Taylor polynomial

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Motivation. Let f be a real fucntion with the folowing properties.

- The value $f(x_0)$ is known.
- We have no effective method to evaluate the function at the other points, different from x_0.
- The value of the first n derivatives of the function f at the point x_0 is known.

We state the following problem: *Find an n-degree polynomial which approximates the function f in the neighbourhood of the point x_0 with the smallest possible error.*
The solution of this problem is introduced in the following definition.

Definition (Taylor polynomial)**.** Let $n \in \mathbb{N}$ be a positive integer and f be a function defined at x_0 which has derivatives up to the order n at x_0. The polynomial

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0}{n!}(x - x_0)^n$$

is called an n-*degree Taylor polynomial of the function f at x_0.*

The point x_0 is called a *centre* of this polynomial.

Theorem 1 (Taylor)**.** Let f be a function defined in a neighborhood $N(x_0)$ of the point x_0. Let $f'(x_0)$, $f''(x_0)$, ..., $f^{(n)}(x_0)$ be the values of the first n derivatives of the function f at the point x_0. Suppose that the $(n+1)$st derivative is continuous in $N(x_0)$. Then for all $x \in N(x_0)$

$$f(x) = T_n(x) + R_{n+1}(x),$$

holds, where $T_n(x)$ is the n-degree Taylor polynomial of the function f in the point x_0 and $R_{n+1}(x)$ a remainder. The remainder can be written in the form

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \tag{1}$$

where c is some number between x and x_0.

Remark 1. From the formula (1) it follows that the remainder is small if

- $(x - x_0)$ is small, i.e., x is close to x_0,
- $n!$ is large, i.e., n is large,
- $|f^{(n+1)}(x)|$ is numerically small in the neighborhood of x_0

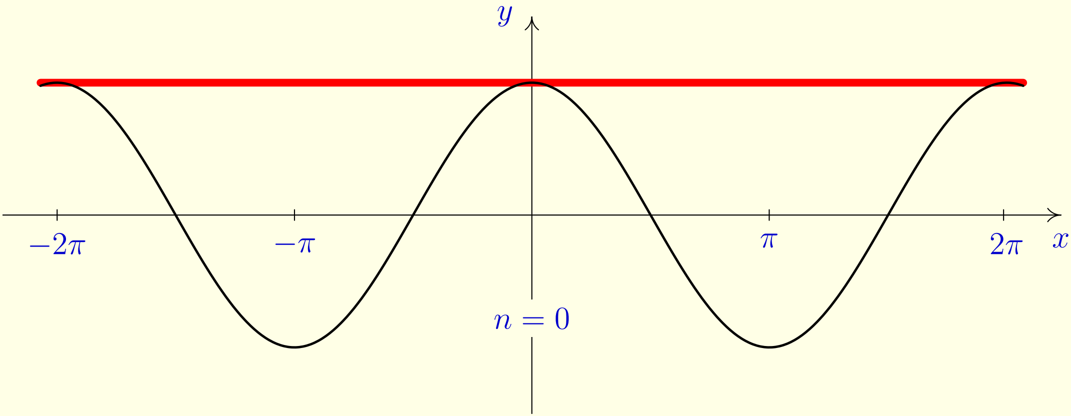
If these conditions are satisfied, we can write

$$f(x) \approx T_n(x) \tag{2}$$

in the neighborhood of x_0 and the error is small.

Taylor approximation of the function $f: \ y = \cos x$ at $x = 0$.

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$



There is no shorter approximation

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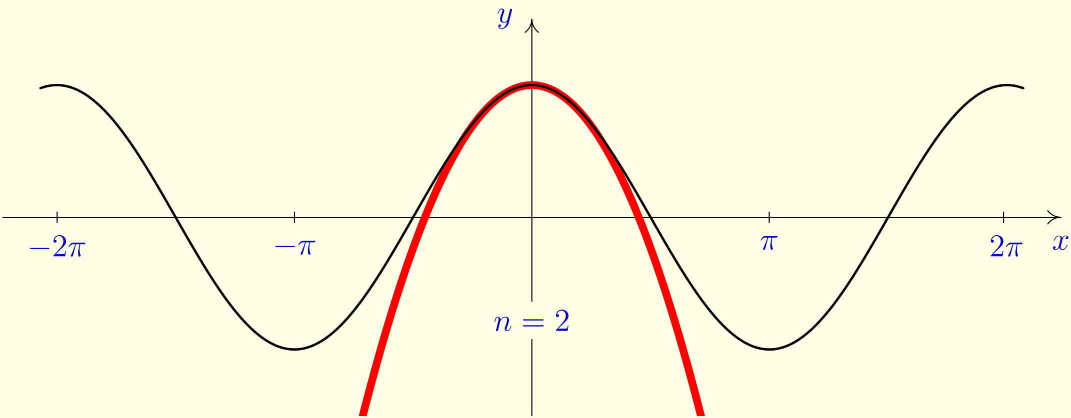
$$\cos x \approx 1 = T_0(x)$$

Taylor approximation of the function $f:\ y = \cos x$ at $x = 0$.

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$



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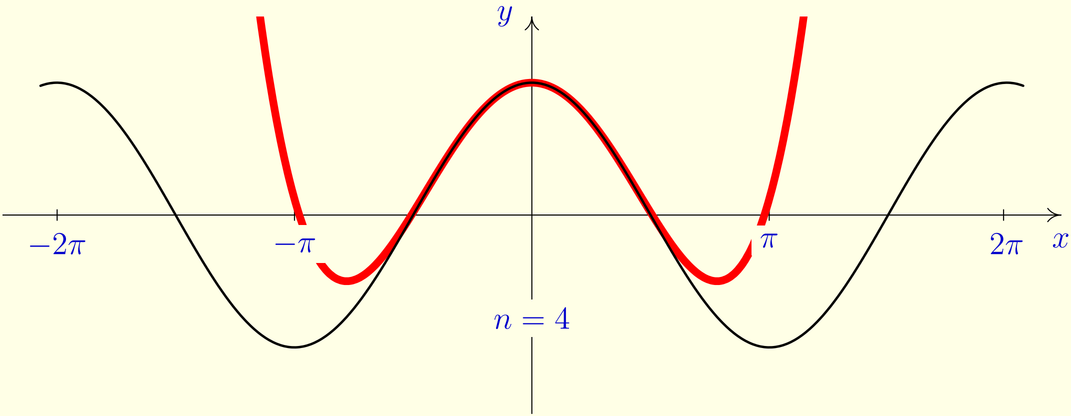
$$\cos x \approx 1 - \frac{x^2}{2} = T_2(x)$$

Taylor approximation of the function $f:\ y=\cos x$ at $x=0$.

$$T_n(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$



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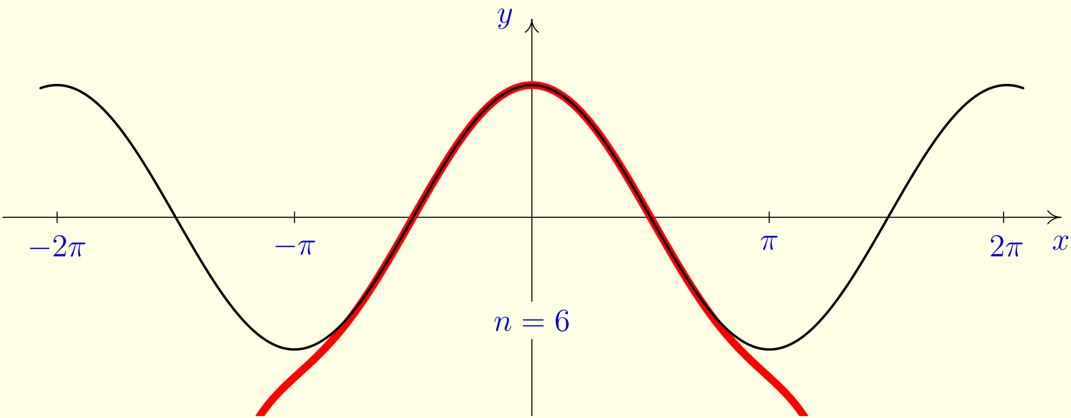
$$\cos x\approx 1-\frac{x^2}{2}+\frac{x^4}{24}=T_4(x)$$

Taylor approximation of the function $f: \ y = \cos x$ at $x = 0$.

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$



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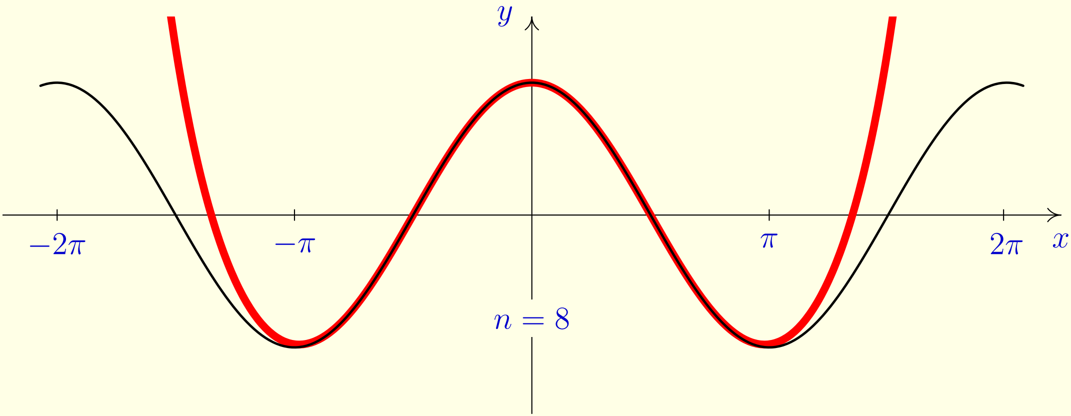
$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} = T_6(x)$$

Taylor approximation of the function $f: \ y = \cos x$ at $x = 0$.

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$



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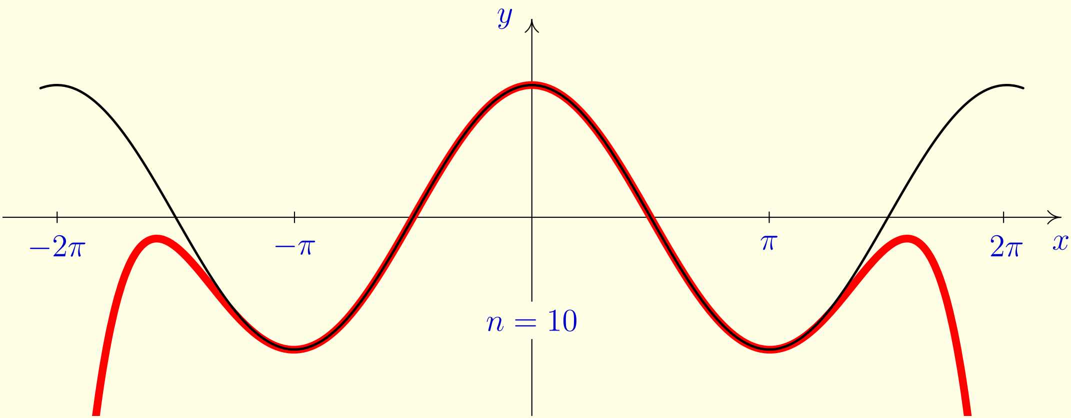
$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} = T_8(x)$$

Taylor approximation of the function $f: \; y = \cos x$ at $x = 0$.

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$



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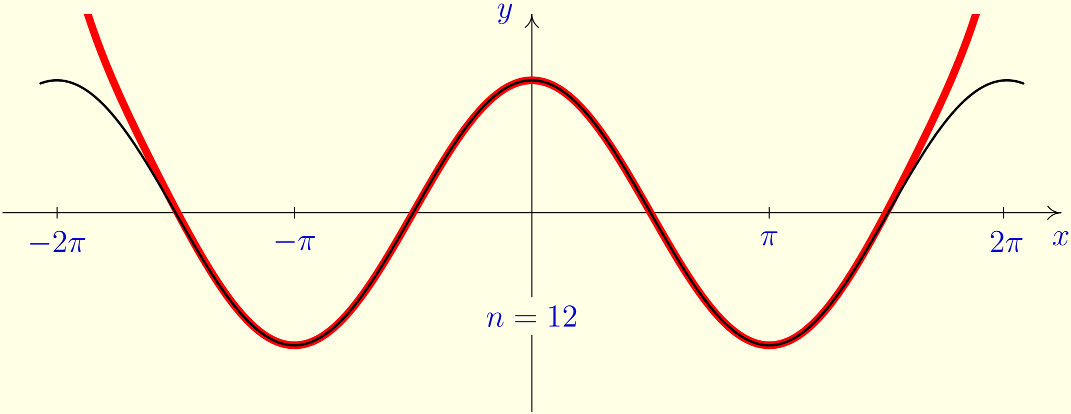
$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} = T_{10}(x)$$

Taylor approximation of the function $f:\ y=\cos x$ at $x=0$.

$$T_n(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$



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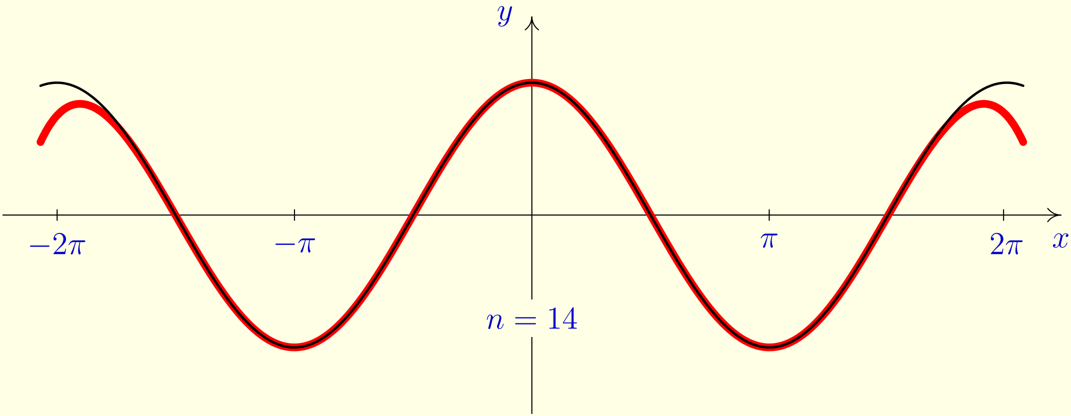
$$\cos x\approx 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\frac{x^{12}}{12!}=T_{12}(x)$$

Taylor approximation of the function $f: \ y = \cos x$ at $x = 0$.

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$



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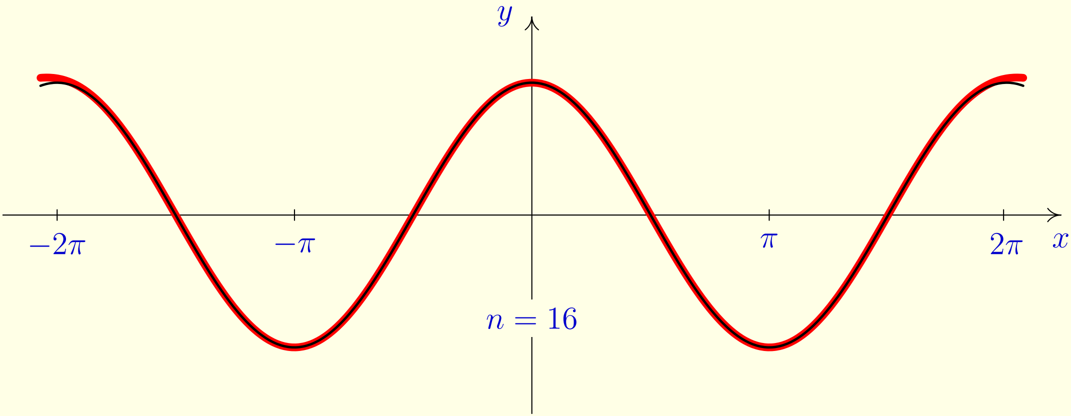
$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} = T_{14}(x)$$

Taylor approximation of the function $f:\ y=\cos x$ at $x=0$.

$$T_n(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$



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Better approximation is too long.

$$\cos x\approx 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\frac{x^{12}}{12!}-\frac{x^{14}}{14!}+\frac{x^{16}}{16!}=T_{16}(x)$$

Applications

- Polynomial approximation for Einstein's formula for total energy of moving object – einstein.pdf – the last part.
- Polynomial approximation for $\int e^{-x^2}\,\mathrm{d}x$ (the error function) – gauss-int.pdf.

Further reading

- http://archives.math.utk.edu/visual.calculus/6/power.3/

Applets

- http://www2.norwich.edu/frey/TaylorPolynomials/
- http://math.furman.edu/~dcs/java/taylor.html