(x, C_1, C_2) = C_1 y_1(x) + C_2 y_2(x) + y_p(x), \qquad C_1 \in \mathbb{R},\ C_2 \in \mathbb{R} \tag{3}$$

is a general solution of the nonhomogeneous LDE (1).

Theorem 2 (solution of the nonhomogeneous LDE with constant coefficients) *Consider the second order LDE with constant coefficients*

$$y'' + py' + qy = f(x). \tag{4}$$

Let $y_1(x)$ and $y_2(x)$ be a fundamental system of solutions of the associated homogeneous equation. Let $A(x)$ and $B(x)$ be differentiable functions with derivatives $A'(x)$ and $B'(x)$ which satisfy the system of equations

$$\begin{aligned} A'(x)y_1(x) + B'(x)y_2(x) &= 0, \\ A'(x)y_1'(x) + B'(x)y_2'(x) &= f(x). \end{aligned} \tag{5}$$

The function $y_p(x)$ defined by the relation

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) \tag{6}$$

is a particular solution of (4). The function

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x), \quad C_1 \in \mathbb{R}, C_2 \in \mathbb{R},$$

is a general solution of (4).

2. Tests

On the following pages you find some tests which guide you throug the process of solving second order LDE via variation of constants.

- The functions from the fundamental system are not unique. If you choose the fundamental system as on the lectures, your answer is unique up to the order of the functions.
- If your answer on the fundamental system is correct (or if you scratch your answer and asks for help by pressing "?"), follow the hint which tells you which one of these two functions is $y_1(x)$ and which one is $y_2(x)$.
- In the linear system for A', B' choose y_1 and y_2 as suggested by the quiz. Then the answers to the questions on A' and B' are unique.
- The functions A and B are unique up to an additive constant.
- The particular solution is unique up to an additive factor which is a linear combination of the functions from the fundamental system. This test is able to grade the users answer on particular solution correctly. Thus both answers $y = 1$ and $y = 1 + \sin(x) + 2\cos(x)$ are correct for the equation $y'' + y = 1$.
- In the field with general solution all functions $y = 1 + A\sin(x) + B\cos(x)$, $y = 1 + \sin(x) + A\cos(x) + 3B\sin(x)$ or $y = 1 + A\sin(x) - B(\cos(x) - \sin(x))$ are equivalent, as excepted from the fact that there are many posibilities how to write the general solution.

Quiz Solve $y'' + y = 1$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

[ ] ?

2. Find the fundamental system (two functions separated by commas).

[ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

where y_1=... and y_2=... Write the system for A' and B'

$[\ ]\cdot A'(x) + [\ ]\cdot B'(x) = 0,$

$[\ ]\cdot A'(x) + [\ ]\cdot B'(x) = [\ ]$?

4. Solving this system we get

$A'(x) =$ [ ] ?

$B'(x) =$ [ ] ?

5. Integrating we get

$A(x) =$ [ ] ?

$B(x) =$ [ ] ?

6. Find particular solution.

$y_p =$ [ ] ?

7. Find general solution, use constants A and B.

$y =$ [ ] ?

Quiz Solve $y'' + y = x$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).
   [ ] ?
2. Find the fundamental system (two functions separated by commas).
   [ ] ?
3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$
   where y_1=... and y_2=...
   Write the system for A' and B'

   $[\ ] \cdot A'(x) + [\ ] \cdot B'(x) = 0,$
   $[\ ] \cdot A'(x) + [\ ] \cdot B'(x) = [\ ]$?
4. Solving this system we get
   $A'(x) =$ [ ] ?
   $B'(x) =$ [ ] ?
5. Integrating we get
   $A(x) =$ [ ] ?
   $B(x) =$ [ ] ?
6. Find particular solution.
   $y_p =$ [ ] ?
7. Find general solution, use constants A and B.
   $y =$ [ ] ?

Quiz Solve $y'' - 3y' + 2y = e^x$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

   [ ] ?

2. Find the fundamental system (two functions separated by commas).

   [ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

   where y_1=... and y_2=...

   Write the system for A' and B'

   $[\ ] \cdot A'(x) + [\ ] \cdot B'(x) = 0,$

   $[\ ] \cdot A'(x) + [\ ] \cdot B'(x) = [\ ]$?

4. Solving this system we get

   $A'(x) =$ [ ] ?

   $B'(x) =$ [ ] ?

5. Integrating we get

   $A(x) =$ [ ] ?

   $B(x) =$ [ ] ?

6. Find particular solution.

   $y_p =$ [ ] ?

7. Find general solution, use constants A and B.

   $y =$ [ ] ?

Quiz Solve $y'' + 4y = x$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

[ ] ?

2. Find the fundamental system (two functions separated by commas).

[ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

where y_1=... and y_2=... Write the system for A' and B'

$$[\ ]\cdot A'(x) + [\ ]\cdot B'(x) = 0,$$
$$[\ ]\cdot A'(x) + [\ ]\cdot B'(x) = [\ ]$$

?

4. Solving this system we get

$A'(x) =$ [ ] ?

$B'(x) =$ [ ] ?

5. Integrating we get

$A(x) =$ [ ] ?

$B(x) =$ [ ] ?

6. Find particular solution.

$y_p =$ [ ] ?

7. Find general solution, use constants A and B.

$y =$ [ ] ?

Quiz Solve $y'' + y = x + 2$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

   [ ] ?

2. Find the fundamental system (two functions separated by commas).

   [ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

   where y_1=... and y_2=... Write the system for A' and B'

   [ ] $\cdot A'(x) +$ [ ] $\cdot B'(x) = 0,$

   [ ] $\cdot A'(x) +$ [ ] $\cdot B'(x) =$ [ ] ?

4. Solving this system we get

   $A'(x) =$ [ ] ?

   $B'(x) =$ [ ] ?

5. Integrating we get

   $A(x) =$ [ ] ?

   $B(x) =$ [ ] ?

6. Find particular solution.

   $y_p =$ [ ] ?

7. Find general solution, use constants A and B.

   $y =$ [ ] ?

Quiz Solve $y'' - 2y' + y = \dfrac{e^x}{\sqrt{4 - x^2}}$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

[ ] ?

2. Find the fundamental system (two functions separated by commas).

[ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

where y_1=... and y_2=... Write the system for A' and B'

[ ] $\cdot\, A'(x) +$ [ ] $\cdot\, B'(x) = 0,$

[ ] $\cdot\, A'(x) +$ [ ] $\cdot\, B'(x) =$ [ ] ?

4. Solving this system we get

$A'(x) =$ [ ] ?

$B'(x) =$ [ ] ?

5. Integrating we get

$A(x) =$ [ ] ?

$B(x) =$ [ ] ?

6. Find particular solution.

$y_p =$ [ ] ?

7. Find general solution, use constants A and B.

$y =$ [ ] ?

Quiz Solve $y'' - 2y' + y = \frac{e^x}{x}$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

[ ] ?

2. Find the fundamental system (two functions separated by commas).

[ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

where y_1=... and y_2=... Write the system for A' and B'

[ ] $\cdot A'(x) +$ [ ] $\cdot B'(x) = 0,$

[ ] $\cdot A'(x) +$ [ ] $\cdot B'(x) =$ [ ] ?

4. Solving this system we get

$A'(x) =$ [ ] ?

$B'(x) =$ [ ] ?

5. Integrating we get

$A(x) =$ [ ] ?

$B(x) =$ [ ] ?

6. Find particular solution.

$y_p =$ [ ] ?

7. Find general solution, use constants A and B.

$y =$ [ ] ?

Quiz Solve $y'' - 5y' + 6y = e^{2x}$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

   [ ] ?

2. Find the fundamental system (two functions separated by commas).

   [ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

   where y_1=... and y_2=... Write the system for A' and B'

   $[\ ] \cdot A'(x) + [\ ] \cdot B'(x) = 0,$

   $[\ ] \cdot A'(x) + [\ ] \cdot B'(x) = [\ ]$?

4. Solving this system we get

   $A'(x) =$ [ ] ?

   $B'(x) =$ [ ] ?

5. Integrating we get

   $A(x) =$ [ ] ?

   $B(x) =$ [ ] ?

6. Find particular solution.

   $y_p =$ [ ] ?

7. Find general solution, use constants A and B.

   $y =$ [ ] ?

Quiz Solve $y'' + y = \dfrac{\cos x}{\sin x}$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

[ ] ?

2. Find the fundamental system (two functions separated by commas).

[ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

where y_1=... and y_2=... Write the system for A' and B'

$$[\ ]\cdot A'(x) + [\ ]\cdot B'(x) = 0,$$
$$[\ ]\cdot A'(x) + [\ ]\cdot B'(x) = [\ ]$$

?

4. Solving this system we get

$A'(x) =$ [ ] ?

$B'(x) =$ [ ] ?

5. Integrating we get

$A(x) =$ [ ] ?

$B(x) =$ [ ] ?

6. Find particular solution.

$y_p =$ [ ] ?

7. Find general solution, use constants A and B.

$y =$ [ ] ?

Quiz Solve $y'' + 2y' - 3y = e^x$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

[ ] ?

2. Find the fundamental system (two functions separated by commas).

[ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

where y_1=... and y_2=... Write the system for A' and B'

$[\ ] \cdot A'(x) + [\ ] \cdot B'(x) = 0,$

$[\ ] \cdot A'(x) + [\ ] \cdot B'(x) = [\ ]$?

4. Solving this system we get

$A'(x) =$ [ ] ?

$B'(x) =$ [ ] ?

5. Integrating we get

$A(x) =$ [ ] ?

$B(x) =$ [ ] ?

6. Find particular solution.

$y_p =$ [ ] ?

7. Find general solution, use constants A and B.

$y =$ [ ] ?

Quiz Solve $y'' - 4y = e^{2x}$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).

   ?

2. Find the fundamental system (two functions separated by commas).

   ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$

   where y_1=... and y_2=...

   Write the system for A' and B'

   $\square \cdot A'(x) + \square \cdot B'(x) = 0,$

   $\square \cdot A'(x) + \square \cdot B'(x) = \square$

   ?

4. Solving this system we get

   $A'(x) =$?

   $B'(x) =$?

5. Integrating we get

   $A(x) =$?

   $B(x) =$?

6. Find particular solution.

   $y_p =$?

7. Find general solution, use constants A and B.

   $y =$?

Quiz Solve $y'' - 4y = xe^x$.

1. Find the characteristic equation for the associated homogeneous equation (use z as a variable).
   [ ] ?

2. Find the fundamental system (two functions separated by commas).
   [ ] ?

3. Look for the particular solution in the form $y_p = A(x)y_1(x) + B(x)y_2(x)$
   where y_1=... and y_2=... Write the system for A' and B'

   [ ] $\cdot A'(x) +$ [ ] $\cdot B'(x) = 0,$
   [ ] $\cdot A'(x) +$ [ ] $\cdot B'(x) =$ [ ] ?

4. Solving this system we get
   $A'(x) =$ [ ] ?
   $B'(x) =$ [ ] ?

5. Integrating we get
   $A(x) =$ [ ] ?
   $B(x) =$ [ ] ?

6. Find particular solution.
   $y_p =$ [ ] ?

7. Find general solution, use constants A and B.
   $y =$ [ ] ?