Riemann integral

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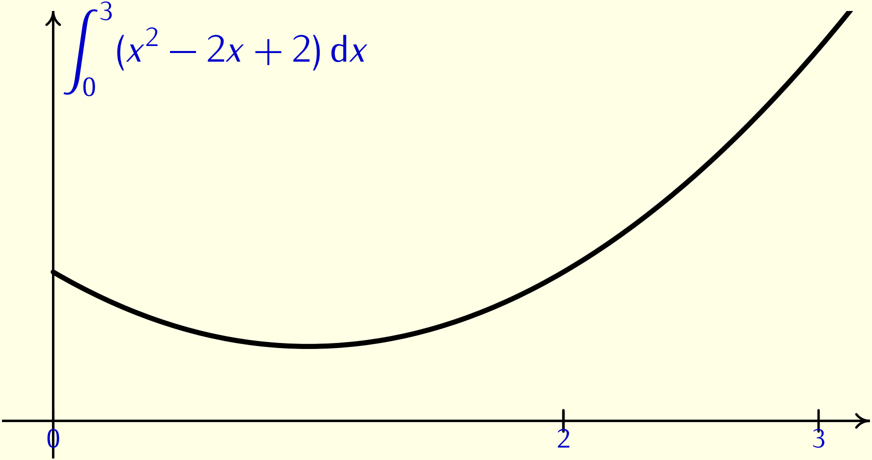
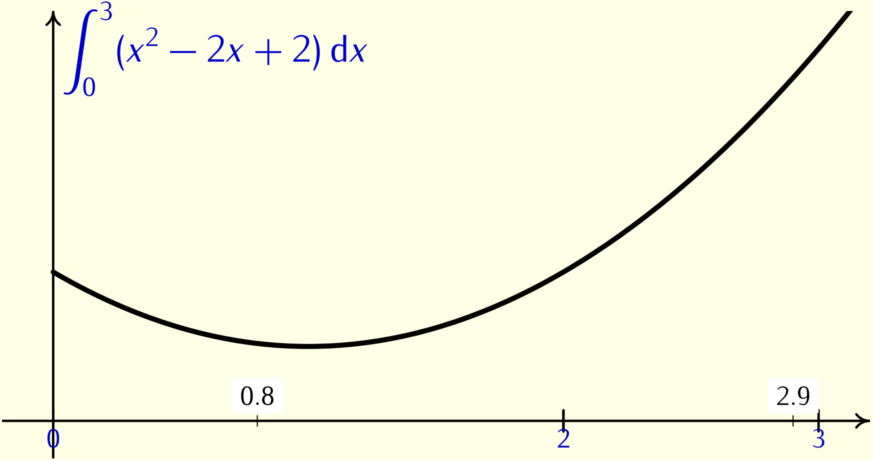
1 Introduction and geometric ideas

The following pages describe main ideas connected to the definition of the Riemann integral.



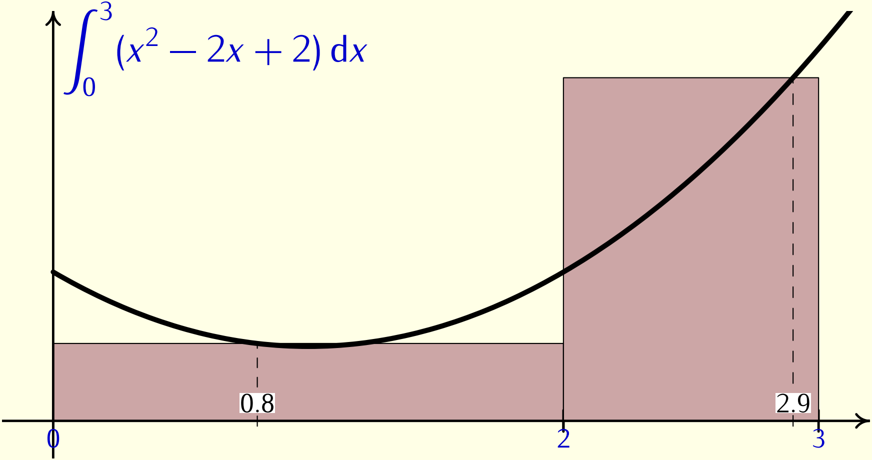
The curve on the picture is $y = x^2 - 2x + 2$
We divide the interval. The norm is 2.



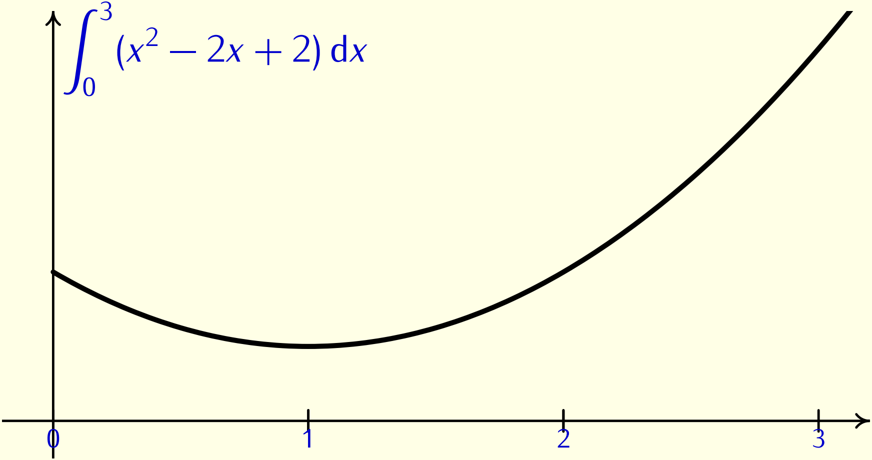
The curve on the picture is $y = x^2 - 2x + 2$

We divide the interval. The norm is 2.

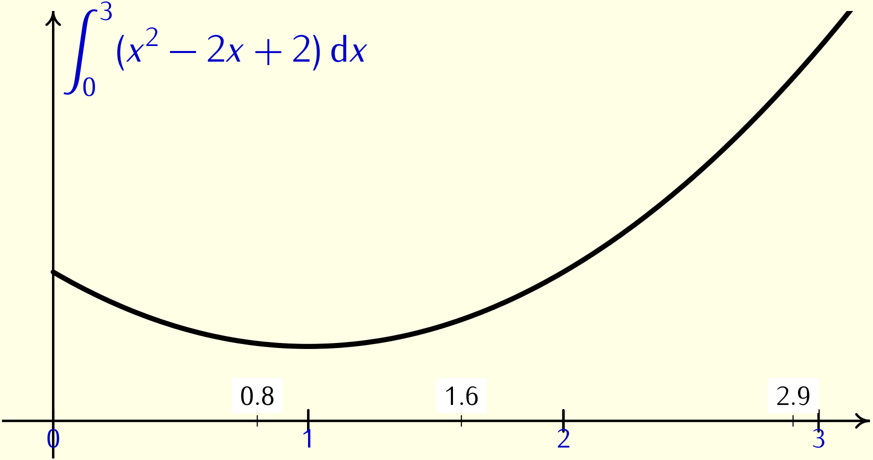
We consider one number in each interval.



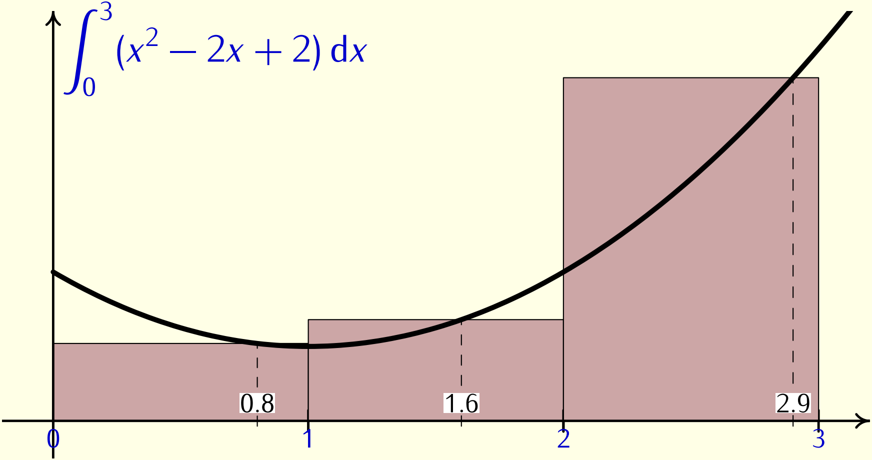
The curve on the picture is $y = x^2 - 2x + 2$
We divide the interval. The norm is 2.
We consider one number in each interval.
We find the corresponding integral sum – the red area.



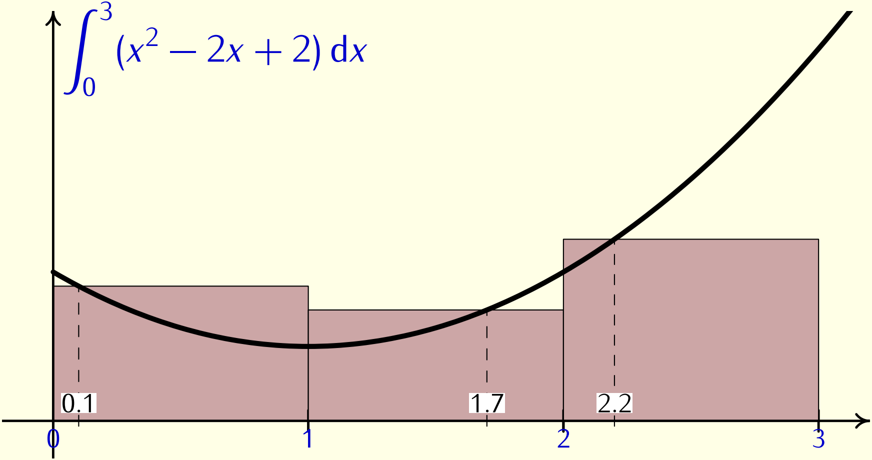
We consider finer division. The norm is 1.



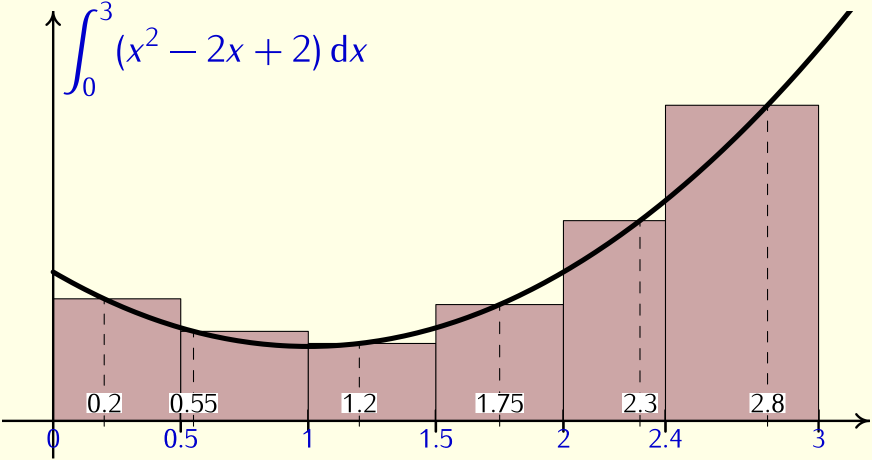
We consider finer division. The norm is 1.
We choose one number in each subinterval.



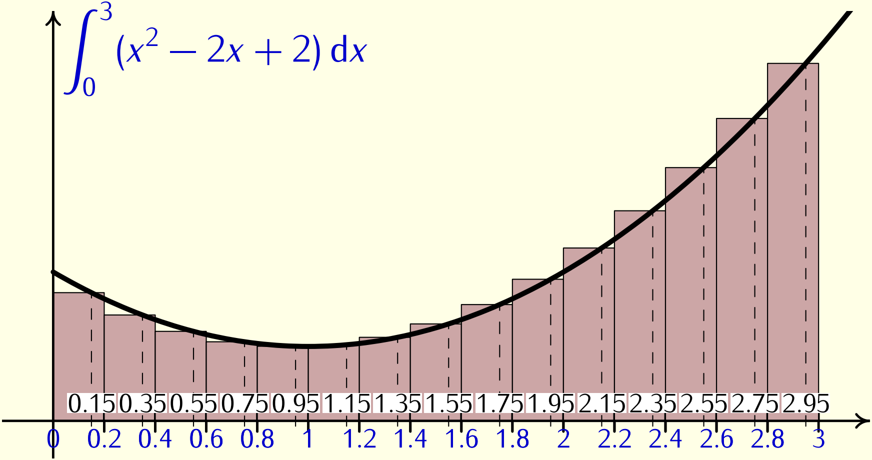
We consider finer division. The norm is 1.
We choose one number in each subinterval.
We find the integral sum – the red area.



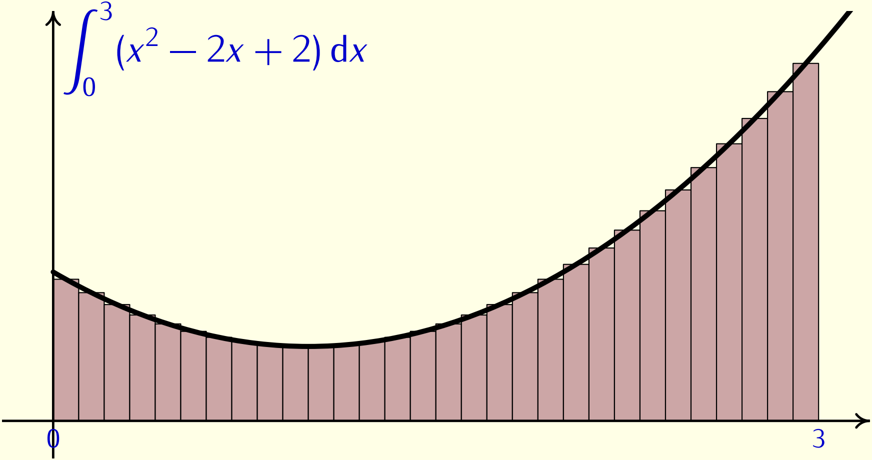
The division remains. The norm is 1.
We consider another numbers from the subintervals.
We find the corresponding integral sum – the red area.
The integral sum depends on the division an on the particular choice of the numbers from subintervals.



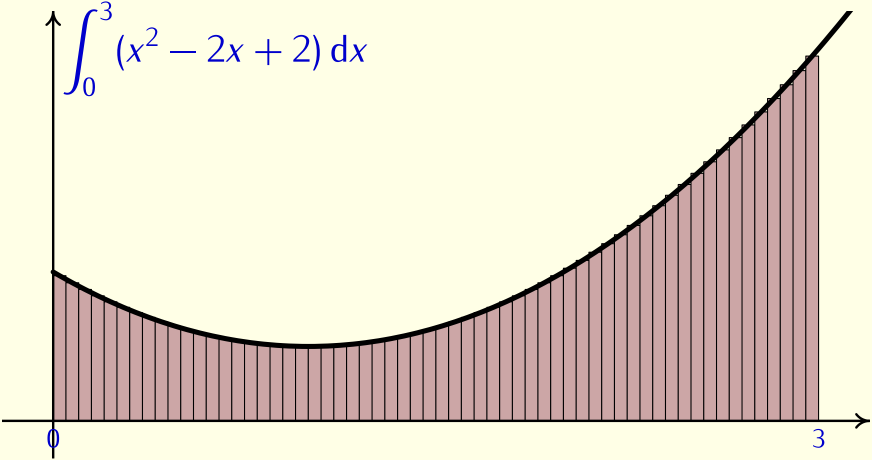
We consider fined division. The norm of this division is 0.6. (Why?)
We choose one number in each subinterval and evaluate the corresponding integral sum.



We consider fined and finer division. The norm of this division is 0.2
We choose the numbers in subintervals and find the integral sum.



We consider finer and finer division. The norm is 0.1.



We consider finer and finer division. The norm of this division is 0.05.
If the integral sum converges to a real number I as the norm of the division approaches zero and the limit value does not depend on the particular choice of the numbers in subintervals, the function is said to be integrable (in the sense of Riemann) and the value I is the definite integral.

2 Riemann integral

Definition (partition of the interval). Let $[a,b]$ be a closed interval. Under a *partition of the interval* $[a,b]$ we understand the finite sequence $D = \{x_0, x_1, \dots, x_n\}$ of points from the interval $[a,b]$ with property

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

Under a *norm of the partition* D we understand the maximal of the distances of two consecutive points in the partition. The norm of the partition D is denoted by $\nu(D)$. Hence $\nu(D) = \max\{x_i - x_{i-1}, 1 \leq i \leq n\}$.

Definition (integral sum). Let $[a, b]$ be a closed interval and f be a function defined and bounded on $[a, b]$. Let D be a partition of the interval $[a, b]$. Let $R = \{\xi_1, \dots, \xi_n\}$ be a finite sequence of points from the interval $[a, b]$ satisfying $x_{i-1} \leq \xi_i \leq x_i$ for $i = 1..n$ (i.e. R contains one arbitrary number from each of the subintervals formed by the partition D). The sum

$$\sigma(f, D, R) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

is said to be an *integral sum of the function* f associated to the *partition* D and the *choice of the numbers* ξ_i *in* R.

Definition (Riemann integral). Let $[a, b]$ be a closed interval and f be a function defined and bounded on the interval $[a, b]$. Let D_n be a sequence of partitions of the interval $[a, b]$ which satisfies $\nu(D) \to 0$ for $n \to \infty$ and R_n be a sequence of the corresponding choices of numbers ξ from this interval. The function f is said to be *integrable in the sense of Riemann on the interval* $[a, b]$ if there exists a real number $I \in \mathbb{R}$ with property

$$\lim_{n\to\infty} \sigma(f, D_n, R_n) = I$$

for every sequence D_n, which satisfies $\lim_{n\to\infty} \nu(D_n) = 0$ and for arbitrary particular choice of the points ξ in R_n. The number I is said to be a *Riemann integral of the function* f *on the interval* $[a, b]$. We write

$$I = \int_a^b f(x)\,\mathrm{d}x.$$

Definition (extension of Riemann integral). For $a > b$ we define $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$. Further we define $\int_a^a f(x)\,\mathrm{d}x = 0$.

Theorem 1 (sufficient conditions for integrability).

1. The function which is continuous on $[a, b]$ is integrable on $[a, b]$ (in the sense of Riemann).
2. The function which is bounded on $[a, b]$ and contains at most finite number of discontinuities on this interval is integrable (in the sense of Riemann).

Theorem 2 (linearity with respect to the function). Let f, g be functions integrable on $[a, b]$ and c be a real number. The following relations are valid.

$$\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x,$$

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x.$$

Theorem 3 (additivity with respect to the domain of the integration). Let f be a function integrable on $[a, b]$. Let $c \in (a, b)$ be arbitrary. Then the functions f is integrable on both intervals $[a, c]$ and $[c, b]$ and

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

holds.

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$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

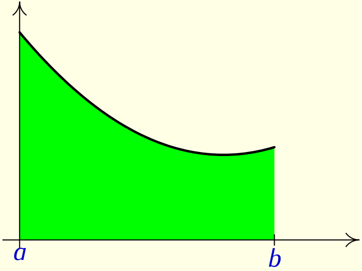
holds.

Theorem 4 (monotonicity with respect to the function). Let f and g be functions integrable on $[a, b]$ such that $f(x) \leq g(x)$ for $x \in (a, b)$. Then $\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$ holds.

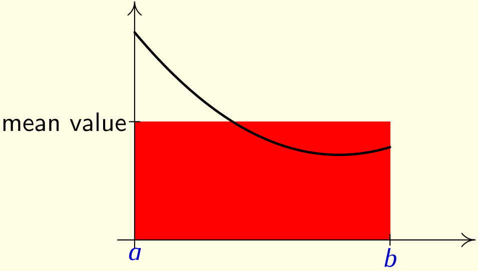
Theorem 5 (mean value theorem). Let f be continuous on the closed interval $[a, b]$. There exists at least one number $\mu \in [a, b]$ with the property $f(\mu)(b - a) = \int_a^b f(x)\,\mathrm{d}x$.

Definition (mean value). The value $f(\mu)$ of the function f in the point μ from the preceding theorem is said to be a *mean value of the function f on the interval* $[a, b]$.

Function



Mean value



3 Newton – Leibniz Theorem

Theorem 6 (Newton–Leibniz). Let $f(x)$ be integrable in the sense of Riemann on $[a, b]$. Let $F(x)$ be a function continuous on $[a, b]$ which is an antiderivative of the function f on the interval (a, b). Then

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$$

holds.

Example.

$$\begin{aligned}\int_0^3 (x^2 - 2x + 2)\,\mathrm{d}x &= \left[\frac{x^3}{3} - x^2 + 2x\right]_0^3 \\ &= \frac{3^3}{3} - 3^2 + 2.3 - \left[\frac{0^3}{3} - 0^2 + 2.0\right] \\ &= 3^2 - 3^2 + 6 - 0 \\ &= 6\end{aligned}$$

Integration by parts in the definite integral

$$\begin{aligned}\int_a^b u(x)v'(x)\,\mathrm{d}x &= \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x)\,\mathrm{d}x,\\ &= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,\mathrm{d}x.\end{aligned} \tag{1}$$

Substitution in definite integral. When integrating by parts in the definite integral, we hace two possibilities:

- We evaluate the indefinite integral separately and use the Newton–Leibniz formula.
- We use the formulas below and transform the limits of the integral. The advantage is that the back substitution is no more necessary.

$$\int_a^b f(\varphi(x))\varphi'(x)\,\mathrm{d}x \;\boxed{\begin{array}{c} t=\varphi(x)\\ \mathrm{d}t=\varphi'(x)\,\mathrm{d}x\\ c=\varphi(a)\\ d=\varphi(b)\end{array}} = \int_c^d f(t)\,\mathrm{d}t \tag{2}$$

$$\int_a^b f(x)\,\mathrm{d}x \;\boxed{\begin{array}{c} x=\varphi(t)\\ \mathrm{d}x=\varphi'(t)\,\mathrm{d}t\\ t=\varphi^{-1}(x)\\ c=\varphi^{-1}(a)\\ d=\varphi^{-1}(b)\end{array}} = \int_c^d f(\varphi(t))\varphi'(t)\,\mathrm{d}t \tag{3}$$

Find $\int_0^2 xe^{x^2}\,\mathrm{d}x$

$$\int_0^2 xe^{-x^2}\,\mathrm{d}x =$$

The funciton $-x^2$ is the inside part of the composite function and the derivative $-2x$ is (up to the constant multiple 2) factor of the integrated function. Thus it is natural to substitute $t = -x^2$.

Find $\int_0^2 x e^{x^2}\,\mathrm{d}x$

$$\int_0^2 x e^{-x^2}\,\mathrm{d}x = \boxed{\begin{aligned} t &= -x^2 \\ \mathrm{d}t &= -2x\,\mathrm{d}x \\ -\frac{1}{2}\,\mathrm{d}t &= x\,\mathrm{d}x \end{aligned}}$$

We find the relationship between $\mathrm{d}x$ and $\mathrm{d}t$.

Find $\displaystyle\int_0^2 x e^{x^2}\,\mathrm{d}x$

$$\int_0^2 x e^{-x^2}\,\mathrm{d}x = \boxed{\begin{aligned} t &= -x^2 \\ \mathrm{d}t &= -2x\,\mathrm{d}x \\ -\frac{1}{2}\,\mathrm{d}t &= x\,\mathrm{d}x \\ x = 0 \Rightarrow t &= -0^2 = 0 \\ x = 2 \Rightarrow t &= -2^2 = -4 \end{aligned}}$$

We have to transform limits of integration as well. Thus, we substitute upper and lower limits to the relationship between x and t.

Find $\int_0^2 x e^{x^2}\,\mathrm{d}x$

$$\int_0^2 x e^{-x^2}\,\mathrm{d}x = -\int_0^{-4} \frac{1}{2} e^{t}\,\mathrm{d}t$$

$$\begin{aligned} t &= -x^2 \\ \mathrm{d}t &= -2x\,\mathrm{d}x \\ -\frac{1}{2}\,\mathrm{d}t &= x\,\mathrm{d}x \\ x = 0 \Rightarrow t &= -0^2 = 0 \\ x = 2 \Rightarrow t &= -2^2 = -4 \end{aligned}$$

We convert the definite integral into t variable. The new upper limit of integration arises from the upper limit in variable x (and similarly for lower limits).

Find $\int_0^2 xe^{x^2}\,\mathrm{d}x$

$$\int_0^2 xe^{-x^2}\,\mathrm{d}x = \boxed{\begin{aligned} t &= -x^2 \\ \mathrm{d}t &= -2x\,\mathrm{d}x \\ -\frac{1}{2}\,\mathrm{d}t &= x\,\mathrm{d}x \\ x = 0 \Rightarrow t &= -0^2 = 0 \\ x = 2 \Rightarrow t &= -2^2 = -4 \end{aligned}}$$

$$= -\int_0^{-4} \frac{1}{2}e^t\,\mathrm{d}t$$

$$= -\left[\frac{1}{2}e^t\right]_0^{-4} = -\left[\frac{1}{2}e^{-4} - \frac{1}{2}e^0\right] = \frac{1}{2} - \frac{1}{2}e^{-4}$$

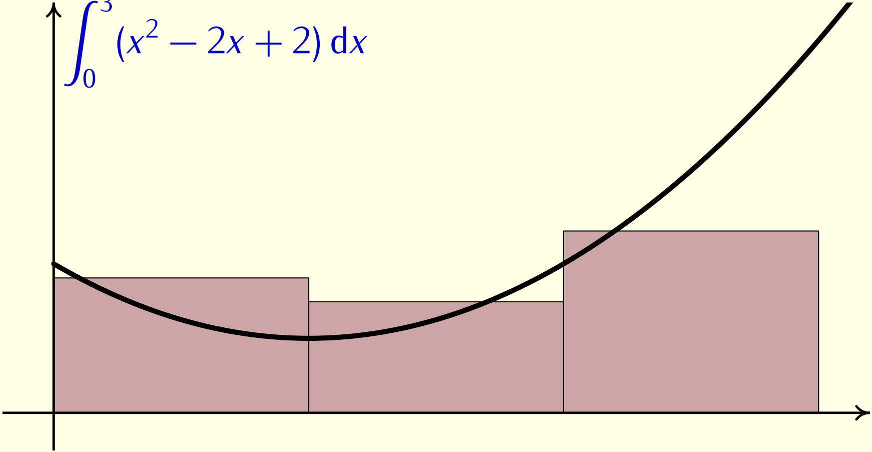
We evaluate the integral in t variable by Newton–Leibniz formula.

4 Numerical estimate — Trapezoidal rule

The definite integral on the interval $[a, b]$ is a real-valued function defined on the set of integrable function. The result is a number and a natural question arises: *Is it possible to miss the indefinite integral, which may be hard to find, and estimate the value of the definite integral by a simple numerical method?*
The answer is positive. To introduce the main ideas of such a method we

- return to the definition of Riemann integral,
- use trapezoids instead of rectangles,
- use subintervals of equal length.

$$\int_0^3 (x^2 - 2x + 2)\,\mathrm{d}x$$

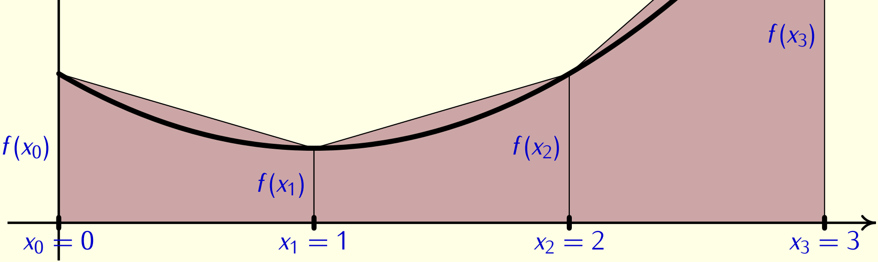


This is the division and integral sum

$$\int_0^3 (x^2 - 2x + 2)\,\mathrm{d}x$$

$$I \approx \frac{h}{2}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\Big)$$

$h = 1$ is the length of subintervals on x-axis



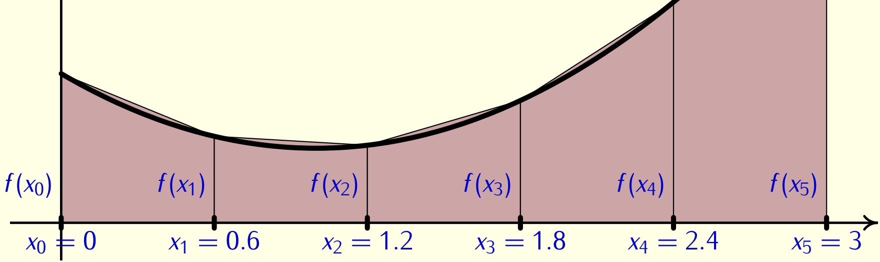
To estimate the area below the curve, we use trapezoids rather than rectangles.

$$I \approx \frac{h[f(x_0) + f(x_1)]}{2} + \frac{h[f(x_1) + f(x_2)]}{2} + \frac{h[f(x_2) + f(x_3)]}{2}$$

$$\int_0^3 (x^2 - 2x + 2)\,\mathrm{d}x$$

$$I \approx \frac{h}{2}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\Big)$$

$h = 0.6$ is the length of subintervals on x-axis



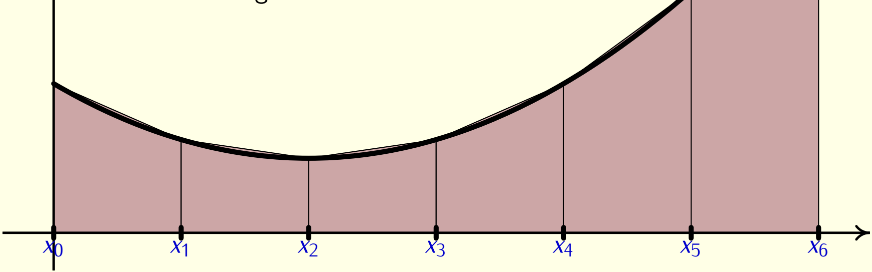
We consider finer and finer division. The approximation is better, but we have to do more calculations.

$$I \approx \frac{h[f(x_0) + f(x_1)]}{2} + \frac{h[f(x_1) + f(x_2)]}{2} + \frac{h[f(x_2) + f(x_3)]}{2} + \frac{h[f(x_3) + f(x_4)]}{2} + \frac{h[f(x_4) + f(x_5)]}{2}$$

$$\int_0^3 (x^2 - 2x + 2)\,\mathrm{d}x$$

$$I \approx \frac{h}{2}\Big(f(x_0) + 2f(x_1) + \cdots + 2f(x_5) + f(x_6)\Big)$$

$h = 0.5$ is the length of subintervals on x-axis



General formula

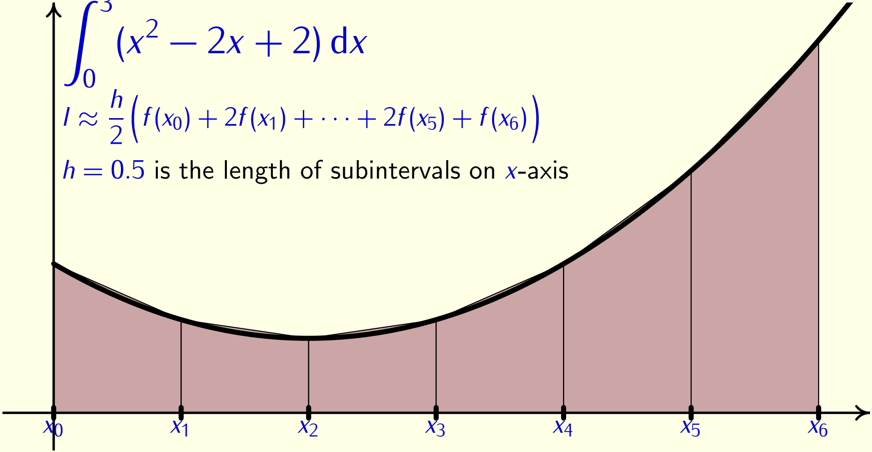
$$I \approx \frac{h}{2}\Big(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\Big) = \frac{h}{2}\sum_{i=0}^{n} m_i f(x_i)$$

where $m_0 = m_n = 1$ and $m_i = 2$ for $0 < i < n$, $h = \dfrac{b-a}{n}$, $x_i = a + ih$.

$$\int_0^3 (x^2 - 2x + 2)\,\mathrm{d}x$$

$$I \approx \frac{h}{2}\Big(f(x_0) + 2f(x_1) + \cdots + 2f(x_5) + f(x_6)\Big)$$

$h = 0.5$ is the length of subintervals on x-axis



The error is small if

- we use sufficiently fine division,
- we don't use too fine division (round off errors are insignificant),
- the function is not "to much nonlinear".

Example. Find $\int_1^2 \frac{\sin x}{x}\,\mathrm{d}x$.

Example. Find $\int_1^2 \frac{\sin x}{x}\,dx$. $n = 10,\ h = 0.1.$

i	$x_i = a + hi$	$y_i = \frac{\sin x_i}{x_i}$	m_i	$m_i y_i$
0	1			
1	1.1			
2	1.2			
3	1.3			
4	1.4			
5	1.5			
6	1.6			
7	1.7			
8	1.8			
9	1.9			
10	2			

- We divide into 10 subinterval, $n = 10$. The length of one subinterval is $h = \frac{b-a}{n} = \frac{2-1}{10} = 0.1$.
- We record the computations in the following table.

Example. Find $\int_1^2 \frac{\sin x}{x}\,\mathrm{d}x$. $n = 10$, $h = 0.1$.

i	$x_i = a + hi$	$y_i = \frac{\sin x_i}{x_i}$	m_i	$m_i y_i$
0	1	0.841471		
1	1.1	0.810189		
2	1.2	0.776699		
3	1.3	0.741199		
4	1.4	0.703893		
5	1.5	0.664997		
6	1.6	0.624734		
7	1.7	0.583332		
8	1.8	0.541026		
9	1.9	0.498053		
10	2	0.454649		

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i	$x_i = a + hi$	$y_i = \frac{\sin x_i}{x_i}$	m_i	$m_i y_i$
0	1	0.841471	1	0.841471
1	1.1	0.810189	2	1.620377
2	1.2	0.776699	2	1.553398
3	1.3	0.741199	2	1.482397
4	1.4	0.703893	2	1.407785
5	1.5	0.664997	2	1.329993
6	1.6	0.624734	2	1.249467
7	1.7	0.583332	2	1.166664
8	1.8	0.541026	2	1.082053
9	1.9	0.498053	2	0.996105
10	2	0.454649	1	0.454649

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The sum in the last column is $S = 13.184361$ and hence

$$\int_1^2 \frac{\sin x}{x}\,\mathrm{d}x \approx \frac{h\,S}{2} = \frac{S}{20} = 0.659218.$$

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1	1.1	0.810189	2	1.620377
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3	1.3	0.741199	2	1.482397
4	1.4	0.703893	2	1.407785
5	1.5	0.664997	2	1.329993
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10	2	0.454649	1	0.454649

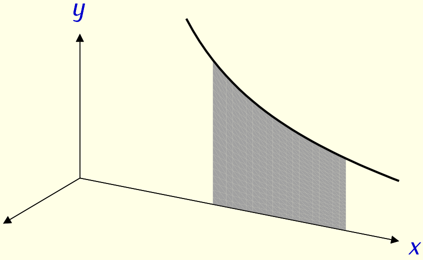
The sum in the last column is $S = 13.184361$ and hence

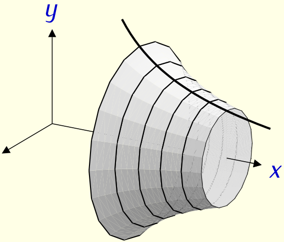
$$\int_1^2 \frac{\sin x}{x}\,\mathrm{d}x \approx \frac{h\,S}{2} = \frac{S}{20} = 0.659218.$$

More advanced techniques show the better approximation $I \doteq 0.659329906435512$ (all digits are correct).

5 Aplication – areas and volumes.

The curvilinear trapezoid and the volume of the solid of revolution

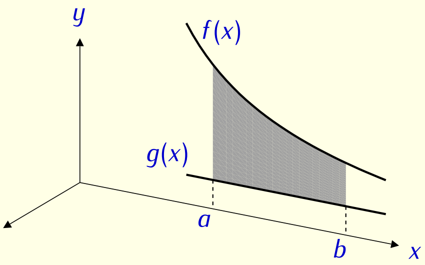


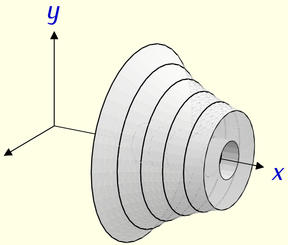


$$S = \int_a^b f(x)\,\mathrm{d}x$$

$$V = \pi \int_a^b f^2(x)\,\mathrm{d}x$$

Area between curves and the solid of revolution.

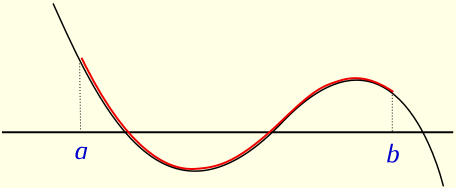




$$S = \int_a^b [f(x) - g(x)]\,\mathrm{d}x$$

$$V = \pi \int_a^b \left[f^2(x) - g^2(x)\right]\mathrm{d}x$$

The length of a smooth (differentiable) curve



$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

Find area between curves $y = 1 - (x - 1)^2$ and $x + y = 0$.

Find area between curves $y = 1 - (x-1)^2$ and $x + y = 0$.

$$1 - (x-1)^2 = -x$$

- The first curve is a parabola, the second one is a line $y = -x$.
- The curves intersect at the point where

  $$1 - (x-1)^2 = -x$$

  holds

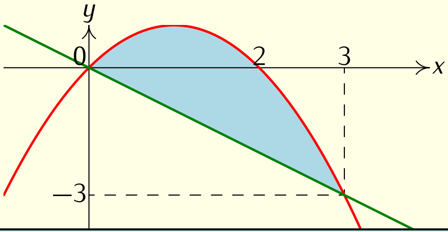
Find area between curves $y = 1 - (x-1)^2$ and $x + y = 0$.

$$\begin{aligned} 1 - (x-1)^2 &= -x \\ 1 - (x^2 - 2x + 1) &= -x \\ 1 - x^2 + 2x - 1 &= -x \\ 3x - x^2 &= 0 \\ (3 - x)x &= 0 \end{aligned}$$

The intercepts are $[0, 0]$ ans $[3, -3]$.

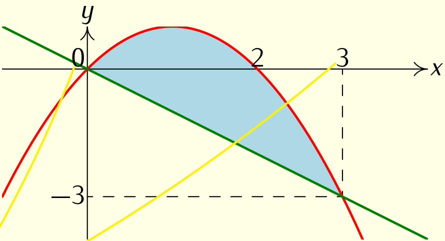
Find area between curves $y = 1 - (x-1)^2$ and $x + y = 0$.

$$\begin{aligned} 1-(x-1)^2 &= -x \\ 1-(x^2-2x+1) &= -x \\ 1-x^2+2x-1 &= -x \\ 3x-x^2 &= 0 \\ (3-x)x &= 0 \end{aligned}$$



$$y = 1-(x-1)^2 = 1-(x^2-2x+1) = 2x-x^2 = x(2-x)$$

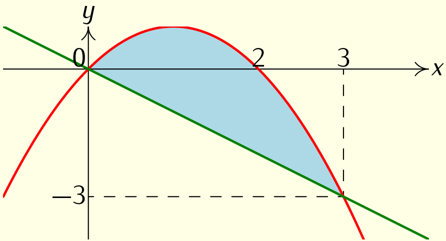
Find area between curves $y = 1 - (x-1)^2$ and $x + y = 0$.



$$S = \int_0^3 1 - (x-1)^2 - (-x)\,\mathrm{d}x$$

$$x + y = 0 \iff y = -x$$

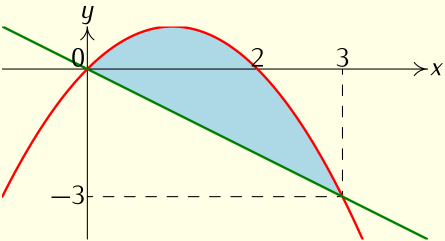
Find area between curves $y = 1 - (x-1)^2$ and $x + y = 0$.



$$S = \int_0^3 1 - (x-1)^2 - (-x)\,\mathrm{d}x = \int_0^3 1 - (x^2 - 2x + 1) + x\,\mathrm{d}x$$

We simplify.

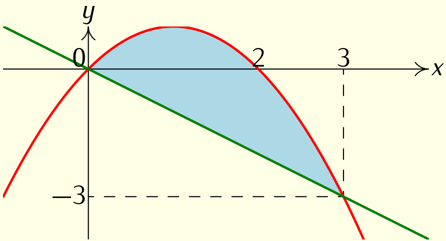
Find area between curves $y = 1 - (x-1)^2$ and $x + y = 0$.



$$S = \int_0^3 1 - (x-1)^2 - (-x)\,\mathrm{d}x = \int_0^3 1 - (x^2 - 2x + 1) + x\,\mathrm{d}x$$
$$= \int_0^3 -x^2 + 3x\,\mathrm{d}x$$

We simplify.

Find area between curves $y = 1 - (x-1)^2$ and $x + y = 0$.

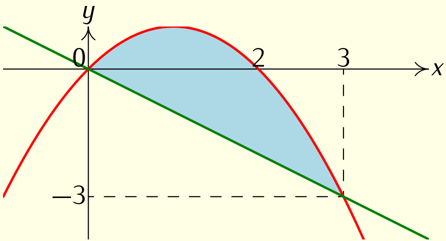


$$S = \int_0^3 1 - (x-1)^2 - (-x)\,\mathrm{d}x = \int_0^3 1 - (x^2 - 2x + 1) + x\,\mathrm{d}x$$

$$= \int_0^3 -x^2 + 3x\,\mathrm{d}x = \left[-\frac{x^3}{3} + 3\frac{x^2}{2}\right]_0^3$$

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$$

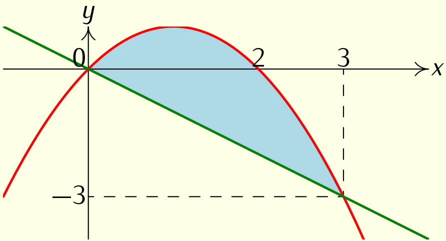
Find area between curves $y = 1 - (x-1)^2$ and $x + y = 0$.



$$S = \int_0^3 1 - (x-1)^2 - (-x)\,\mathrm{d}x = \int_0^3 1 - (x^2 - 2x + 1) + x\,\mathrm{d}x$$
$$= \int_0^3 -x^2 + 3x\,\mathrm{d}x = \left[-\frac{x^3}{3} + 3\frac{x^2}{2}\right]_0^3 = \left[-\frac{3^3}{3} + 3\frac{3^2}{2}\right] - \left[-\frac{0^3}{3} + 3\frac{0^2}{2}\right]$$

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$$

Find area between curves $y = 1 - (x-1)^2$ and $x + y = 0$.



$$S = \int_0^3 1 - (x-1)^2 - (-x)\,\mathrm{d}x = \int_0^3 1 - (x^2 - 2x + 1) + x\,\mathrm{d}x$$

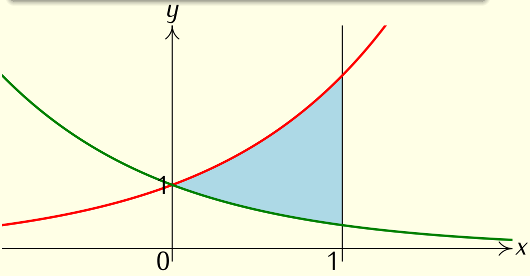
$$= \int_0^3 -x^2 + 3x\,\mathrm{d}x = \left[-\frac{x^3}{3} + 3\frac{x^2}{2}\right]_0^3 = \left[-\frac{3^3}{3} + 3\frac{3^2}{2}\right] - \left[-\frac{0^3}{3} + 3\frac{0^2}{2}\right]$$

$$= -9 + \frac{27}{2} = \frac{9}{2}$$

Finished.

Find the area between curves $y = e^x$ and $y = e^{-x}$ for $x \in [0, 1]$ and find the volume of the corresponding solid of revolution (consider revolution about x-axis).

$y = e^x$, $y = e^{-x}$, $x \in [0, 1]$, $S = ?$, $V = ?$.

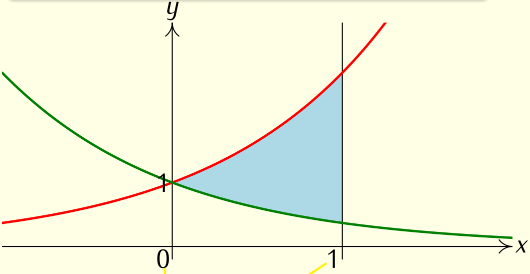


$$S = \int_a^b \Big(f(x) - g(x) \Big)\, \mathrm{d}x$$

$$V = \pi \int_a^b \Big(f^2(x) - g^2(x) \Big)\, \mathrm{d}x$$

We draw the picture.

$y = e^x$, $y = e^{-x}$, $x \in [0, 1]$, $S = ?$, $V = ?$.



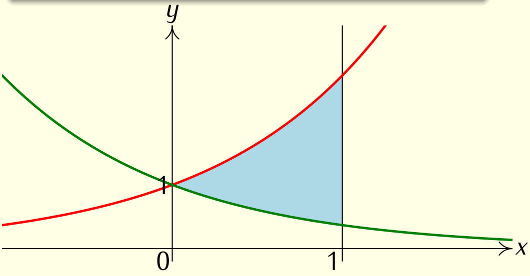
$$S = \int_a^b \Big(f(x) - g(x) \Big) \,\mathrm{d}x$$

$$V = \pi \int_a^b \Big(f^2(x) - g^2(x) \Big) \,\mathrm{d}x$$

$$S = \int_0^1 e^x - e^{-x} \,\mathrm{d}x$$

We write the area as an integral.

$y = e^x$, $y = e^{-x}$, $x \in [0, 1]$, $S = ?$, $V = ?$.



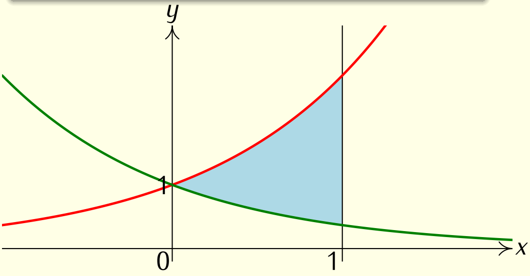
$$S = \int_a^b \Big(f(x) - g(x) \Big)\, \mathrm{d}x$$

$$V = \pi \int_a^b \Big(f^2(x) - g^2(x) \Big)\, \mathrm{d}x$$

$$S = \int_0^1 e^x - e^{-x}\, \mathrm{d}x = \left[e^x + e^{-x} \right]_0^1$$

We find the indefinite integral.

$y = e^x$, $y = e^{-x}$, $x \in [0, 1]$, $S = ?$, $V = ?$.



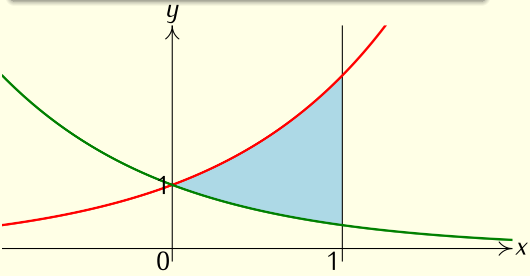
$$S = \int_a^b \Big(f(x) - g(x) \Big) \, dx$$

$$V = \pi \int_a^b \Big(f^2(x) - g^2(x) \Big) \, dx$$

$$S = \int_0^1 e^x - e^{-x} \, dx = [e^x + e^{-x}]_0^1 = e^1 + e^{-1} - [e^0 + e^0]$$

We find the definite integral.

$y = e^x$, $y = e^{-x}$, $x \in [0,1]$, $S = ?$, $V = ?$.



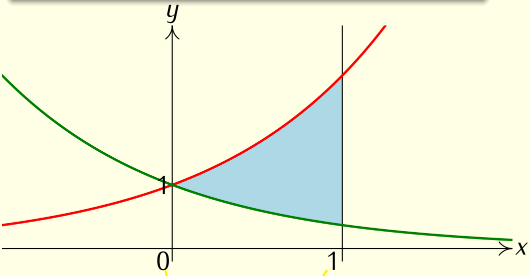
$$S = \int_a^b \Big(f(x) - g(x)\Big)\,\mathrm{d}x$$

$$V = \pi \int_a^b \Big(f^2(x) - g^2(x)\Big)\,\mathrm{d}x$$

$$S = \int_0^1 e^x - e^{-x}\,\mathrm{d}x = [e^x + e^{-x}]_0^1 = e^1 + e^{-1} - \left[e^0 + e^0\right] = e + \frac{1}{e} - 2$$

We have the area.

$y = e^x$, $y = e^{-x}$, $x \in [0,1]$, $S = ?$, $V = ?$.



$$S = \int_a^b \Big(f(x) - g(x) \Big)\,\mathrm{d}x$$

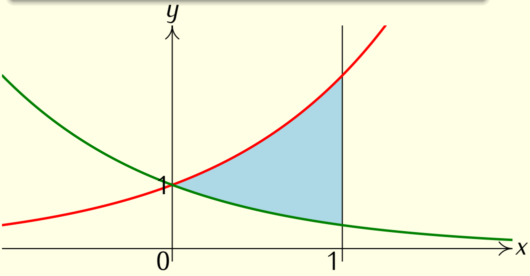
$$V = \pi \int_a^b \Big(f^2(x) - g^2(x) \Big)\,\mathrm{d}x$$

$$S = \int_0^1 e^x - e^{-x}\,\mathrm{d}x = [e^x + e^{-x}]_0^1 = e^1 + e^{-1} - [e^0 + e^0] = e + \frac{1}{e} - 2$$

$$V = \pi \int_0^1 (e^x)^2 - (e^{-x})^2\,\mathrm{d}x$$

We write the volume as an integral.

$y = e^x$, $y = e^{-x}$, $x \in [0,1]$, $S = ?$, $V = ?$.



$$S = \int_a^b \Big(f(x) - g(x) \Big) \, dx$$

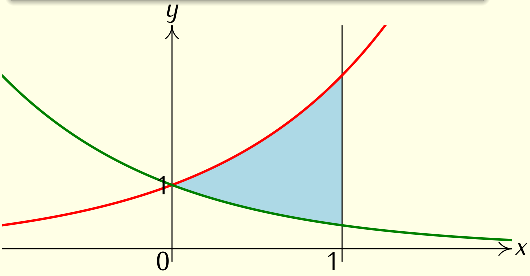
$$V = \pi \int_a^b \Big(f^2(x) - g^2(x) \Big) \, dx$$

$$S = \int_0^1 e^x - e^{-x} \, dx = \left[e^x + e^{-x} \right]_0^1 = e^1 + e^{-1} - \left[e^0 + e^0 \right] = e + \frac{1}{e} - 2$$

$$V = \pi \int_0^1 (e^x)^2 - (e^{-x})^2 \, dx = \pi \int_0^1 e^{2x} - e^{-2x} \, dx$$

We simplify.

$y = e^x$, $y = e^{-x}$, $x \in [0,1]$, $S = ?$, $V = ?$.



$$S = \int_a^b \Big(f(x) - g(x) \Big) \, \mathrm{d}x$$

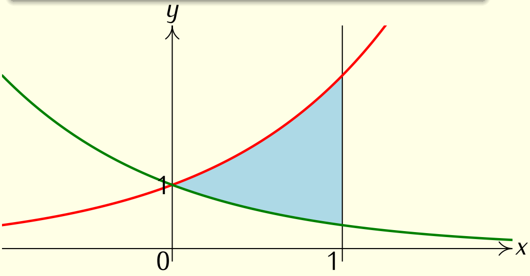
$$V = \pi \int_a^b \Big(f^2(x) - g^2(x) \Big) \, \mathrm{d}x$$

$$S = \int_0^1 e^x - e^{-x} \, \mathrm{d}x = [e^x + e^{-x}]_0^1 = e^1 + e^{-1} - \left[e^0 + e^0\right] = e + \frac{1}{e} - 2$$

$$V = \pi \int_0^1 (e^x)^2 - (e^{-x})^2 \, \mathrm{d}x = \pi \int_0^1 e^{2x} - e^{-2x} \, \mathrm{d}x = \pi \left[\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x} \right]_0^1$$

We evaluate the indefinite integral.

$y = e^x$, $y = e^{-x}$, $x \in [0,1]$, $S = ?$, $V = ?$.



$$S = \int_a^b \Big(f(x) - g(x) \Big) \, \mathrm{d}x$$

$$V = \pi \int_a^b \Big(f^2(x) - g^2(x) \Big) \, \mathrm{d}x$$

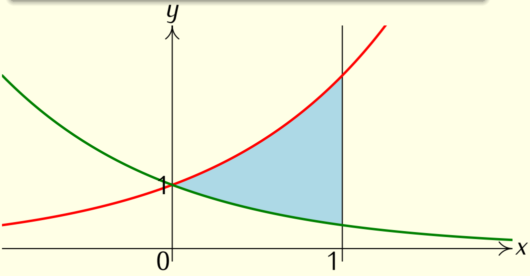
$$S = \int_0^1 e^x - e^{-x} \, \mathrm{d}x = [e^x + e^{-x}]_0^1 = e^1 + e^{-1} - \left[e^0 + e^0\right] = e + \frac{1}{e} - 2$$

$$V = \pi \int_0^1 (e^x)^2 - (e^{-x})^2 \, \mathrm{d}x = \pi \int_0^1 e^{2x} - e^{-2x} \, \mathrm{d}x = \pi \left[\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x} \right]_0^1$$

$$= \pi \left[\frac{1}{2} e^2 + \frac{1}{2} e^{-2} - \left(\frac{1}{2} e^0 + \frac{1}{2} e^0 \right) \right]$$

We evaluate the definite integral.

$y = e^x$, $y = e^{-x}$, $x \in [0, 1]$, $S = ?$, $V = ?$.



$$S = \int_a^b \Big(f(x) - g(x) \Big)\, \mathrm{d}x$$

$$V = \pi \int_a^b \Big(f^2(x) - g^2(x) \Big)\, \mathrm{d}x$$

$$S = \int_0^1 e^x - e^{-x}\, \mathrm{d}x = [e^x + e^{-x}]_0^1 = e^1 + e^{-1} - [e^0 + e^0] = e + \frac{1}{e} - 2$$

$$V = \pi \int_0^1 (e^x)^2 - (e^{-x})^2\, \mathrm{d}x = \pi \int_0^1 e^{2x} - e^{-2x}\, \mathrm{d}x = \pi \left[\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x} \right]_0^1$$

$$= \pi \left[\frac{1}{2} e^2 + \frac{1}{2} e^{-2} - \left(\frac{1}{2} e^0 + \frac{1}{2} e^0 \right) \right] = \pi \left[\frac{1}{2} e^2 + \frac{1}{2e^2} - 1 \right]$$

We simplify.

Find the volume of the solid of revolution which arises by revolving the area below the curve $y = e^{\sqrt{x}}$ for $x \in [0, 1]$ about x-axis.

$V = ?$, $x \in [0, 1]$, $y = e^{\sqrt{x}}$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

We use the formula for the volume.

$$V = ?,\ x \in [0,1],\ y = e^{\sqrt{x}}$$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

$$\int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

We evaluate the indefinite integral first.

$$V = ?,\ x \in [0,1],\ y = e^{\sqrt{x}}$$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

$$\int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x = \int e^{2\sqrt{x}} \mathrm{d}x$$

We simplify.

$$V = ?,\ x \in [0,1],\ y = e^{\sqrt{x}}$$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

$$\int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x = \int e^{2\sqrt{x}}\,\mathrm{d}x$$

$$\boxed{\begin{aligned} 2\sqrt{x} &= t \\ 4x &= t^2 \\ 4\,\mathrm{d}x &= 2t\,\mathrm{d}t \\ \mathrm{d}x &= \frac{1}{2}t\,\mathrm{d}t \end{aligned}}$$

We substitute.

$$V = ?,\ x \in [0,1],\ y = e^{\sqrt{x}}$$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

$$\int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x = \int e^{2\sqrt{x}}\,\mathrm{d}x \quad \boxed{\begin{aligned} 2\sqrt{x} &= t \\ 4x &= t^2 \\ 4\,\mathrm{d}x &= 2t\,\mathrm{d}t \\ \mathrm{d}x &= \frac{1}{2}t\,\mathrm{d}t \end{aligned}} = \frac{1}{2}\int t \cdot e^t\,\mathrm{d}t$$

We substitute.

$$V = ?,\ x \in [0,1],\ y = e^{\sqrt{x}}$$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

$$\int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x = \int e^{2\sqrt{x}}\,\mathrm{d}x \ \boxed{\begin{aligned} 2\sqrt{x} &= t \\ 4x &= t^2 \\ 4\,\mathrm{d}x &= 2t\,\mathrm{d}t \\ \mathrm{d}x &= \frac{1}{2} t\,\mathrm{d}t \end{aligned}} = \frac{1}{2} \int t \cdot e^t\,\mathrm{d}t$$

$$\boxed{\begin{matrix} u = t & u' = 1 \\ v' = e^t & v = e^t \end{matrix}} = \frac{1}{2}\left(t \cdot e^t - \int 1 \cdot e^t\,\mathrm{d}t\right)$$

We integrate by part

$$\int u \cdot v'\,\mathrm{d}x = u \cdot v - \int u' \cdot v\,\mathrm{d}x$$

$$\boxed{V = ?,\ x \in [0,1],\ y = e^{\sqrt{x}}}$$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

$$\int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x = \int e^{2\sqrt{x}} \mathrm{d}x \ \boxed{\begin{aligned} 2\sqrt{x} &= t \\ 4x &= t^2 \\ 4\,\mathrm{d}x &= 2t\,\mathrm{d}t \\ \mathrm{d}x &= \frac{1}{2}t\,\mathrm{d}t \end{aligned}} = \frac{1}{2}\int t \cdot e^t \,\mathrm{d}t$$

$$\boxed{\begin{matrix} u = t & u' = 1 \\ v' = e^t & v = e^t \end{matrix}} = \frac{1}{2}\left(t \cdot e^t - \int 1 \cdot e^t \,\mathrm{d}t\right) = \frac{1}{2}\left(te^t - e^t\right)$$

We finish the integration.

$$V = ?,\ x \in [0,1],\ y = e^{\sqrt{x}}$$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

$$\int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x = \int e^{2\sqrt{x}}\,\mathrm{d}x \boxed{\begin{aligned} 2\sqrt{x} &= t \\ 4x &= t^2 \\ 4\,\mathrm{d}x &= 2t\,\mathrm{d}t \\ \mathrm{d}x &= \frac{1}{2}t\,\mathrm{d}t \end{aligned}} = \frac{1}{2}\int t \cdot e^t\,\mathrm{d}t$$

$$\boxed{\begin{matrix} u = t & u' = 1 \\ v' = e^t & v = e^t \end{matrix}} = \frac{1}{2}\left(t \cdot e^t - \int 1 \cdot e^t\,\mathrm{d}t\right) = \frac{1}{2}\left(te^t - e^t\right)$$

$$= \frac{1}{2} \cdot e^t \cdot (t-1)$$

Factor.

$$\boxed{V = ?,\ x \in [0,1],\ y = e^{\sqrt{x}}}$$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

$$\int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x = \int e^{2\sqrt{x}}\, \mathrm{d}x \boxed{\begin{gathered} 2\sqrt{x} = t \\ 4x = t^2 \\ 4\,\mathrm{d}x = 2t\,\mathrm{d}t \\ \mathrm{d}x = \frac{1}{2}t\,\mathrm{d}t \end{gathered}} = \frac{1}{2}\int t \cdot e^t\, \mathrm{d}t$$

$$\boxed{\begin{matrix} u = t & u' = 1 \\ v' = e^t & v = e^t \end{matrix}} = \frac{1}{2}\left(t \cdot e^t - \int 1 \cdot e^t\, \mathrm{d}t\right) = \frac{1}{2}\left(te^t - e^t\right)$$

$$= \frac{1}{2} \cdot e^t \cdot (t-1) = \frac{1}{2} \cdot e^{2\sqrt{x}} \cdot \left(2\sqrt{x} - 1\right)$$

We use the back susbstitution. The constant of integration is arbitrary, we choose zero constant of integration.

$$V = ?,\ x \in [0,1],\ y = e^{\sqrt{x}}$$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x$$

$$\int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x = \frac{1}{2} \cdot e^{2\sqrt{x}} \cdot (2\sqrt{x} - 1)$$

$$V = \pi \left[\frac{1}{2} e^{2\sqrt{x}} (2\sqrt{x} - 1)\right]_0^1$$

$$= \pi \left[\frac{1}{2} e^2 (2 - 1) - \frac{1}{2} e^0 (0 - 1)\right]$$

We evaluate the definite integral by Newton–Leibniz theorem.

$V = ?$, $x \in [0, 1]$, $y = e^{\sqrt{x}}$

$$V = \pi \int_0^1 \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x \qquad \int \left(e^{\sqrt{x}}\right)^2 \mathrm{d}x = \frac{1}{2} \cdot e^{2\sqrt{x}} \cdot \left(2\sqrt{x} - 1\right)$$

$$\begin{aligned} V &= \pi \left[\frac{1}{2} e^{2\sqrt{x}} \left(2\sqrt{x} - 1\right)\right]_0^1 \\ &= \pi \left[\frac{1}{2} e^2 (2 - 1) - \frac{1}{2} e^0 (0 - 1)\right] \\ &= \pi \left[\frac{e^2}{2} + \frac{1}{2}\right] \\ &= \pi \frac{e^2 + 1}{2} \end{aligned}$$

We simplify.

Further reading

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