Integration by substitution

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Look at three or four or twenty my quizzes and then fill in my questionnaire please!

1. Theory

Theorem 1 (the 1st method of substitution) *Let $f(t)$ be a function continuous on I, $\phi(x)$ be a function differentiable on J and $\phi(J) = I$ holds. Then for $x \in J$*

$$\int f(\phi(x))\phi'(x)dx = \boxed{\begin{array}{c}\phi(x) = t\\ \phi'(x)dx = dt\end{array}} \Rightarrow \int f(t)dt \tag{1}$$

holds, where we substitute $t = \phi(x)$ on the right hand side.

Theorem 2 (the 2nd method of substitution) *Let $f(x)$ be a function continuous on the open interval I, $\phi(t)$ be a differentiable function which has no stationary point on the interval J and $\phi(J) = I$. Then*

$$\int f(x)dx = \boxed{\begin{array}{c}x = \phi(t)\\ dx = \phi'(t)dt\end{array}} \Rightarrow \int f(\phi(t))\phi'(t)dt \tag{2}$$

holds on I, where we substitute $t = \phi^{-1}(x)$, on the right-hand side. Here $\phi^{-1}(x)$ denotes the inverse function to the function $\phi(x)$.

2. Test

- Integrate by substitution. Follow the substitution suggested for the integral and find the relationship between differentials, transform the integral into the intehgral in new varaible, integrate wioth respect to this new variable and use back substitution to find the integral in the original variable x.
  - The red field expects an equation involving differentials $\mathrm{d}x$ and $\mathrm{d}t$.
  - The green fields expect a function in variable t.
  - The gray field expects a function in variable x.
- The correct answers to the first set of questions are $2x\mathrm{d}x = \mathrm{d}t$, $\frac{1}{2}e^t$, $\frac{1}{2}e^t$ and $\frac{1}{2}e^{x^2}$, respectively.
- As usual, you can see the answer by pressing ? button. But don't use this button too much, please. All (or at least almost all) computations are easy. We have to learn the technique in these quizzes. The problems on exam are harder[1]!
- As usual: If you have any comments or suggestions concerning this test, let me know, please!

[1] this means slightly longer computation of derivatives and integrals and so on

Quiz 1.

1. $\int xe^{x^2}\mathrm{d}x =$ $x^2 = t$ $\Rightarrow \int \qquad \mathrm{d}t$

$= \qquad = \qquad + C$?

2. $\int xe^{-x^2}\mathrm{d}x =$ $-x^2 = t$ $\Rightarrow \int \qquad \mathrm{d}t$

$= \qquad = \qquad + C$?

3. $\int xe^{4-x^2}\mathrm{d}x =$ $4 - x^2 = t$ $\Rightarrow \int \qquad \mathrm{d}t$

$= \qquad = \qquad + C$?

4. $\int xe^{-x^2}\mathrm{d}x =$ $x^2 = t$ $\Rightarrow \int \qquad \mathrm{d}t$

$= \qquad = \qquad + C$?

5. $\int x^2 e^{1-x^3} \mathrm{d}x = \boxed{1-x^3=t} \Rightarrow \int \boxed{} \mathrm{d}t$

$= \boxed{} = \boxed{} + C$?

6. $\int \sin x \cos x \mathrm{d}x = \boxed{\cos x = t} \Rightarrow \int \boxed{} \mathrm{d}t$

$= \boxed{} = \boxed{} + C$?

7. $\int \sin x \cos x \mathrm{d}x = \boxed{\sin x = t} \Rightarrow \int \boxed{} \mathrm{d}t$

$= \boxed{} = \boxed{} + C$?

8. $\int \frac{1}{x} \ln x \mathrm{d}x = \boxed{\ln x = t} \Rightarrow \int \boxed{} \mathrm{d}t$

$= \boxed{} = \boxed{} + C$?

9. $\int \frac{x}{x^4+1}\mathrm{d}x = \boxed{x^2 = t} \Rightarrow \int \quad \mathrm{d}t$

$= \quad = \quad + C$?

10. $\int \frac{e^{\sqrt{x+1}}}{x+1}\mathrm{d}x = \boxed{\sqrt{x+1} = t} \Rightarrow \int \quad \mathrm{d}t$

$= \quad = \quad + C$?

11. $\int \sin^2 x \cos x \mathrm{d}x = \boxed{\sin x = t} \Rightarrow \int \quad \mathrm{d}t$

$= \quad = \quad + C$?

12. $\int \sin^2 x \cos^3 x \mathrm{d}x = \boxed{\sin x = t} \Rightarrow \int \quad \mathrm{d}t$

$= \quad = \quad + C$?

13. $\int 3x\sin(x^2+1)\,\mathrm{d}x = \boxed{\begin{array}{c} x^2+1=t \\ \boxed{\qquad\qquad} \end{array}} \Rightarrow \int \boxed{\qquad\qquad\qquad} \,\mathrm{d}t$

$= \boxed{\qquad\qquad} = \boxed{\qquad\qquad} + C$?

14. $\int x\sqrt{x^2+1}\,\mathrm{d}x = \boxed{\begin{array}{c} x^2+1=t \\ \boxed{\qquad\qquad} \end{array}} \Rightarrow \int \boxed{\qquad\qquad\qquad} \,\mathrm{d}t$

$= \boxed{\qquad\qquad} = \boxed{\qquad\qquad} + C$?

15. $\int x\sqrt{x^2+1}\,\mathrm{d}x = \boxed{\begin{array}{c} x^2+1=t^2 \\ \boxed{\qquad\qquad} \end{array}} \Rightarrow \int \boxed{\qquad\qquad\qquad} \,\mathrm{d}t$

$= \boxed{\qquad\qquad} = \boxed{\qquad\qquad} + C$?

16. $\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \boxed{\begin{array}{c} x+1=t^2 \\ \boxed{\qquad\qquad} \end{array}} \Rightarrow \int \boxed{\qquad\qquad\qquad} \,\mathrm{d}t$

$= \boxed{\qquad\qquad} = \boxed{\qquad\qquad} + C$?

17. $\displaystyle\int \frac{x}{\sqrt{x+1}+1}\,\mathrm{d}x =$ $x+1=t^2$ $\Rightarrow \displaystyle\int \quad \mathrm{d}t$

$= \quad = \quad + C$?

18. $\displaystyle\int \frac{x}{1+\sqrt{x^2+1}}\,\mathrm{d}x =$ $x^2+1=t^2$ $\Rightarrow \displaystyle\int \quad \mathrm{d}t$

$= \quad = \quad + C$?

19. $\displaystyle\int \frac{\sqrt{x+2}+1}{\sqrt{x+2}-1}\,\mathrm{d}x =$ $x+2=t^2$ $\Rightarrow \displaystyle\int \quad \mathrm{d}t$

$= \quad = \quad + C$?

20. $\displaystyle\int \frac{\sin^2 x\cos x}{1+\sin x}\,\mathrm{d}x =$ $\sin x = t$ $\Rightarrow \displaystyle\int \quad \mathrm{d}t$

$= \quad = \quad + C$?

21. $\int \frac{\sin x \cos x}{1+\cos x}\,\mathrm{d}x =$ $\cos x = t$ $\Rightarrow \int \quad \mathrm{d}t$

$= \quad = \quad + C$?

22. $\int \frac{e^x}{1+e^{2x}}\,\mathrm{d}x =$ $e^x = t$ $\Rightarrow \int \quad \mathrm{d}t$

$= \quad = \quad + C$?

23. $\int \frac{\sqrt{x+1}}{x+2}\,\mathrm{d}x =$ $x+1 = t^2$ $\Rightarrow \int \quad \mathrm{d}t$

$= \quad = \quad + C$?

24. $\int \frac{1}{1+e^x}\,\mathrm{d}x =$ $e^x = t$ $\Rightarrow \int \quad \mathrm{d}t$

$= \quad = \quad + C$?