The Derivative and Optimalization

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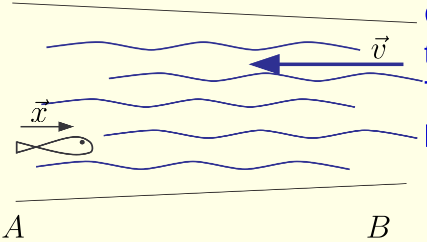
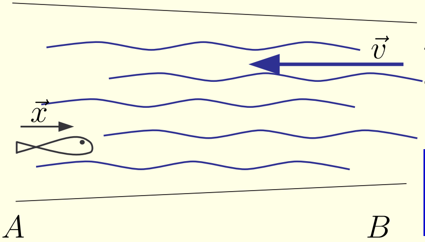
- The swimmer swimms in river against the current.
- If he is too fast, he loose a lot of energy.
- If he is to slow, he is pulled down by the current.
- What is the optimal speed to reach the point B from the point A (optimal in the sense of the minimal loss of energy).



Cossing convenient units we can suppose that $\vec{v} = 1$.
The energy necessary to reach the point B from A has to be minimal

- Consider a fish in the river. The speed of the current and the speed of the fish is considered with respect to the observer on the river bank.
- A fish an excelent swimmer – it swimms at the speed which ensures as small loss of energy as possible.
- The energy loss of the fish swimming is proprotional to the cube root of the velocity with respect to the water, i.e. $(x + v)^3$.



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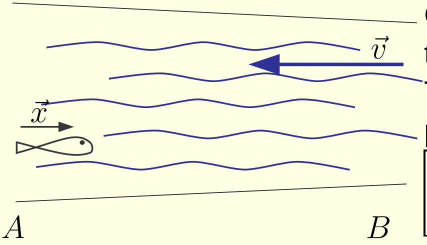
The energy necessary to reach the point B from A has to be minimal :

$$\boxed{\frac{(x+1)^3}{x} \to \text{minimum}}$$

- The loss of energy per a unit time is proportional to $(x+1)^3$.
- The fish will swimm $\dfrac{\text{length}}{\text{speed}}$ time units and the time is propotional to $\dfrac{1}{x}$.



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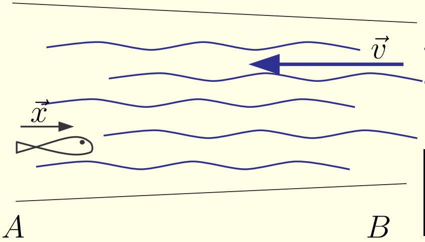
$$\left(\frac{(x+1)^3}{x}\right)' = \frac{3(x+1)^2 \cdot x - (x+1)^3 \cdot 1}{x^2}$$

$$= \frac{(x+1)^2\Big[3x - (x+1)\Big]}{x^2} = \frac{(x+1)^2\Big[2x-1\Big]}{x^2}$$

We differentiate.



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Derivative vanishes if $x = \dfrac{1}{2}$, i.e. the speed of the fish is equal to one half of the speed of the current. From the nature of the problem it follows, that this stationary point is a minimum.

The stationary point $x = -1$ has no practical meaning in this problem.

From **Calculus, an introduction to applied mathematics**, H.P. Greenspan, D. J. Benney, J. E. Turner, nakl. McGraw Hill (1986), page. 128.

Migrating fish know the solution of this problem!