Integrals – refcard

Integrals by parts

(P1) $\int \boxed{P(x)} \cdot \sin x \,\mathrm{d}x$

(P2) $\int \boxed{P(x)} \cdot \cos x \,\mathrm{d}x$

(P3) $\int \boxed{P(x)} \cdot e^x \,\mathrm{d}x$

(P4) $\int P(x) \cdot \boxed{\ln x} \,\mathrm{d}x$

(P5) $\int P(x) \cdot \boxed{\operatorname{arctg} x} \,\mathrm{d}x$

The marked part is to be killed by differentiation.

Substitution

(S1)	$\int R(\cos x) \cdot \sin x \,\mathrm{d}x$	$\boxed{\cos x = t}$
(S2)	$\int R(\sin x) \cdot \cos x \,\mathrm{d}x$	$\boxed{\sin x = t}$
(S3)	$\int R(x, \sqrt{ax+b}) \,\mathrm{d}x$	$\boxed{ax + b = t^2}$
(S4)	$\int R(x, \sqrt[k]{ax+b}) \,\mathrm{d}x$	$\boxed{ax + b = t^k}$
(S5)	$\int R(x^2, \sqrt{x^2+b}) x \,\mathrm{d}x$	$\boxed{x^2 + b = t^2}$
(S6)	$\int R(\ln x) \frac{1}{x} \,\mathrm{d}x$	$\boxed{\ln x = t}$
(S7)	$\int R(e^x) \,\mathrm{d}x$	$\boxed{e^x = t}$

Rational functions

(R1) $\int \frac{P_n(x)}{x^m} \,\mathrm{d}x$ → divide each term separately

(R2) $\int \frac{P_n(x)}{Q_m(x)} \,\mathrm{d}x,\ n \geq m$ → divide, then (R3)

(R3) $\int \frac{P_n(x)}{Q_m(x)} \,\mathrm{d}x,\ n < m$ → partial fractions

(R4) $\int \frac{1}{ax - c} \,\mathrm{d}x = \frac{1}{a} \ln |ax - c|$

(R5) $\int \frac{1}{(ax - c)^n} \,\mathrm{d}x = \frac{1}{a} \cdot \frac{1}{1-n} \cdot \frac{1}{(ax-c)^{n-1}}$

(R6) $\int \frac{1}{x^2 + c^2} \,\mathrm{d}x = \frac{1}{c} \operatorname{arctg} \frac{x}{c}$

(R7) $\int \frac{x}{x^2 \pm c^2} = \frac{1}{2} \ln |x^2 \pm c^2|$

(R8) $\int \frac{\alpha x + \beta}{x^2 + c^2} \,\mathrm{d}x$ → linear combination of (R6), (R7)

(R9) $\int \frac{1}{ax^2 + bx + c} \,\mathrm{d}x$ → complete square in denominator

(R10) $\int \frac{2ax + b}{ax^2 + bx + c} \,\mathrm{d}x = \ln |ax^2 + bx + c|$

(R11) $\int \frac{\alpha x + \beta}{ax^2 + bx + c} \,\mathrm{d}x$ → linear combination of (R9), (R10)

Differential equations – refcard

Nonlinear equations

y is a general solution

- The basic problem of integral calculus: $\boxed{y' = f(x)}$

  $y = \int f(x) \,\mathrm{d}x + C$

- Separated variables: $\boxed{y' = f(x)g(y)}$

  1. Constant solutions $y = y_0$ for every y_0 which satisfies $g(y_0) = 0$.
  2. $\int \frac{1}{g(y)} \,\mathrm{d}y = \int f(x) \,\mathrm{d}x + C \qquad C \in \mathbb{R}$

Linear differential equations

y_{GH}: general solution of hom. eq.
y_{PN}: particular solution of nonhom. eq.
y_{GN}: general solution of nonhom. eq.

- 1-st order homogeneous: $\boxed{y' + a(x)y = 0}$

  $y_{GH} = Ce^{-\int a(x)\,\mathrm{d}x}, \qquad C \in \mathbb{R}$

- 1-st order nonhomogeneous: $\boxed{y' + a(x)y = b(x)}$

  $$y_{GN} = y_{GH} + y_{PN} = e^{-\int a(x)\,\mathrm{d}x} \left[C + \int b(x) e^{\int a(x)\,\mathrm{d}x} \,\mathrm{d}x \right], \qquad C \in \mathbb{R}$$

- 2-nd order homogeneous, with constant coeficients: $\boxed{y'' + py' + qy = 0}$

  $y_{GH} = C_1 y_1 + C_2 y_2, \qquad C_1 \in \mathbb{R},\ C_2 \in \mathbb{R}$ where

  - ◇ $y_1 = e^{z_1 x}$ $\quad y_2 = e^{z_2 x}$ $\quad$ if $z_{1,2} \in \mathbb{R},\ z_1 \neq z_2$,
  - ◇ $y_1 = e^{z_1 x}$ $\quad y_2 = x e^{z_1 x}$ $\quad$ if $z_{1,2} \in \mathbb{R},\ z_1 = z_2$,
  - ◇ $y_1 = e^{\alpha x} \cos(\beta x)$ $\quad y_2 = e^{\alpha x} \sin(\beta x)$ $\quad$ if $z_{1,2} \notin \mathbb{R},\ z_{1,2} = \alpha \pm i\beta$,

  $z_{1,2}$ beeing solutions of the characteristic equation $z^2 + pz + q = 0$.

- 2-nd order nonhomogeneous, constant coeficients: $\boxed{y'' + py' + qy = f(x)}$

  $$y_{GN} = y_{GH} + y_{PN} = \left(C_1 + \int \frac{W_1}{W} \,\mathrm{d}x \right) y_1 + \left(C_2 + \int \frac{W_2}{W} \,\mathrm{d}x \right) y_2$$

  $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1',$$

  $$W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} = -f y_2, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} = f y_1$$