First order linear differential equation
Variation of constant
Interactive tests

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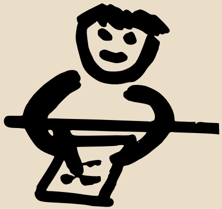
Look at three or four or twenty my quizzes and then fill in my questionnaire please!



1. Theory

Definition 1 (first order linear ODE) *Let a, b be continuous function on I. The equation*

$$y' + a(x)y = b(x) \tag{1}$$

is said to be the first order linear ordinary differential equation (shortly LDE). If $b(x) \equiv 0$ on I, then the equation is called homogeneous and nonhomogeneous otherwise.

Definition 2 (associated homogeneous equation) *Consider nonhomogeneous equation (1). Homogeneous equation*

$$y' + a(x)y = 0 \tag{2}$$

with the left-hand side identical to the left-hand side of (1) is said to be a homogeneous equation associated with the nonhomogeneous equation (1).

In this file, the form (1) is refferred as a *normal form* of the first order linear differential equation.

Theorem 1 (general solution of LDE)

- *If $y_p(x)$ is a particular solution of nonhomogeneous LDE and $y_0(x)$ is a general solution of the associated homogeneous LDE, then the function*

  $$y(x) = y_p(x) + y_0(x)$$

  is a general solution of nonhomogeneous LDE.
- *If $y_p(x)$ is a particular solution of nonhomogeneous LDE and $y_0(x)$ a nontrivial particular solution of the associated homogeneous LDE, then the function*

  $$y(x) = y_p(x) + Cy_0(x)$$

  is a general solution of nonhomogeneous LDE.

If $y_{p0}(x)$ is a nontrivial particular solution of the homogeneous LDE, then the general solution of this equation is $y(x,C) = Cy_{p0}(x)$. We will look for the particular solution of the nonhomogeneous equation in the form

$$y_p(x) = K(x)y_{p0}(x), \tag{3}$$

where $K(x)$ is a differentiable function. Since this function replaces the constant in the formula for general solution of the nonhomogeneous equation, this method is called *variation of the constant*. We claim that $y_p(x)$ is a solution of (1) and we have to find $K(x)$. Differentiating (3) we obtain

$$y_p'(x) = K'(x)y_{p0}(x) + K(x)y_{p0}'(x),$$

and substitution into (1) yields

$$K'(x)y_{p0}(x) + K(x)y_{p0}'(x) + a(x)K(x)y_{p0}(x) = b(x),$$

and

$$K'(x)y_{p0}(x) + K(x)\left[y_{p0}'(x) + a(x)y_{p0}(x)\right] = b(x).$$

Since $y_{p0}(x)$ is a solution of the homogeneous LDE, the expression in brackets vanishes and hence

$$K'(x)y_{p0}(x) = b(x). \tag{4}$$

We solve the last equation for $K'(x)$ and after integration we obtain $K(x)$. Substitution of the function $K(x)$ into (3) gives a particular solution of the nonhomogeneous equation and by Theorem 1 we can write the general solution of the nonhomogeneous equation (1).

2. Tests

- On the following pages you are given linear equations (one equation per page) and the general solution of the associated homogeneous equation. This suggests the form of the particular solution of the nonhomogeneous equation. You have to find this particular solution and then the general solution (of the original nonhomogeneous equation, of course).
- You have to find the derivative. Consider K to be a function and K' to be its derivative. Don't write $K(x)$ — this is parsed as $K*x$, i.e. "the function K multiplied with x". When you find the derivative and write it into the corresponding field, this derivative will be substituted into original nonhomogeneous equation automatically.
- Solve the equation from the last step with respect to K' (K should cancel!) and find K by integration.
- From K find the particular and then general solution of the nonhomogenous equation.‘
- As usual, you can see the answer by pressing ? button. But don't use this button too much, please. All (or at least almost all) computations are easy. We have to learn the technique in these quizzes. The problems on exam are harder[1]!
- As usual: If you have any comments or suggestions concerning this test, let me know, please!

[1]this means slightly longer computation of derivatives and integrals and so on

Quiz 1. Solve the 1-st order linear ODE

$$y' - \frac{1}{x} y = 1.$$

The general solution of the associated homogeneous equation is $y_0(x) = Cx$, $C \in \mathbb{R}$. We look for the particular solution in the form $y_p(x) = Kx$, where K is a function of x.

1. We differentiate the expression

$$y_p(x) = Kx.$$

The term K is supposed to be a function of x. Write the derivative to the next field. Write `K` and `K'` for the function K and its derivative, respectively.

$y_p' =$ [ ] ?

2. Substitution into original equation gives (you should see the replacement for y', if your last answer is correct).

$$\underbrace{\text{y'}}_{y'} - \frac{1}{x} \underbrace{K \cdot x}_{y} = 1$$

Isolating K' we get: $K' =$ [ ] ?

3. Integrating we get $K =$ [ ] ?

4. Particular solution of nonhomogeneous equation

$y_p(x) =$ [ ] ?

5. General solution of nonhomogeneous equation, use C as an arbitrary real consatnt

$y(x) = y_p(x) + y_0(x) =$ [ ] ?

Quiz 2. Solve the 1-st order linear ODE

$$y' - \frac{2}{x}y = 1.$$

The general solution of the associated homogeneous equation is $y_0(x) = Cx^2$, $C \in \mathbb{R}$. We look for the particular solution in the form $y_p(x) = Kx^2$, where K is a function of x.

1. We differentiate the expression

$$y_p(x) = Kx^2.$$

The term K is supposed to be a function of x. Write the derivative to the next field. Write `K` and `K'` for the function K and its derivative, respectively.

$y_p' =$ [ ] ?

2. Substitution into original equation gives (you should see the replacement for y', if your last answer is correct).

$$\underbrace{\text{y'}}_{y'} - \frac{2}{x}\underbrace{K \cdot x^2}_{y} = 1$$

Isolating K' we get: $K' =$ [ ] ?

3. Integrating we get $K =$ [ ] ?

4. Particular solution of nonhomogeneous equation

$y_p(x) =$ [ ] ?

5. General solution of nonhomogeneous equation, use C as an arbitrary real consatnt

$y(x) = y_p(x) + y_0(x) =$ [ ] ?

Quiz 3. Solve the 1-st order linear ODE

$$y' + \frac{1}{x}y = x.$$

The general solution of the associated homogeneous equation is $y_0(x) = C\frac{1}{x}$, $C \in \mathbb{R}$. We look for the particular solution in the form $y_p(x) = K\frac{1}{x}$, where K is a function of x.

1. We differentiate the expression

$$y_p(x) = K\frac{1}{x}.$$

The term K is supposed to be a function of x. Write the derivative to the next field. Write `K` and `K'` for the function K and its derivative, respectively.

$y_p' =$ [ ] ?

2. Substitution into original equation gives (you should see the replacement for y', if your last answer is correct).

$$\underbrace{\text{y'}}_{y'} + \frac{1}{x}\underbrace{K \cdot \frac{1}{x}}_{y} = x$$

Isolating K' we get: $K' =$ [ ] ?

3. Integrating we get $K =$ [ ] ?

4. Particular solution of nonhomogeneous equation

$y_p(x) =$ [ ] ?

5. General solution of nonhomogeneous equation, use C as an arbitrary real consatnt

$y(x) = y_p(x) + y_0(x) =$ [ ] ?

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Quiz 4. Solve the 1-st order linear ODE

$$y' - 2xy = xe^{x^2}.$$

The general solution of the associated homogeneous equation is $y_0(x) = Ce^{x^2}$, $C \in \mathbb{R}$. We look for the particular solution in the form $y_p(x) = Ke^{x^2}$, where K is a function of x.

1. We differentiate the expression

$$y_p(x) = Ke^{x^2}.$$

The term K is supposed to be a function of x. Write the derivative to the next field. Write `K` and `K'` for the function K and its derivative, respectively.

$y_p' =$?

2. Substitution into original equation gives (you should see the replacement for y', if your last answer is correct).

$$\underbrace{\text{y'}}_{y'} - 2x \underbrace{K \cdot e^{x^2}}_{y} = xe^{x^2}$$

Isolating K' we get: $K' =$?

3. Integrating we get $K =$?

4. Particular solution of nonhomogeneous equation

$y_p(x) =$?

5. General solution of nonhomogeneous equation, use C as an arbitrary real consatnt

$y(x) = y_p(x) + y_0(x) =$?

Quiz 5. Solve the 1-st order linear ODE

$$y' + 3y = 1.$$

The general solution of the associated homogeneous equation is $y_0(x) = Ce^{-3x}$, $C \in \mathbb{R}$. We look for the particular solution in the form $y_p(x) = Ke^{-3x}$, where K is a function of x.

1. We differentiate the expression

$$y_p(x) = Ke^{-3x}.$$

The term K is supposed to be a function of x. Write the derivative to the next field. Write `K` and `K'` for the function K and its derivative, respectively.

$y_p' =$ [ ] ?

2. Substitution into original equation gives (you should see the replacement for y', if your last answer is correct).

$$\underbrace{\text{y'}}_{y'} + 3\underbrace{K \cdot e^{-3x}}_{y} = 1$$

Isolating K' we get: $K' =$ [ ] ?

3. Integrating we get $K =$ [ ] ?

4. Particular solution of nonhomogeneous equation
$y_p(x) =$ [ ] ?

5. General solution of nonhomogeneous equation, use C as an arbitrary real consatnt
$y(x) = y_p(x) + y_0(x) =$ [ ] ?

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Quiz 6. Solve the 1-st order linear ODE

$$y' - \frac{2x}{x^2+1}y = (x^2+1)^2.$$

The general solution of the associated homogeneous equation is $y_0(x) = C(x^2+1)$, $C \in \mathbb{R}$. We look for the particular solution in the form $y_p(x) = K(x^2+1)$, where K is a function of x.

1. We differentiate the expression

$$y_p(x) = K(x^2+1).$$

The term K is supposed to be a function of x. Write the derivative to the next field. Write K and K' for the function K and its derivative, respectively.

$y_p' =$ [ ] ?

2. Substitution into original equation gives (you should see the replacement for y', if your last answer is correct).

$$\underbrace{\text{y'}}_{y'} - \frac{2x}{x^2+1}\underbrace{K\cdot(x^2+1)}_{y} = (x^2+1)^2$$

Isolating K' we get: $K' =$ [ ] ?

3. Integrating we get $K =$ [ ] ?

4. Particular solution of nonhomogeneous equation

$y_p(x) =$ [ ] ?

5. General solution of nonhomogeneous equation, use C as an arbitrary real consatnt

$y(x) = y_p(x) + y_0(x) =$ [ ] ?