Investigating the function

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Contents

Our main object is not to transfer a complete tabulation of functional values to graph paper but to gain a good qualitative (and maybe also quantitative) understanding of the function shape with as little effort as possible. To gain a rapid impression of a function and to explore its main properties we employ various methods of the differential calculus.

When constructing the graph of given function we are interested less in drawings of high accuracy than in drawings which "show trends". From the graph of the function it must be clear where the function is increasing and where decreasing, where there are breaks in the graph, where there are zeros and local extrema of the function,where it is concave up and down and what is its shape generally. An idea to locate a few points of the graph and connect them blindly is usually not helpful.

1. We find the domain of f, decide whether f is odd, even or periodical.

2. We find intercepts of the graph with axes and intervals where the value of the function is positive and/or negative.

3. We find one-sided limits at the points of discontinuity and at $\pm\infty$.

4. We find and simplify the derivative f'. Then we find intervals where f is increasing and/or decreasing and local extrema of the function f.

5. We find and simplify the second derivative f''. Then we find the intervals where f is concave up and/or down and inflection points of the function f.

6. If the limits in $+\infty$ and/or $-\infty$ are not finite, we find inclined asymptotes in these points.

7. We sketch the graph of the function f.

$$\boxed{y = \frac{x}{1 + x^2}}$$

$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$;

- The restriction on domain follows from the denominator of the fraction.
- Hence $x^2 + 1$ must be nonzero. However, this is always true in $\mathbb{R}$.

$\boxed{y = \frac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

- The numerator, x, is an odd function, the denominator, $(1+x^2)$, is an even function.
- The fraction is an odd function.

$\boxed{y = \frac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

$y = 0$

We establish the x-intercepts and the sign of the function.

$\boxed{y = \frac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

$y = 0 \Rightarrow \frac{x}{1+x^2} = 0$

We establish the x-intercepts and the sign of the function.

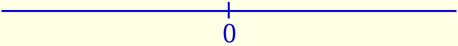
$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

The fraction equals zero only if the numerator equals zero.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

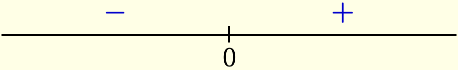
$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$



- We sketch the x-axis with the intercept.
- There is no point of discontinuity.

$\boxed{y = \frac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

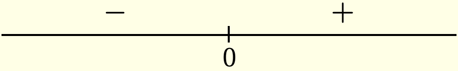
$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$



- The denominator $(1 + x^2)$ is always positive.
- Hence the fraction and the numerator have common sign.
- The function is positive if x is positive and vice versa.

$\boxed{y = \frac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

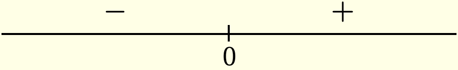


$$\lim_{x\to\pm\infty} \frac{x}{1+x^2}$$

We establish limits at infinity.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

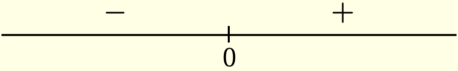


$$\lim_{x\to\pm\infty} \frac{x}{1+x^2} = \lim_{x\to\pm\infty} \frac{1}{x}$$

- We know that in limit of this type the leading terms in numerator and denominator are the only important quantities.
- The green part can be omitted.
- We cancel: $\dfrac{x}{x^2} = \dfrac{1}{x}$.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

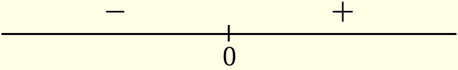


$$\lim_{x\to\pm\infty} \frac{x}{1+x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = \frac{1}{\pm\infty}$$

We substitute.

$\boxed{y = \frac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

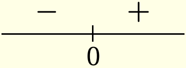


$$\lim_{x\to\pm\infty} \frac{x}{1+x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = \frac{1}{\pm\infty} = 0$$

- Both $\frac{1}{\infty}$ and $\frac{1}{-\infty}$ equal zero.
- The function possesses the horizontal asymptote $y = 0$ at both $\pm\infty$.

$$\boxed{y = \frac{x}{1+x^2}}$$

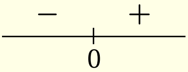
$Dom(f) = \mathbb{R}$; odd;



$$y' = \frac{1(1+x^2) - x(0+2x)}{(1+x^2)^2}$$

- We evaluate the derivative.
- We differentiate the quotient by the quotient rule.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

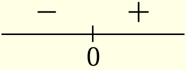


$$y' = \frac{1(1+x^2) - x(0+2x)}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

We simplify.

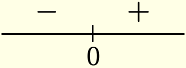
$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;



$$y' = \frac{1(1+x^2) - x(0+2x)}{(1+x^2)^2}$$
$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$
$$= \frac{1-x^2}{(1+x^2)^2}$$

We simplify.

$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$; odd;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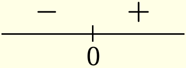
$$y' = \frac{1-x^2}{(1+x^2)^2}\ ;$$

$$y' = 0$$

We solve $y' = 0$.

$\boxed{y = \frac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;



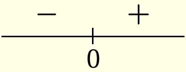
$$y' = \frac{1-x^2}{(1+x^2)^2}\ ;$$

$$y' = 0$$

$$\frac{1-x^2}{(1+x^2)^2} = 0$$

We solve $y' = 0$. We substitute for y'.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

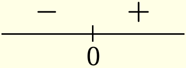


$y' = \dfrac{1-x^2}{(1+x^2)^2}$;

$$\begin{aligned} y' &= 0 \\ \frac{1-x^2}{(1+x^2)^2} &= 0 \\ 1-x^2 &= 0 \end{aligned}$$

The fraction equals zero iff the numerator equals zero.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

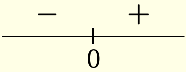


$$y' = \frac{1-x^2}{(1+x^2)^2} \ ;$$

$$\begin{aligned} y' &= 0 \\ \frac{1-x^2}{(1+x^2)^2} &= 0 \\ 1-x^2 &= 0 \\ x^2 &= 1 \end{aligned}$$

We isolate x^2.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

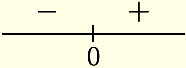


$$y' = \frac{1-x^2}{(1+x^2)^2}\ ;$$

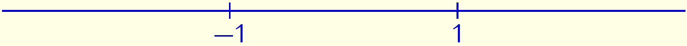
$$\begin{aligned} y' &= 0 \\ \frac{1-x^2}{(1+x^2)^2} &= 0 \\ 1-x^2 &= 0 \\ x^2 &= 1 \\ x_1 &= 1 \\ x_2 &= -1 \end{aligned}$$

We find x. There are two stationary points.

$\boxed{y=\frac{x}{1+x^2}}$ $Dom(f)=\mathbb{R}$; odd;

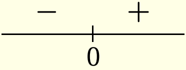


$$y'=\frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2}=\pm 1$$

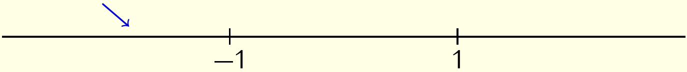


- We draw the x-axis with stationary points.
- There are no points of discontinuity.

$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$; odd;



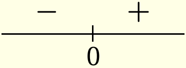
$$y' = \frac{1-x^2}{(1+x^2)^2} \;; \qquad x_{1,2} = \pm 1$$



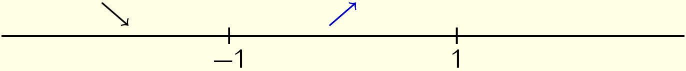
We test $x = -2$. We have

$$y'(-2) = \frac{1-4}{\text{positive}} < 0.$$

$\boxed{y = \frac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;



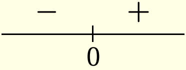
$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$



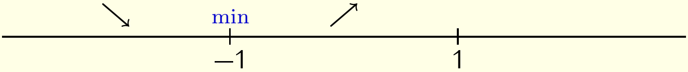
We test $x = 0$. We have

$$y'(0) = \frac{1}{1} > 0.$$

$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$; odd;



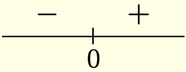
$$y' = \frac{1-x^2}{(1+x^2)^2} \; ; \qquad x_{1,2} = \pm 1$$



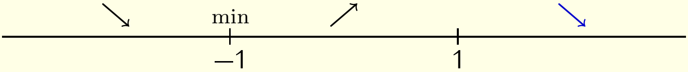
The function possesses a local minimum at $x = -1$. The value of the function is

$$y(-1) = \frac{-1}{1+(-1)^2} = -\frac{1}{2}.$$

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;



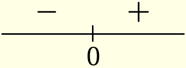
$$y' = \frac{1-x^2}{(1+x^2)^2} \;; \qquad x_{1,2} = \pm 1$$



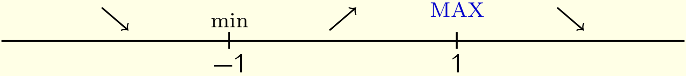
We test $x = 2$. We have

$$y'(2) = \frac{1-4}{\text{positive}} < 0.$$

$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$; odd;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

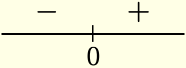


The function possesses a local maximum at $x = 1$. The value of the function is

$$y(1) = -y(-1) = \frac{1}{2},$$

since the function is known to be odd and $y(-1)$ was already established.

$\boxed{y=\dfrac{x}{1+x^2}}$ $Dom(f)=\mathbb{R}$; odd;

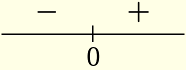


$$y'=\frac{1-x^2}{(1+x^2)^2}\ ;\qquad x_{1,2}=\pm1$$

$$y''=\left(\frac{1-x^2}{(1+x^2)^2}\right)'$$

We continue with the second derivative.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

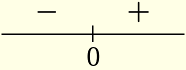


$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

$$y'' = \left(\frac{1-x^2}{(1+x^2)^2}\right)'$$
$$= \frac{-2x(1+x^2)^2-(1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4}$$

- We differentiate the quotient by the quotient rule.
- The denominator is differentiated as a composite function. This allows to factor and cancel in the next step.

$$\boxed{y=\frac{x}{1+x^2}}$$ $Dom(f)=\mathbb{R}$; odd;

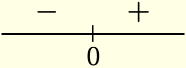


$$y'=\frac{1-x^2}{(1+x^2)^2}\ ;\qquad x_{1,2}=\pm 1$$

$$\begin{aligned}
y''&=\left(\frac{1-x^2}{(1+x^2)^2}\right)'\\
&=\frac{-2x(1+x^2)^2-(1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4}\\
&=\frac{-2x(1+x^2)[(1+x^2)+(1-x^2)2]}{(1+x^2)^4}
\end{aligned}$$

We take out the red repeating factor

$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$; odd;

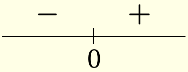


$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

$$\begin{aligned}
y'' &= \left(\frac{1-x^2}{(1+x^2)^2}\right)' \\
&= \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4} \\
&= \frac{-2x(1+x^2)[\ 1+x^2\ \ + (1-x^2)2]}{(1+x^2)^4} \\
&= \frac{-2x[3-x^2]}{(1+x^2)^3}
\end{aligned}$$

Green parts cancel. We simplify inside the brackets.

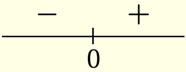
$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;



$y' = \dfrac{1-x^2}{(1+x^2)^2}$; $x_{1,2} = \pm 1$

$$\begin{aligned} y'' &= \left(\frac{1-x^2}{(1+x^2)^2}\right)' \\ &= \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4} \\ &= \frac{-2x(1+x^2)[\,1+x^2\ \ +(1-x^2)2]}{(1+x^2)^4} \\ &= \frac{-2x[3-x^2]}{(1+x^2)^3} \\ &= 2\frac{x(x^2-3)}{(1+x^2)^3} \end{aligned}$$

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;

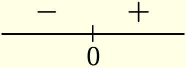


$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0$$

We solve $y'' = 0$.

$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$; odd;

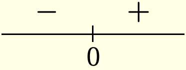


$$y' = \frac{1-x^2}{(1+x^2)^2} \;; \qquad x_{1,2} = \pm 1$$

$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

The fraction equals zero iff the numerator equals zero.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

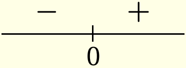
$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$

We have two possibilities: either $x = 0$ or $x^2 - 3 = 0$. The latter possibility gives

$$x^2 = 3$$
$$x = \pm\sqrt{3}.$$

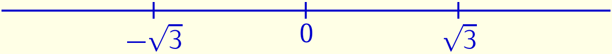
$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$; odd;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

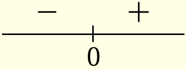
$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$



We mark the point on the x-axis. There is no point of discontinuity.

$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$; odd;



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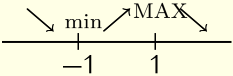

$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

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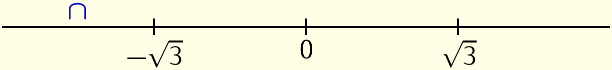
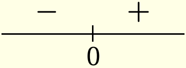


We test $x = -2$. We get

$$y''(-2) = 2\frac{-2(4-3)}{\text{positive}} < 0.$$

$$\boxed{y = \frac{x}{1+x^2}}$$ $Dom(f) = \mathbb{R}$; odd;



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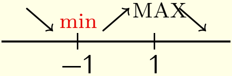

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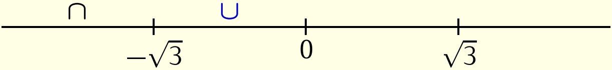
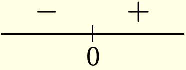


We test $x = -1$. The function is concave up, since there is a local minimum at -1.

$\boxed{y = \dfrac{x}{1+x^2}}$ $Dom(f) = \mathbb{R}$; odd;



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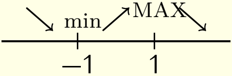

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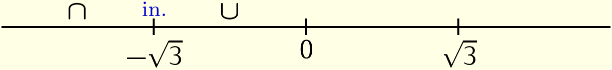
$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$



Inflection at $x = -\sqrt{3}$. The value of the function is

$$y(-\sqrt{3}) = \frac{-\sqrt{3}}{1+3} \approx -0.43.$$

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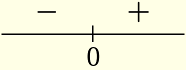

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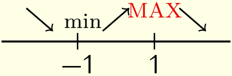

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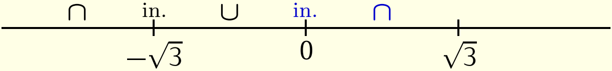
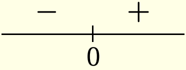
$x_3=0,\ x_4=\sqrt{3},\ x_5=-\sqrt{3}$



We test $x=1$. The function is concave down, since there is a local maximum at 1. There is an inflection at $x=0$.

$$\boxed{y=\frac{x}{1+x^2}}$$ $Dom(f)=\mathbb{R}$; odd;



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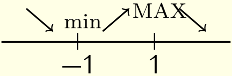

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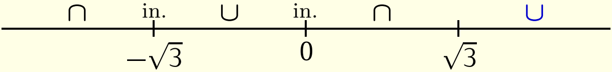

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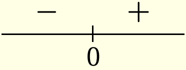
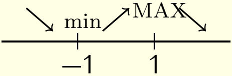
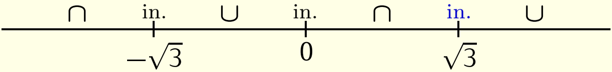


$y'=\frac{1-x^2}{(1+x^2)^2}$; $x_{1,2}=\pm 1$



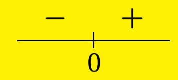
$$y''=2\frac{x(x^2-3)}{(1+x^2)^3} \quad\Rightarrow\quad 2\frac{x(x^2-3)}{(1+x^2)^3}=0 \quad\Rightarrow\quad x(x^2-3)=0$$

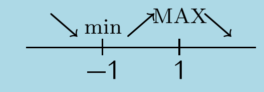
$x_3=0,\ x_4=\sqrt{3},\ x_5=-\sqrt{3}$

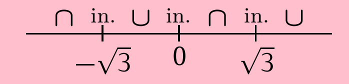


Inflection at $x=\sqrt{3}$. The value of the function is

$$y(\sqrt{3})=\frac{\sqrt{3}}{1+3}\approx 0.43.$$





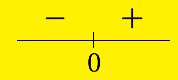


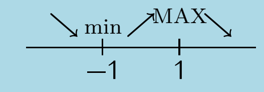
$f(0) = 0$
$f(\pm\infty) = 0$

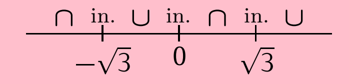
$$f(\pm 1) = \pm\frac{1}{2}$$

$$f(\pm\sqrt{3}) \approx \pm 0.433$$

We summarize all important computations.







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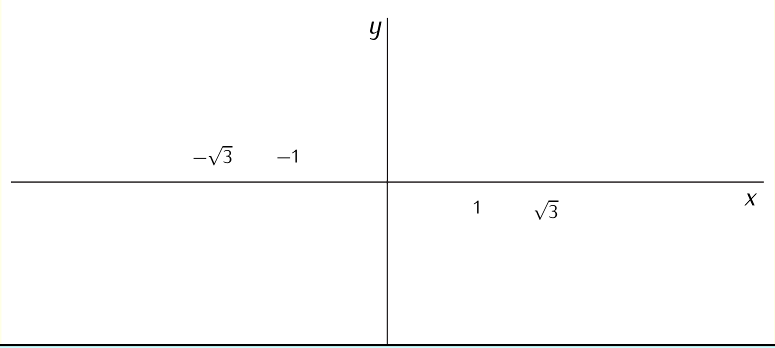
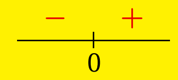
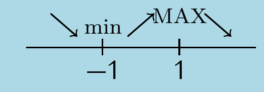
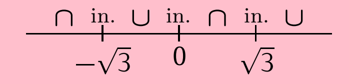


We draw the coordinate system.







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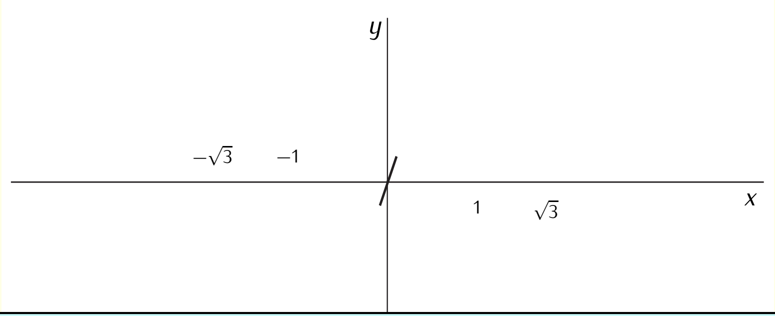
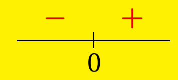
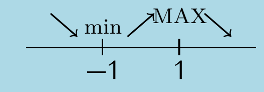
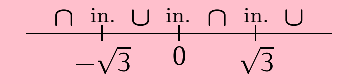


There is the x-intercept $x=0$. The sign of the function changes from negative to positive.







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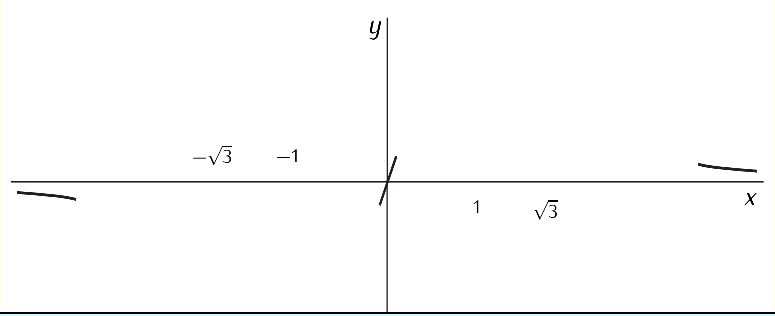
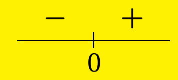
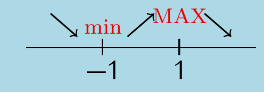
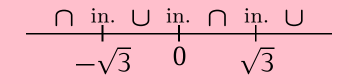


We record the information about asymptote at $\pm\infty$. We are aware of the signs of the function.







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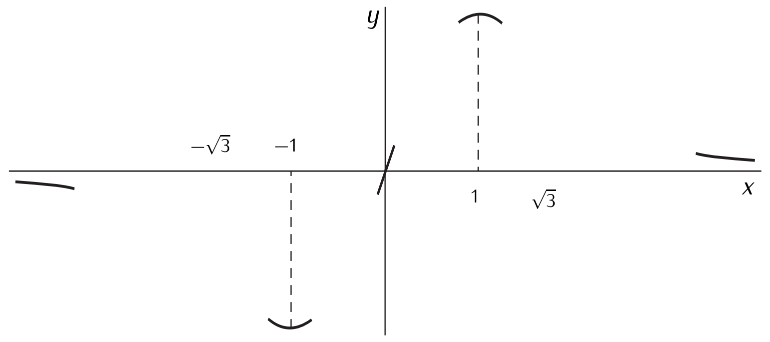

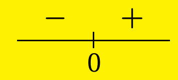

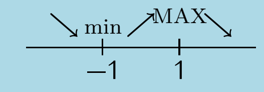

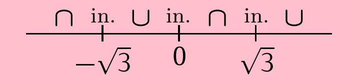

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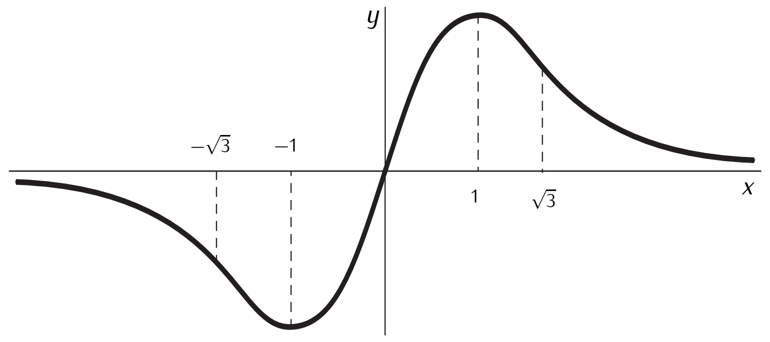


$$\boxed{y = \frac{3x+1}{x^3}}$$

$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

We establish the domain. The denominator must be nonzero.

$$\boxed{y = \frac{3x+1}{x^3}}\quad Dom(f) = \mathbb{R} \setminus \{0\}\ ;$$

$$y = 0$$

We establish the x-intercepts as a solution of the relation $y = 0$.

$\boxed{y = \frac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

$$y = 0$$
$$\frac{3x+1}{x^3} = 0$$

$$\boxed{y = \frac{3x+1}{x^3}} \quad Dom(f) = \mathbb{R} \setminus \{0\} \ ;$$

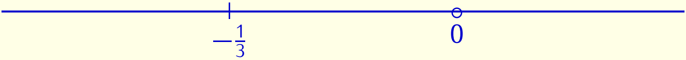
$$\begin{aligned} y &= 0 \\ \frac{3x+1}{x^3} &= 0 \\ 3x+1 &= 0 \end{aligned}$$

$\boxed{y = \dfrac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

$$\begin{gathered} y = 0 \\ \frac{3x+1}{x^3} = 0 \\ 3x+1 = 0 \\ x = -\frac{1}{3} \end{gathered}$$

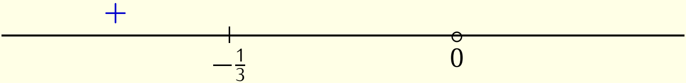
The function possesses a unique x-intercept $x = -\dfrac{1}{3}$

$\boxed{y=\frac{3x+1}{x^3}}$ $Dom(f)=\mathbb{R}\setminus\{0\}$;



- We establish the sign of the function.
- We divide the x-axis by the x-intercept $x=-\dfrac{1}{3}$ and the point of discontinuity $x=0$ into three subintervals.

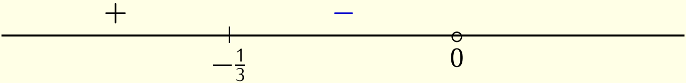
$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



We consider the leftmost interval and choose $x = -1$ from this interval. An evaluation shows

$$y(-1) = \frac{-3+1}{-1} = 2 > 0.$$

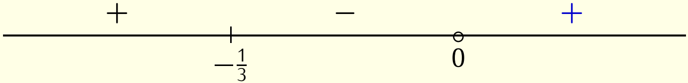
$\boxed{y = \dfrac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



We consider the middle interval and choose $x = -\dfrac{1}{4}$ from this interval. An evaluation shows

$$y(-\frac{1}{4}) = \frac{-\frac{3}{4}+1}{-\frac{1}{64}} = \frac{\frac{1}{4}}{-\frac{1}{64}} = -16 < 0.$$

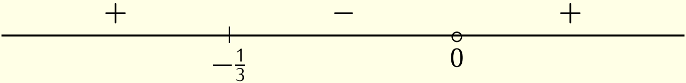
$$\boxed{y = \frac{3x+1}{x^3}}\quad Dom(f) = \mathbb{R} \setminus \{0\}\ ;$$



We consider the last interval and choose $x = 1$ from this interval. An evaluation shows

$$y(1) = \frac{3+1}{1} = 4 > 0.$$

$$\boxed{y = \frac{3x+1}{x^3}}\quad Dom(f) = \mathbb{R} \setminus \{0\}\ ;$$

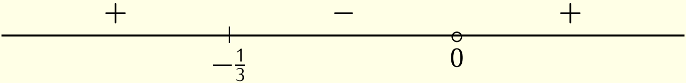


$$\lim_{x\to 0^+} \frac{3x+1}{x^3} =$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} =$$

We evaluate one-sided limits at the point of discontinuity

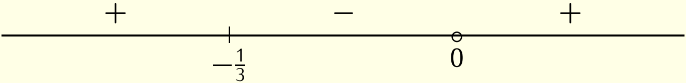
$$\boxed{y = \frac{3x+1}{x^3}}\quad Dom(f) = \mathbb{R} \setminus \{0\}\ ;$$



$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{0}$$
$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{0}$$

The substitution $x = 0$ reveals that both limits are like $\frac{\text{nonzero}}{\text{zero}}$.

$\boxed{y = \frac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

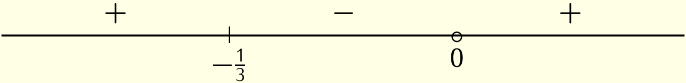


$$\lim_{x \to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x \to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

- By a theorem from lecture we know, that the one-sided limits are infinite.
- The diagram of the signs enables to decide which limit is equal to plus infinity and which to minus infinity.

$$\boxed{y=\frac{3x+1}{x^3}}\quad Dom(f)=\mathbb{R}\setminus\{0\}\ ;$$



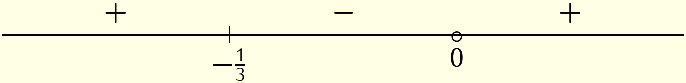
$$\lim_{x\to 0^+}\frac{3x+1}{x^3}=\frac{1}{+0}=\infty$$

$$\lim_{x\to 0^-}\frac{3x+1}{x^3}=\frac{1}{-0}=-\infty$$

$$\lim_{x\to\pm\infty}\frac{3x+1}{x^3}$$

We evaluate limits at infinity

$$\boxed{y = \frac{3x+1}{x^3}} \quad Dom(f) = \mathbb{R} \setminus \{0\} \ ;$$



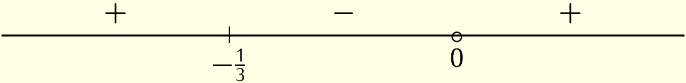
$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3}$$

It is known that only the leading terms have influence to the limit and the other terms can be omitted.

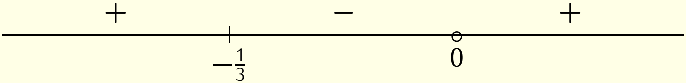
$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$
$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$
$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3} = \lim_{x\to \pm\infty} \frac{3}{x^2}$$

We cancel x.

$$\boxed{y = \frac{3x+1}{x^3}} \quad Dom(f) = \mathbb{R} \setminus \{0\} \ ;$$



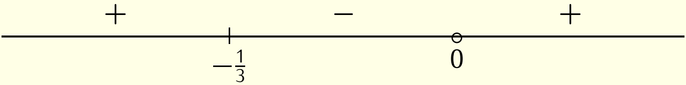
$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3} = \lim_{x\to \pm\infty} \frac{3}{x^2} = \frac{3}{\infty}$$

We substitute

$$\boxed{y = \frac{3x+1}{x^3}} \quad Dom(f) = \mathbb{R} \setminus \{0\} \;;$$



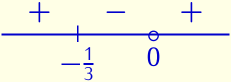
$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

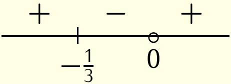
$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3} = \lim_{x\to \pm\infty} \frac{3}{x^2} = \frac{3}{\infty} = 0$$

The limit is evaluated.
The function possesses the horizontal asymptote $y = 0$ at $\pm\infty$.

$\boxed{y = \frac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

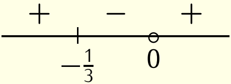


$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2}$$

We differentiate the quotient.

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$\boxed{y = \frac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

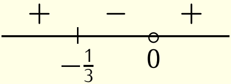


$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

We factor.

$$\boxed{y = \frac{3x+1}{x^3}}$$

$Dom(f) = \mathbb{R} \setminus \{0\}$;



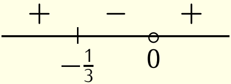
$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

$$= 3\frac{x - 3x - 1}{x^4}$$

We cancel the fraction.

$$\boxed{y = \frac{3x+1}{x^3}}$$

$Dom(f) = \mathbb{R} \setminus \{0\}$;

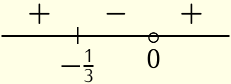


$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

$$= 3\frac{x - 3x - 1}{x^4} = 3\frac{-2x-1}{x^4}$$

We simplify the numerator.

$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

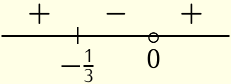


$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

$$= 3\frac{x - 3x - 1}{x^4} = 3\frac{-2x-1}{x^4} = -3\frac{2x+1}{x^4}$$

The derivative is found.

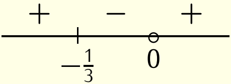
$\boxed{y = \dfrac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



$y'(x) = -3\dfrac{2x+1}{x^4}$;

The derivative is found.

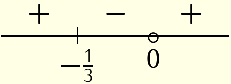
$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



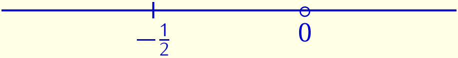
$$y'(x) = -3\frac{2x+1}{x^4} \ ; \ x_1 = -\frac{1}{2}$$

The solution of $y' = 0$ is the solution of $2x + 1 = 0$.

$$\boxed{y=\frac{3x+1}{x^3}}\quad Dom(f)=\mathbb{R}\setminus\{0\}\ ;$$

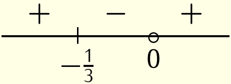


$$y'(x)=-3\frac{2x+1}{x^4}\ ;\quad x_1=-\frac{1}{2}$$

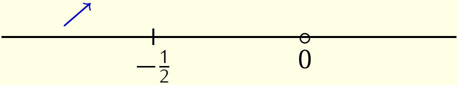


We mark the stationary point and the point of discontinuity on the x-axis.

$$\boxed{y=\frac{3x+1}{x^3}}$$ $Dom(f)=\mathbb{R}\setminus\{0\}$;

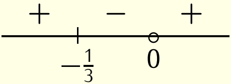


$$y'(x)=-3\frac{2x+1}{x^4}\ ;\quad x_1=-\frac{1}{2}$$

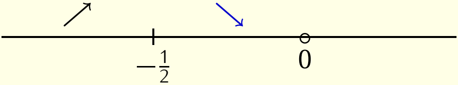


$$y'(-1)=-3\frac{-2+1}{1}=3>0$$

$$\boxed{y=\frac{3x+1}{x^3}}$$ $Dom(f)=\mathbb{R}\setminus\{0\}$;

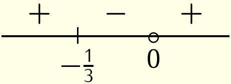


$$y'(x)=-3\frac{2x+1}{x^4}\ ;\quad x_1=-\frac{1}{2}$$

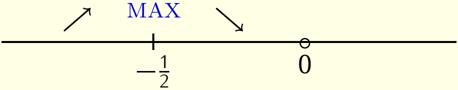


$y'(-\frac{1}{3})<0$ since the function changes the sign from positive to negative.

$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

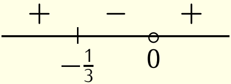


$$y'(x) = -3\frac{2x+1}{x^4} \;;\;\; x_1 = -\frac{1}{2}$$

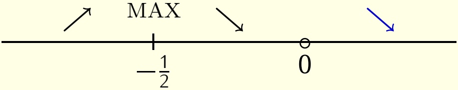


The function possesses a local maximum at $x = -\dfrac{1}{2}$. The value of the function is $y(-\dfrac{1}{2}) = \dfrac{-\frac{3}{2}+1}{-\frac{1}{8}} = \dfrac{-\frac{1}{2}}{-\frac{1}{8}} = 4$.

$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

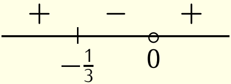


$$y'(x) = -3\frac{2x+1}{x^4} \ ; \ \ x_1 = -\frac{1}{2}$$

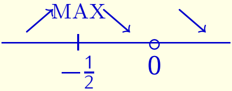


$$y'(1) = -3\frac{3}{1} = -9 < 0$$

$\boxed{y = \frac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



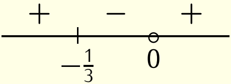
$y'(x) = -3\frac{2x+1}{x^4}$;



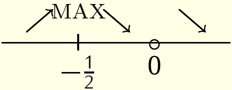
$$y'' = -3\left(\frac{2x+1}{x^4}\right)'$$

$$\boxed{y=\frac{3x+1}{x^3}}$$

$Dom(f)=\mathbb{R}\setminus\{0\}$;

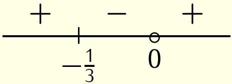


$y'(x)=-3\frac{2x+1}{x^4}$;

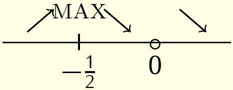


$$y''=-3\left(\frac{2x+1}{x^4}\right)'=-3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$\boxed{y=\frac{3x+1}{x^3}}$ $Dom(f)=\mathbb{R}\setminus\{0\}$;



$y'(x)=-3\frac{2x+1}{x^4}$;

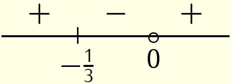


$$y''=-3\left(\frac{2x+1}{x^4}\right)'=-3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

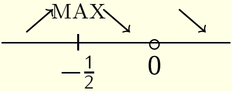
$$=-3\frac{2x^4-8x^4-4x^3}{x^8}$$

$$\boxed{y = \frac{3x+1}{x^3}}$$

$Dom(f) = \mathbb{R} \setminus \{0\}$;



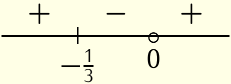
$$y'(x) = -3\frac{2x+1}{x^4} \ ;$$



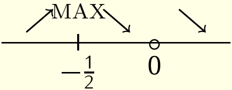
$$y'' = -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$$= -3\frac{2x^4-8x^4-4x^3}{x^8} = -3\frac{-6x^4-4x^3}{x^8}$$

$\boxed{y = \dfrac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



$y'(x) = -3\dfrac{2x+1}{x^4}$;

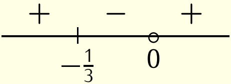


$$y'' = -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

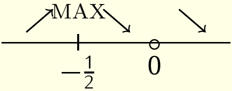
$$= -3\frac{2x^4-8x^4-4x^3}{x^8} = -3\frac{-6x^4-4x^3}{x^8}$$

$$= 6\frac{3x^4+2x^3}{x^8}$$

$$\boxed{y=\frac{3x+1}{x^3}}$$ $Dom(f)=\mathbb{R}\setminus\{0\}$;

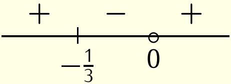


$$y'(x)=-3\frac{2x+1}{x^4}\ ;$$

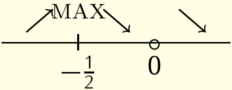


$$\begin{aligned}
y'' &= -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}\\
&= -3\frac{2x^4-8x^4-4x^3}{x^8} = -3\frac{-6x^4-4x^3}{x^8}\\
&= 6\frac{3x^4+2x^3}{x^8} = 6\frac{(3x+2)x^3}{x^8}
\end{aligned}$$

$\boxed{y=\frac{3x+1}{x^3}}$ $Dom(f)=\mathbb{R}\setminus\{0\}$;

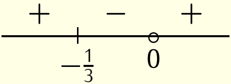


$y'(x)=-3\frac{2x+1}{x^4}$;

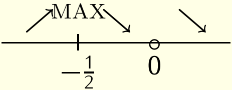


$$
\begin{aligned}
y'' &= -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}\\
&= -3\frac{2x^4-8x^4-4x^3}{x^8} = -3\frac{-6x^4-4x^3}{x^8}\\
&= 6\frac{3x^4+2x^3}{x^8} = 6\frac{(3x+2)x^3}{x^8}\\
&= 6\frac{3x+2}{x^5}
\end{aligned}
$$

$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;

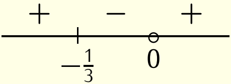


$$y'(x) = -3\frac{2x+1}{x^4}\ ;$$

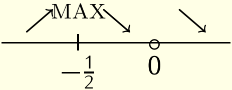


$$y'' = 6\frac{3x+2}{x^5}\ ;$$

$\boxed{y = \frac{3x+1}{x^3}}$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



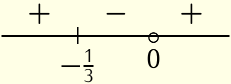
$y'(x) = -3\frac{2x+1}{x^4}$;



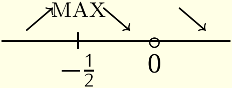
$y'' = 6\frac{3x+2}{x^5}$; $x_2 = -\frac{2}{3}$

$y'' = 0$ for $3x + 2 = 0$, i.e. $x = -\frac{2}{3}$.

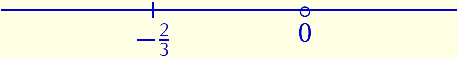
$$\boxed{y=\frac{3x+1}{x^3}}$$ $Dom(f)=\mathbb{R}\setminus\{0\}$;



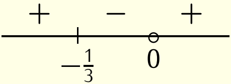
$$y'(x)=-3\frac{2x+1}{x^4}\ ;$$



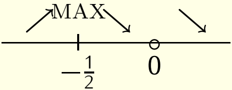
$$y''=6\frac{3x+2}{x^5}\ ;\ x_2=-\frac{2}{3}$$



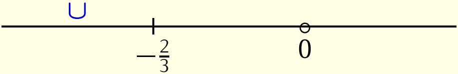
$$\boxed{y = \frac{3x+1}{x^3}}$$ $Dom(f) = \mathbb{R} \setminus \{0\}$;



$$y'(x) = -3\frac{2x+1}{x^4}$$;

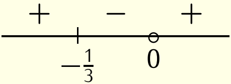


$$y'' = 6\frac{3x+2}{x^5} \text{ ; } x_2 = -\frac{2}{3}$$

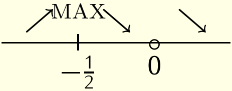


$$y''(-1) = 6\frac{-1}{-1} = 6 > 0$$

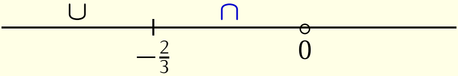
$$\boxed{y=\frac{3x+1}{x^3}}$$ $Dom(f)=\mathbb{R}\setminus\{0\}$;



$$y'(x)=-3\frac{2x+1}{x^4} \;;$$

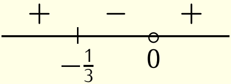


$$y''=6\frac{3x+2}{x^5} \;;\; x_2=-\frac{2}{3}$$



$$y''(-\frac{1}{3})=6\frac{-1+2}{-\frac{1}{3^5}}<0$$

$$\boxed{y=\frac{3x+1}{x^3}}\quad Dom(f)=\mathbb{R}\setminus\{0\}\ ;$$



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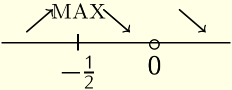
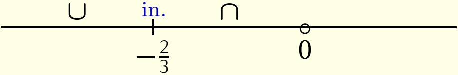
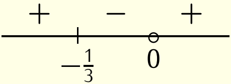


$$y''=6\frac{3x+2}{x^5}\ ;\ x_2=-\frac{2}{3}$$



Inflection at $x=-\frac{2}{3}$. $y(-\frac{2}{3})=\frac{-2+1}{-\frac{2^5}{3^5}}\approx 3.375$

$$\boxed{y=\frac{3x+1}{x^3}}\quad Dom(f)=\mathbb{R}\setminus\{0\}\ ;$$



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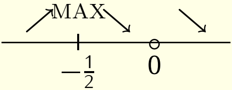
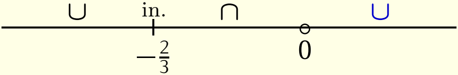
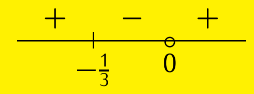
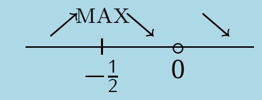


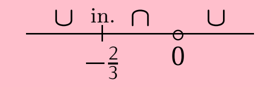
$$y''=6\frac{3x+2}{x^5}\ ;\ x_2=-\frac{2}{3}$$



$$y''(1)=6\frac{5}{1}=30>0$$





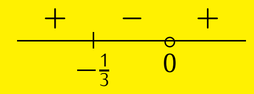


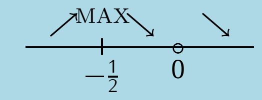
$$f(-\frac{1}{3}) = 0$$
$$f(-\frac{1}{2}) = 4$$

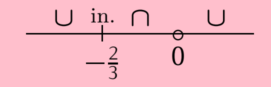
$$f(-\frac{2}{3}) \approx 3.4$$
$$f(\pm\infty) = 0,$$

$$f(0+) = \infty,$$
$$f(0-) = -\infty$$

We summarize our computations.







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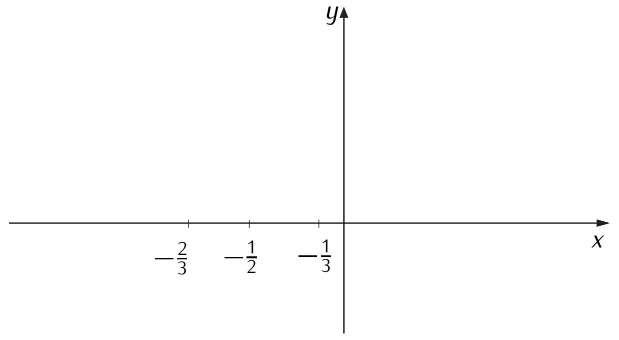

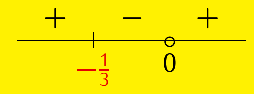

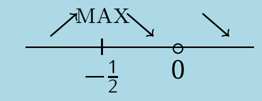

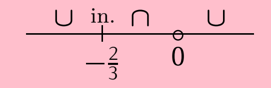

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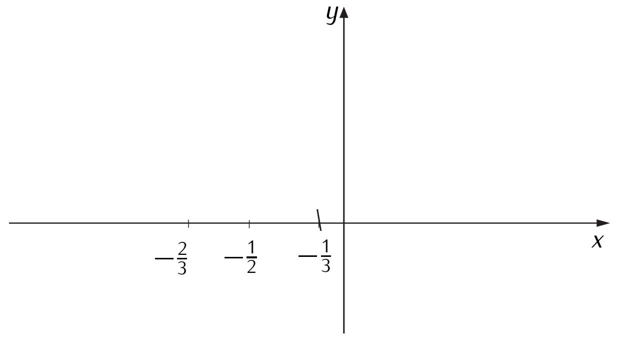

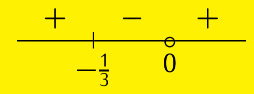

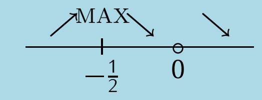

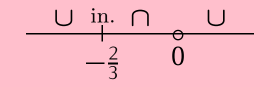

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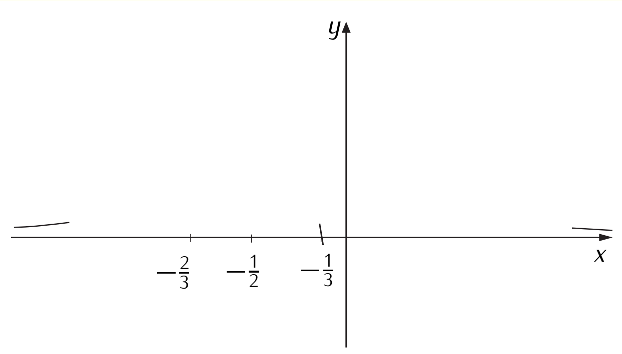

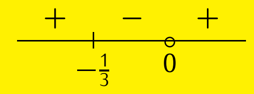

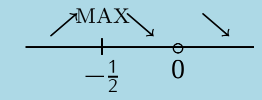

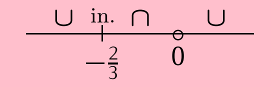

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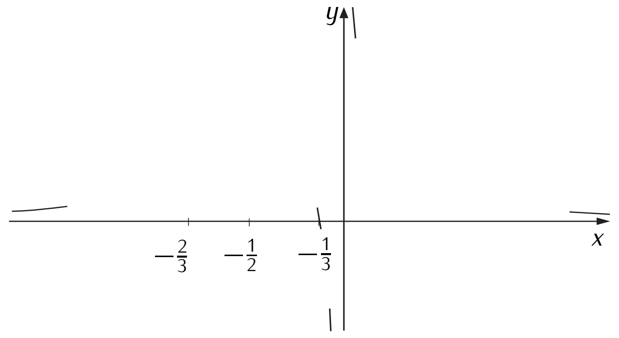

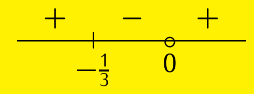

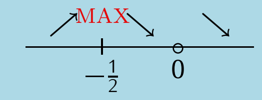

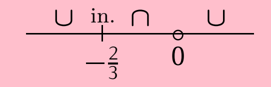

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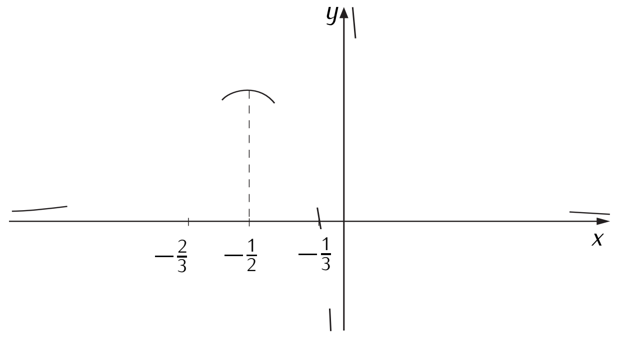

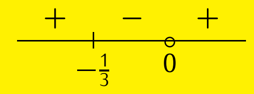

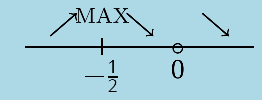

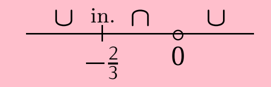

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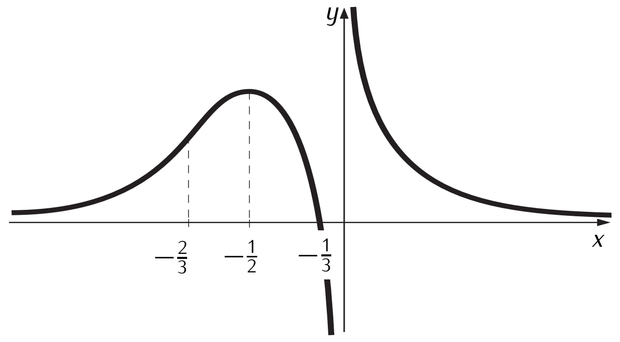


$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$,

We establish the domain. The denominator cannot equal to zero. Hence

$$x - 1 \neq 0$$

and

$$x \neq 1.$$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}} \quad Dom(f) = \mathbb{R} \setminus \{1\},$$

$$y(0) = \frac{2(0 - 0 + 1)}{(0-1)^2} = 2$$

- We establish the y-intercept.
- We substitute $x = 0$ and find $y(0)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$,

$$\frac{2(x^2 - x + 1)}{(x-1)^2} = 0$$

- We look for x-intercepts.
- We put $y = 0$ and solve the equation

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$,

$$\frac{2(x^2 - x + 1)}{(x-1)^2} = 0$$

$$x^2 - x + 1 = 0$$

The fraction equals zero iff the numerator equals zero.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$\frac{2(x^2 - x + 1)}{(x-1)^2} = 0$$
$$x^2 - x + 1 = 0$$

The quadratic equation possesses no solution, since the discriminant from the quadratic formula

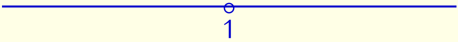
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is negative and the roots are complex numbers.

$$D = b^2 - 4ac = 1 - 4.1.1 = -3 < 0$$

$$y = \frac{2(x^2 - x + 1)}{(x-1)^2}$$

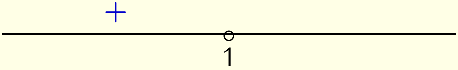
$Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept



We draw the x-axis with the point of discontinuity $x = 1$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

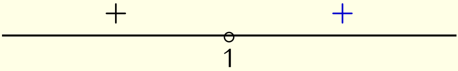
$Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept



We know that $y(0) = 2 > 0$. The function is positive on $(-\infty, 1)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

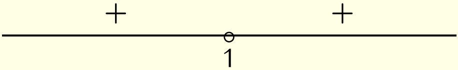


We evaluate

$$y(2) = \frac{2(4-2+1)}{(2-1)^2} > 0.$$

The function is positive on $(1, \infty)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept



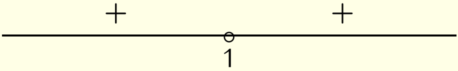
$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2}$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2}$$

We evaluate one-sided limits at the point of discontinuity

$$y = \frac{2(x^2 - x + 1)}{(x-1)^2}$$

$Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

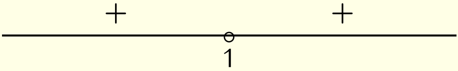


$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{0}$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{0}$$

We substitute $x = 1$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept



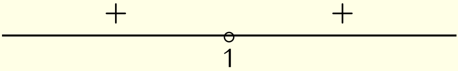
$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

We conclude the result.

$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$

$Dom(f)=\mathbb{R}\setminus\{1\}$, $y(0)=2$, no x-intercept



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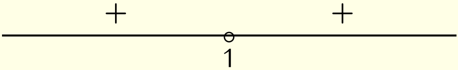
$$\lim_{x\to 1^-}\frac{2(x^2-x+1)}{(x-1)^2}=\frac{2}{+0}=+\infty$$

$$\lim_{x\to\pm\infty}\frac{2(x^2-x+1)}{(x-1)^2}$$

We evaluate limits at $\pm\infty$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept



$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

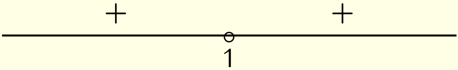
$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to \pm\infty} \frac{2(x^2 - x + 1)}{(x-1)^2} = \lim_{x\to \pm\infty} \frac{2x^2}{x^2} =$$

We consider the leading coefficients only.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept



$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to \pm\infty} \frac{2(x^2 - x + 1)}{(x-1)^2} = \lim_{x\to \pm\infty} \frac{2x^2}{x^2} = \lim_{x\to \pm\infty} \frac{2}{1} = 2$$

The function possesses a finite limit at $\pm\infty$. Hence the line $y = 2$ is a horizontal asymptote to the graph at both $\pm\infty$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$y' = 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)'$$

We evaluate the derivative

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$\begin{aligned} y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\ &= 2\frac{(2x-1)(x-1)^2 - (x^2-x+1)2(x-1)(1-0)}{((x-1)^2)^2} \end{aligned}$$

- We use the quotient rule
$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$
- We use the chain rule to differentiate the factor $(x-1)^2$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$y' = 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)'$$

$$= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2}$$

$$= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4}$$

We take out the repeating factor $(x-1)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$
\begin{aligned}
y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\
&= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\
&= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \\
&= 2\frac{2x^2 - 2x - x + 1 - (2x^2 - 2x + 2)}{(x-1)^3}
\end{aligned}
$$

We multiply the parentheses and cancel the factor $(x-1)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$\begin{aligned}
y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\
&= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\
&= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \\
&= 2\frac{2x^2 - 2x - x + 1 - (2x^2 - 2x + 2)}{(x-1)^3} \\
&= 2\frac{-x-1}{(x-1)^3}
\end{aligned}$$

We simplify the numerator.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$\begin{aligned}
y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\
&= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\
&= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \\
&= 2\frac{2x^2 - 2x - x + 1 - (2x^2 - 2x + 2)}{(x-1)^3} \\
&= 2\frac{-x-1}{(x-1)^3} = -2\frac{x+1}{(x-1)^3}
\end{aligned}$$

The derivative is found.

$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$ $Dom(f)=\mathbb{R}\setminus\{1\}$, $y(0)=2$, no x-intercept

$y'=-2\frac{x+1}{(x-1)^3}$;

$$-2\frac{x+1}{(x-1)^3}=0$$

We solve the equation $y'=0$. This gives stationary points.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$y' = -2\frac{x+1}{(x-1)^3} \quad ;$$

$$-2\frac{x+1}{(x-1)^3} = 0$$

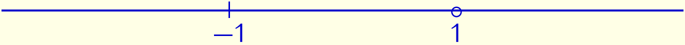
$$x + 1 = 0$$

$$x = -1$$

The fraction equals zero iff the numerator is zero. Hence, there is a unique stationary point $x = -1$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

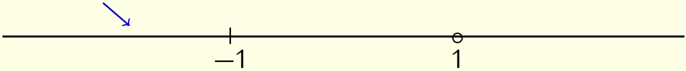
$$y' = -2\frac{x+1}{(x-1)^3} \quad ; \; x_1 = -1$$



We draw the scheme with the stationary point and the point of discontinuity.

$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$y' = -2\frac{x+1}{(x-1)^3}$; $x_1 = -1$

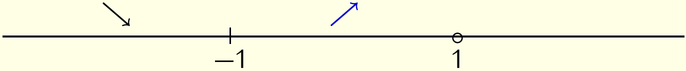


We evaluate $y'(-2)$.

$$y'(-2) = -2\frac{-2+1}{(-2-1)^3} = -2\frac{\text{negative}}{\text{negative}} < 0$$

$\boxed{y = \dfrac{2(x^2 - x + 1)}{(x-1)^2}}$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$y' = -2\dfrac{x+1}{(x-1)^3}$; $x_1 = -1$

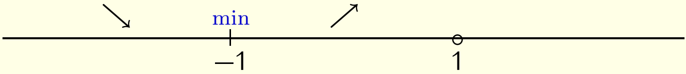


We evaluate $y'(0)$.

$$y'(0) = -2\frac{0+1}{(0-1)^3} = -2\frac{\text{positive}}{\text{negative}} > 0$$

$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$y' = -2\frac{x+1}{(x-1)^3}$; $x_1 = -1$… local minimum, $y(-1) = \frac{3}{2}$

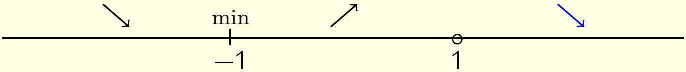


Local minimum at $x = -1$. The value is

$$y(-1) = \frac{2((-1)^2 - (-1) + 1)}{(-1-1)^2} = \frac{2.3}{4} = \frac{3}{2}.$$

$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$ $Dom(f)=\mathbb{R}\setminus\{1\}$, $y(0)=2$, no x-intercept

$y'=-2\frac{x+1}{(x-1)^3}$; $x_1=-1\ldots$ local minimum, $y(-1)=\frac{3}{2}$



$$y'(2)=-2\frac{2+1}{(2-1)^3}=-2\frac{3}{1}<0$$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$y' = -2\frac{x+1}{(x-1)^3}$$; $x_1 = -1\ldots$ local minimum, $y(-1) = \frac{3}{2}$

$$y'' = -2\left(\frac{x+1}{(x-1)^3}\right)'$$

We evaluate the second derivative.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$y' = -2\dfrac{x+1}{(x-1)^3}$; $x_1 = -1 \ldots$ local minimum, $y(-1) = \dfrac{3}{2}$

$$y'' = -2\left(\frac{x+1}{(x-1)^3}\right)'$$

$$= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2}$$

- We differentiate the quotient by the quotient rule.
- The denominator is differentiated as a composite function.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$y' = -2\dfrac{x+1}{(x-1)^3}$; $x_1 = -1\ldots$ local minimum, $y(-1) = \dfrac{3}{2}$

$$\begin{aligned} y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\ &= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \\ &= -2(x-1)^2\frac{(x-1)-(x+1)3}{(x-1)^6} \end{aligned}$$

We take out the common factor $(x-1)^2$ from the numerator.

$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$ $Dom(f)=\mathbb{R}\setminus\{1\}$, $y(0)=2$, no x-intercept

$y'=-2\frac{x+1}{(x-1)^3}$; $x_1=-1\ldots$ local minimum, $y(-1)=\frac{3}{2}$

$$\begin{aligned} y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\ &= -2\frac{1(x-1)^3-(x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \\ &= -2(x-1)^2\frac{(x-1)-(x+1)3}{(x-1)^6} \\ &= -2\frac{-2x-4}{(x-1)^4} \end{aligned}$$

We cancel and simplify.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$y' = -2\frac{x+1}{(x-1)^3}$; $x_1 = -1 \ldots$ local minimum, $y(-1) = \frac{3}{2}$

$$\begin{aligned} y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\ &= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \\ &= -2(x-1)^2\frac{(x-1)-(x+1)3}{(x-1)^6} \\ &= -2\frac{-2x-4}{(x-1)^4} = 4\frac{x+2}{(x-1)^4} \end{aligned}$$

The second derivative is known.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$y' = -2\frac{x+1}{(x-1)^3}$; $x_1 = -1 \ldots$ local minimum, $y(-1) = \frac{3}{2}$

$y'' = 4\frac{x+2}{(x-1)^4}$;

$$4\frac{x+2}{(x-1)^4} = 0$$

We solve the equation $y'' = 0$.

$\boxed{y = \dfrac{2(x^2 - x + 1)}{(x-1)^2}}$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$y' = -2\dfrac{x+1}{(x-1)^3}$; $x_1 = -1 \ldots$ local minimum, $y(-1) = \dfrac{3}{2}$

$y'' = 4\dfrac{x+2}{(x-1)^4}$; $x_2 = -2$

$$4\frac{x+2}{(x-1)^4} = 0$$

$$x + 2 = 0$$

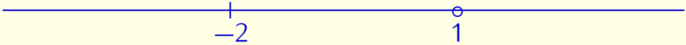
$$x = -2$$

The equation possesses a unique solution $x = -2$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$y' = -2\frac{x+1}{(x-1)^3} \quad ; x_1 = -1 \ldots \text{local minimum, } y(-1) = \frac{3}{2}$$

$$y'' = 4\frac{x+2}{(x-1)^4} \quad ; x_2 = -2$$

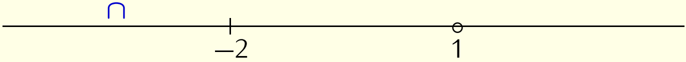


We draw the diagram with intervals of concavity.

$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$ $Dom(f)=\mathbb{R}\setminus\{1\}$, $y(0)=2$, no x-intercept

$y'=-2\dfrac{x+1}{(x-1)^3}$; $x_1=-1\ldots$ local minimum, $y(-1)=\dfrac{3}{2}$

$y''=4\dfrac{x+2}{(x-1)^4}$; $x_2=-2$

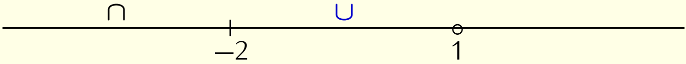


$$y''(-3)=4\frac{-3+2}{\text{positive}}<0$$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$y' = -2\frac{x+1}{(x-1)^3} \quad ; x_1 = -1 \ldots \text{local minimum, } y(-1) = \frac{3}{2}$$

$$y'' = 4\frac{x+2}{(x-1)^4} \quad ; x_2 = -2$$

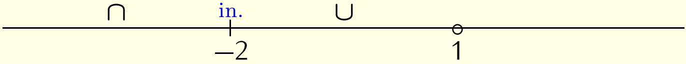


$$y''(0) = 4\frac{0+2}{\text{positive}} > 0$$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $Dom(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, no x-intercept

$$y' = -2\frac{x+1}{(x-1)^3} \quad ; x_1 = -1 \ldots \text{local minimum, } y(-1) = \frac{3}{2}$$

$$y'' = 4\frac{x+2}{(x-1)^4} \quad ; x_2 = -2$$



Inflection at $x = -2$. The value of the function at this point is

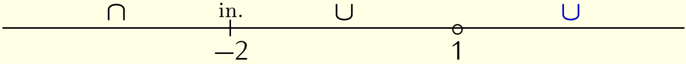
$$y(-2) = \frac{14}{9}.$$

(Try yourself.)

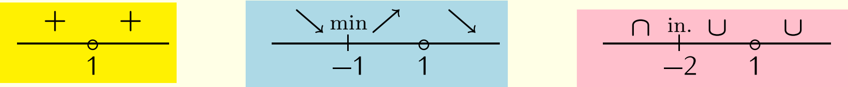
$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$ $Dom(f)=\mathbb{R}\setminus\{1\}$, $y(0)=2$, no x-intercept

$$y'=-2\frac{x+1}{(x-1)^3}\quad;\ x_1=-1\ldots\text{local minimum},\ y(-1)=\frac{3}{2}$$

$$y''=4\frac{x+2}{(x-1)^4}\quad;\ x_2=-2$$



$$y''(2)=4\frac{2+1}{\text{positive}}>0$$



$f(0) = 2$

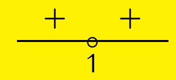
$f(\pm\infty) = 2$

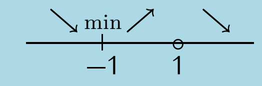
$f(1\pm) = +\infty$

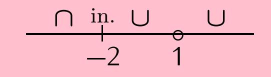
$f(-1) = \dfrac{3}{2}$

$f(-2) = \dfrac{14}{9}$

We summarize the computations.







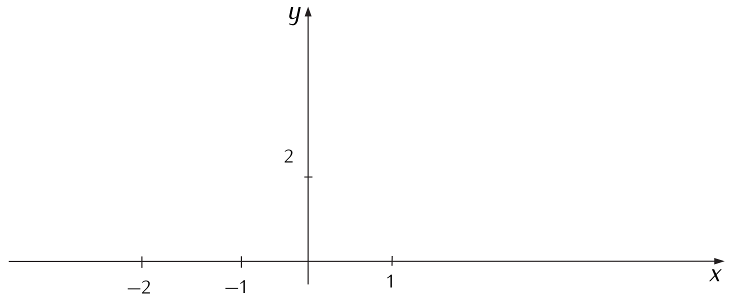
$f(0) = 2$

$f(\pm\infty) = 2$

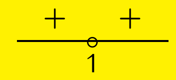
$f(1\pm) = +\infty$

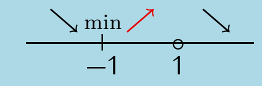
$f(-1) = \dfrac{3}{2}$

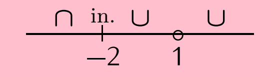
$f(-2) = \dfrac{14}{9}$



We draw the coordinate system.







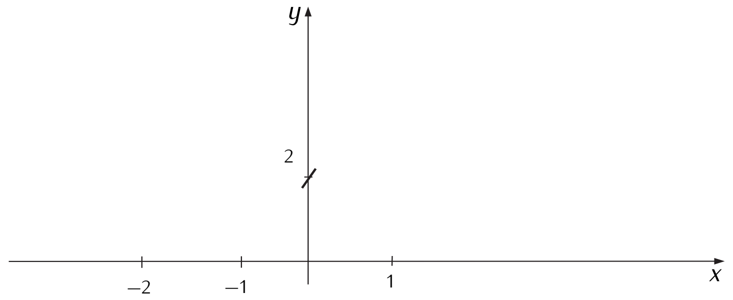
$f(0) = 2$

$f(\pm\infty) = 2$

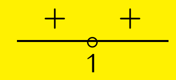
$f(1\pm) = +\infty$

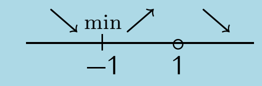
$f(-1) = \frac{3}{2}$

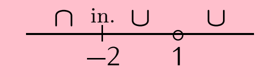
$f(-2) = \frac{14}{9}$



We mark the y-intercept. The function is increasing in this point.







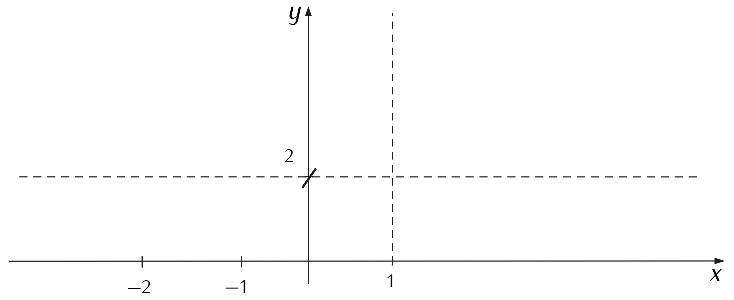
$f(0) = 2$

$f(\pm\infty) = 2$

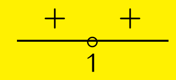
$f(1\pm) = +\infty$

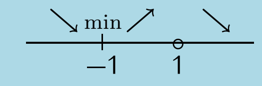
$f(-1) = \dfrac{3}{2}$

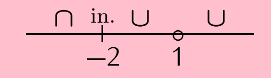
$f(-2) = \dfrac{14}{9}$



We draw the asymptotes.







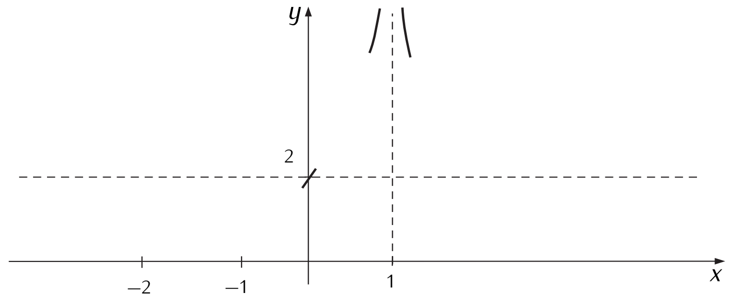
$f(0) = 2$

$f(\pm\infty) = 2$

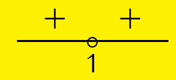
$f(1\pm) = +\infty$

$f(-1) = \dfrac{3}{2}$

$f(-2) = \dfrac{14}{9}$

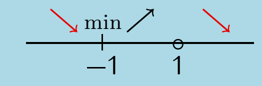


We draw the function close to the vertical asymptote.



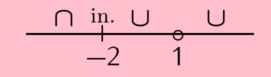
$f(0) = 2$

$f(\pm\infty) = 2$

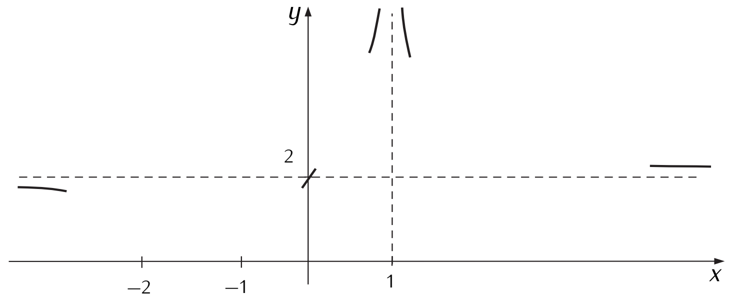


$f(1\pm) = +\infty$

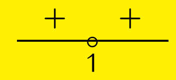
$f(-1) = \dfrac{3}{2}$

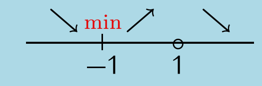


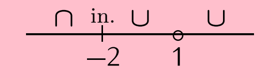
$f(-2) = \dfrac{14}{9}$



We draw the function close to the horizontal asymptote.







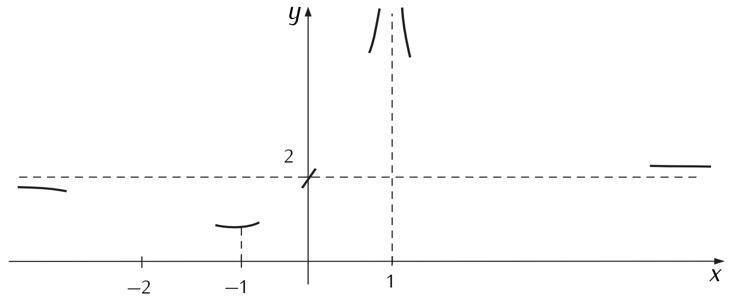
$f(0) = 2$

$f(\pm\infty) = 2$

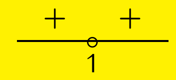
$f(1\pm) = +\infty$

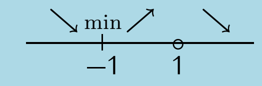
$f(-1) = \frac{3}{2}$

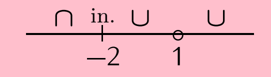
$f(-2) = \frac{14}{9}$



We draw the local minimum of the function.







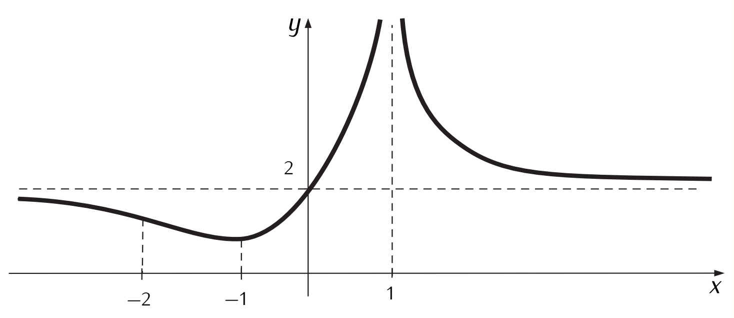
$f(0) = 2$

$f(\pm\infty) = 2$

$f(1\pm) = +\infty$

$f(-1) = \dfrac{3}{2}$

$f(-2) = \dfrac{14}{9}$



Finished.

$$\boxed{y = \frac{x^3}{3 - x^2}}$$

$$\boxed{y = \frac{x^3}{3 - x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$;

The restriction $3 - x^2 \neq 0$ reveals two points of discontinuity: $\pm\sqrt{3}$.

$$\boxed{y = \frac{x^3}{3-x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

The y-intercept is at

$$y(0) = \frac{0}{3-0} = 0.$$

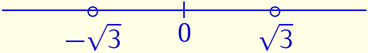
$\boxed{y = \dfrac{x^3}{3 - x^2}}$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

x-intercept: $x = 0$

Solving $\dfrac{x^3}{3 - x^2} = 0$ we get the unique x-intercept at $x = 0$.

$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

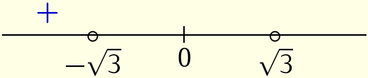
x-intercept: $x = 0$



We mark the x-intercept and points of discontinuity on real axis.

$$\boxed{y = \frac{x^3}{3 - x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\};$ $y(0) = 0$

x-intercept: $x = 0$



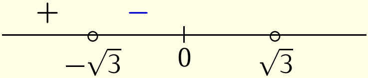
Inequality

$$y(-2) = \frac{-8}{3-4} = 8 > 0$$

holds and the function is positive on $(-\infty, -\sqrt{3})$.

$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



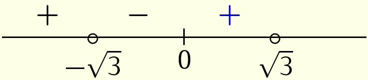
Inequality

$$y(-1) = \frac{-1}{3-1} = -\frac{1}{2} < 0$$

holds and the function is negative on $(-\sqrt{3}, 0)$.

$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



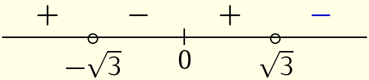
Inequality

$$y(1) = \frac{1}{3 - 1} = \frac{1}{2} > 0$$

holds and the function is positive on $(0, \sqrt{3})$.

$$\boxed{y = \frac{x^3}{3 - x^2}}\quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



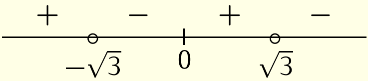
Inequality

$$y(2) = \frac{8}{3-4} = -8 < 0$$

holds and the function is negative on $(\sqrt{3}, \infty)$.

$$\boxed{y = \frac{x^3}{3 - x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

x-intercept: $x = 0$



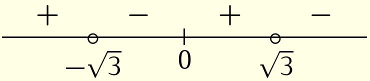
$$\lim_{x \to -\sqrt{3}^-} \frac{x^3}{3 - x^2} \qquad \lim_{x \to \sqrt{3}^-} \frac{x^3}{3 - x^2}$$

$$\lim_{x \to -\sqrt{3}^+} \frac{x^3}{3 - x^2} \qquad \lim_{x \to \sqrt{3}^+} \frac{x^3}{3 - x^2}$$

We investigate one-sided limits at the points of discontinuity.

$$\boxed{y = \frac{x^3}{3-x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

x-intercept: $x = 0$



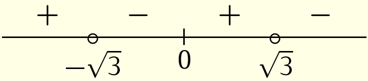
$$\lim_{x\to-\sqrt{3}^-} \frac{x^3}{3-x^2} = \frac{-\sqrt{27}}{0} = \infty \qquad \lim_{x\to\sqrt{3}^-} \frac{x^3}{3-x^2} = \frac{\sqrt{27}}{0} = \infty$$

$$\lim_{x\to-\sqrt{3}^+} \frac{x^3}{3-x^2} = \frac{-\sqrt{27}}{0} = -\infty \qquad \lim_{x\to\sqrt{3}^+} \frac{x^3}{3-x^2} = \frac{\sqrt{27}}{0} = -\infty$$

All limits are of the type $\frac{\text{nonzero}}{0}$ and the limits are $\pm\infty$. The correct sign can be established from the sign chart without any additional computation.

$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



$$\lim_{x\to-\sqrt{3}^-} \frac{x^3}{3-x^2} = \frac{-\sqrt{27}}{0} = \infty \qquad \lim_{x\to\sqrt{3}^-} \frac{x^3}{3-x^2} = \frac{\sqrt{27}}{0} = \infty$$

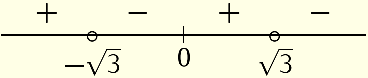
$$\lim_{x\to-\sqrt{3}^+} \frac{x^3}{3-x^2} = \frac{-\sqrt{27}}{0} = -\infty \qquad \lim_{x\to\sqrt{3}^+} \frac{x^3}{3-x^2} = \frac{\sqrt{27}}{0} = -\infty$$

$$\lim_{x\to\pm\infty} \frac{x^3}{3-x^2}$$

We find limits at infinity.

$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



$+ \quad - \quad + \quad -$

$-\sqrt{3} \quad 0 \quad \sqrt{3}$

$$\lim_{x \to -\sqrt{3}^-} \frac{x^3}{3 - x^2} = \frac{-\sqrt{27}}{0} = \infty \qquad \lim_{x \to \sqrt{3}^-} \frac{x^3}{3 - x^2} = \frac{\sqrt{27}}{0} = \infty$$

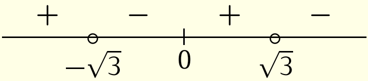
$$\lim_{x \to -\sqrt{3}^+} \frac{x^3}{3 - x^2} = \frac{-\sqrt{27}}{0} = -\infty \qquad \lim_{x \to \sqrt{3}^+} \frac{x^3}{3 - x^2} = \frac{\sqrt{27}}{0} = -\infty$$

$$\lim_{x \to \pm\infty} \frac{x^3}{3 - x^2} = \lim_{x \to \pm\infty} \frac{x^3}{-x^2} = \lim_{x \to \pm\infty} -x = \mp\infty$$

The function is a quotient of two polynomials and the value of the limit can be established from leading terms in numerator and denominator.

$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

x-intercept: $x=0$



$+\quad -\quad +\quad -$

$-\sqrt{3}\quad 0\quad \sqrt{3}$

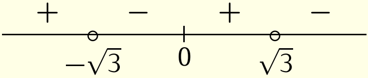
$$y'=\frac{3x^2\cdot(3-x^2)-x^3\cdot(0-2x)}{(3-x^2)^2}$$

We differentiate the quotient $\frac{x^3}{3-x^2}$ using the quotient rule

$$\left(\frac{u}{v}\right)'=\frac{u'\cdot v-u\cdot v'}{v^2}.$$

$$\boxed{y = \frac{x^3}{3-x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

x-intercept: $x = 0$



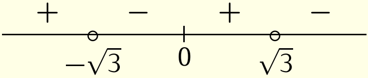
$$y' = \frac{3x^2 \cdot (3-x^2) - x^3 \cdot (0-2x)}{(3-x^2)^2}$$

$$= \frac{x^2\Big(3(3-x^2)+2x^2\Big)}{(3-x^2)^2}$$

We factorize, the common factor is x^2.

$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

x-intercept: $x=0$



$$y'=\frac{3x^2\cdot(3-x^2)-x^3\cdot(0-2x)}{(3-x^2)^2}$$

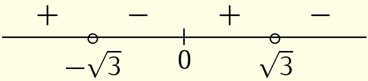
$$=\frac{x^2\Big(3(3-x^2)+2x^2\Big)}{(3-x^2)^2}$$

$$=\frac{x^2\Big(9-x^2\Big)}{(3-x^2)^2}$$

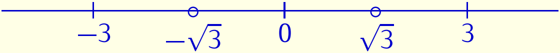
We simplify.

$\boxed{y=\dfrac{x^3}{3-x^2}}$ $D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\}$; $y(0)=0$

x-intercept: $x=0$



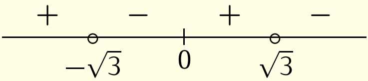
$$y'=\frac{x^2\left(9-x^2\right)}{(3-x^2)^2}\ ;$$



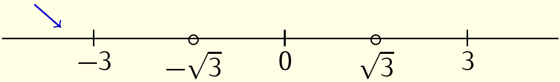
The solution of $x^2(9-x^2)=0$ is $x=0$ and $x=\pm3$. We mark these points and points of discontinuity on the real axis.

$$\boxed{y = \frac{x^3}{3 - x^2}}\quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



$$y' = \frac{x^2\left(9 - x^2\right)}{(3 - x^2)^2}\ ;$$

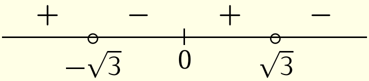


The red marked expressions in the derivative are positive and have no influence to the sign of this derivative. It is sufficient to focus ourselves to the sign of the expression $(9 - x^2)$. For $x = -4$ we have

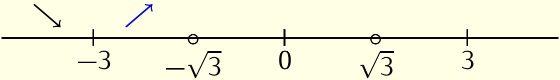
$$9 - x^2 = 9 - (-4)^2 < 0.$$

$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

x-intercept: $x=0$



$$y'=\frac{x^2\left(9-x^2\right)}{(3-x^2)^2}\ ;$$

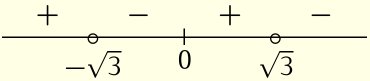


For $x=-2$ we have

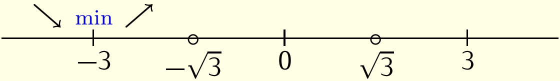
$$9-x^2=9-(-2)^2>0.$$

$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



$$y' = \frac{x^2\left(9-x^2\right)}{(3-x^2)^2} \;;$$

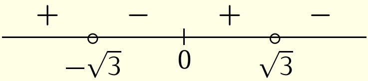


The local minimum appears at $x = -3$. The value of the function at this point is

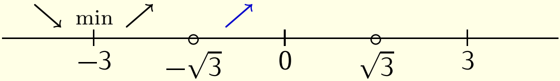
$$y(-3) = \frac{-27}{3-9} = \frac{-27}{-6} = \frac{9}{2}$$

$$\boxed{y = \frac{x^3}{3 - x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

x-intercept: $x = 0$



$$y' = \frac{x^2\left(9 - x^2\right)}{(3 - x^2)^2} ;$$

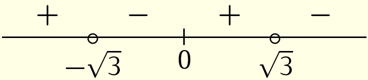


For $x = -1$ we have

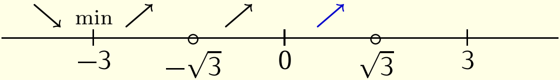
$$9 - x^2 = 9 - (-1)^2 > 0.$$

$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



$$y' = \frac{x^2\left(9-x^2\right)}{(3-x^2)^2}\ ;$$

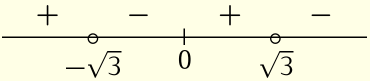


For $x = 1$ we have

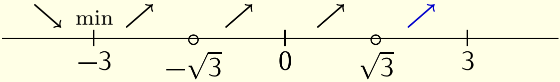
$$9 - x^2 = 9 - 1^2 > 0.$$

$\boxed{y = \dfrac{x^3}{3-x^2}}$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

x-intercept: $x = 0$



$$y' = \frac{x^2\left(9-x^2\right)}{(3-x^2)^2};$$

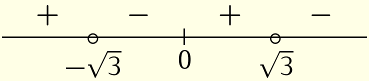


For $x = 2$ we have

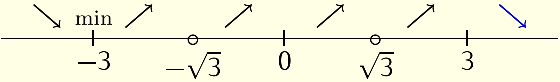
$$9 - x^2 = 9 - 2^2 > 0.$$

$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



$$y' = \frac{x^2\left(9-x^2\right)}{(3-x^2)^2};$$

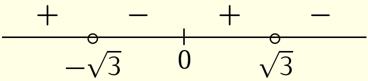


For $x = 4$ we have

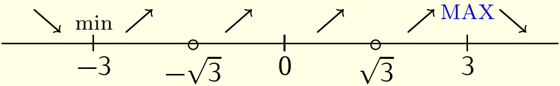
$$9 - x^2 = 9 - 4^2 < 0.$$

$$\boxed{y = \frac{x^3}{3-x^2}}\quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

x-intercept: $x = 0$



$$y' = \frac{x^2\left(9-x^2\right)}{(3-x^2)^2}\ ;$$



A local minimum appears at $x = 3$. The value of the function at this point is

$$y(3) = \frac{27}{3-9} = \frac{27}{-6} = -\frac{9}{2}$$

$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

$$y''=\frac{(18x-4x^3)\cdot(3-x^2)^2-(9x^2-x^4)\cdot 2(3-x^2)(-2x)}{\left((3-x^2)^2\right)^2}$$

We differentiate the function

$$\frac{x^2(9-x^2)}{(3-x^2)^2}=\frac{9x^2-x^4}{(3-x^2)^2}$$

using the quotient rule

$$\left(\frac{u}{v}\right)'=\frac{u'\cdot v-u\cdot v'}{v^2}$$

and find y''.

$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

$$y'' = \frac{(18x - 4x^3)\cdot(3-x^2)^2 - (9x^2 - x^4)\cdot 2(3-x^2)(-2x)}{\left((3-x^2)^2\right)^2}$$

$$= \frac{2x(3-x^2)\cdot\left[(9-2x^2)(3-x^2) + (9x-x^3)(2x)\right]}{(3-x^2)^4}$$

We factorize the repeating terms in numerator.

$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

$$\begin{aligned}
y''&=\frac{(18x-4x^3)\cdot(3-x^2)^2-(9x^2-x^4)\cdot 2(3-x^2)(-2x)}{\Bigl((3-x^2)^2\Bigr)^2}\\
&=\frac{2x(3-x^2)\cdot\Bigl[(9-2x^2)(3-x^2)+(9x-x^3)(2x)\Bigr]}{(3-x^2)^4}\\
&=\frac{2x\cdot\Bigl[27-9x^2-6x^2+2x^4+18x^2-2x^4\Bigr]}{(3-x^2)^3}
\end{aligned}$$

We cancel and simplify.

$$\boxed{y=\frac{x^3}{3-x^2}}\qquad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

$$\begin{aligned}
y'' &= \frac{(18x-4x^3)\cdot(3-x^2)^2-(9x^2-x^4)\cdot 2(3-x^2)(-2x)}{\left((3-x^2)^2\right)^2}\\
&= \frac{2x(3-x^2)\cdot\left[(9-2x^2)(3-x^2)+(9x-x^3)(2x)\right]}{(3-x^2)^4}\\
&= \frac{2x\cdot\left[27-9x^2-6x^2+2x^4+18x^2-2x^4\right]}{(3-x^2)^3}\\
&= \frac{2x\cdot\left[27+3x^2\right]}{(3-x^2)^3}
\end{aligned}$$

Another simplification.

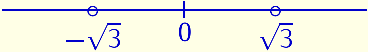
$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

$$y'' = \frac{2x \cdot \left[27 + 3x^2\right]}{(3-x^2)^3} \; ; \; x = 0$$

We solve the equation $y'' = 0$. The unique solution of this equation is $x = 0$, since the expression $(27 + 3x^2)$ is positive for all x.

$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

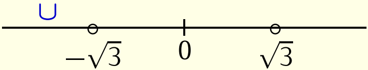
$$y'' = \frac{2x \cdot \left[27 + 3x^2\right]}{(3 - x^2)^3} \ ; \ x = 0$$



We mark the point $x = 0$ and points of discontinuity on real axis.

$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

$$y'' = \frac{2x \cdot \left[27 + 3x^2\right]}{(3 - x^2)^3} \text{ ; } x = 0$$



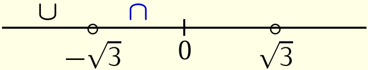
Substituting $x = -2$ we get

$$y''(-2) = \frac{2 \cdot (-2) \cdot [\text{positive}]}{(3 - (-2)^2)^3} = \frac{\text{negative}}{\text{negative}} > 0$$

and the function is concave up on the interval involving -2.

$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

$$y'' = \frac{2x \cdot \left[27 + 3x^2\right]}{(3 - x^2)^3}\ ;\ x = 0$$



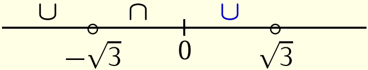
Substituting $x = -1$ we get

$$y''(-1) = \frac{2 \cdot (-1) \cdot [\text{positive}]}{(3 - (-1)^2)^3} = \frac{\text{negative}}{\text{positive}} < 0$$

and the function is concave down on the interval involving -1.

$$\boxed{y = \frac{x^3}{3 - x^2}}\quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

$$y'' = \frac{2x \cdot \left[27 + 3x^2\right]}{(3 - x^2)^3}\ ;\ x = 0$$



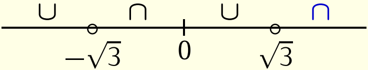
Substituting $x = 1$ we get

$$y''(1) = \frac{2 \cdot 1 \cdot [\text{positive}]}{(3 - 1^2)^3} = \frac{\text{positive}}{\text{positive}} > 0$$

and the function is concave up on the interval involving 1.

$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

$$y''=\frac{2x\cdot\left[27+3x^2\right]}{(3-x^2)^3}\ ;\ x=0$$



Substituting $x=2$ we get

$$y''(2)=\frac{2\cdot 2\cdot[\text{positive}]}{(3-2^2)^3}=\frac{\text{negative}}{\text{negative}}<0$$

and the function is concave down on the interval involving 1.

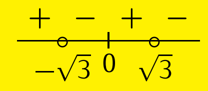
$\boxed{y = \frac{x^3}{3-x^2}}$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $\quad y(0) = 0$

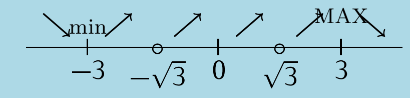
Long division shows

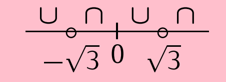
$$\frac{x^3}{3-x^2} = -x + \frac{3x}{3-x^2}$$

The first part is a linear function, the second part approaches zero a x approaches plus or minus infinity.

The asymptote at $\pm\infty$ is $y = -x$.



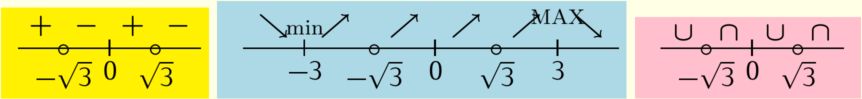




$f(0) = 0;$ $\qquad f(\pm 3) = \mp\frac{9}{2}$

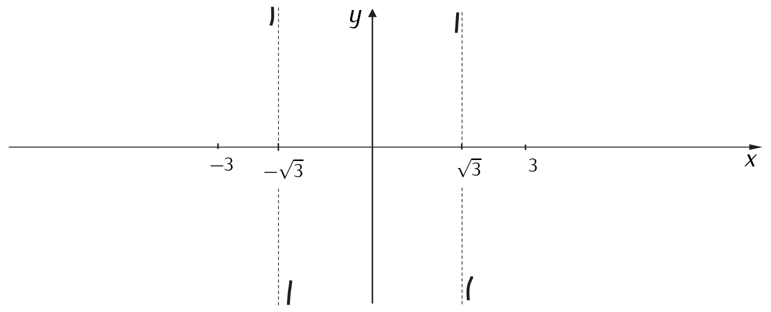
$f(\pm\infty) = \mp\infty;$ $\qquad f(-\sqrt{3}\pm) = \mp\infty;$ $\qquad f(\sqrt{3}\pm) = \mp\infty$

We repeat all important informations.

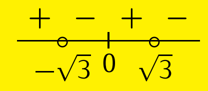


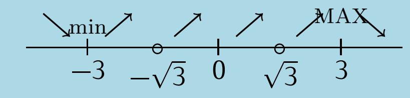
$f(0)=0;$ $f(\pm 3)=\mp\dfrac{9}{2}$

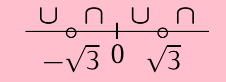
$f(\pm\infty)=\mp\infty;$ $f(-\sqrt{3}\pm)=\mp\infty;$ $f(\sqrt{3}\pm)=\mp\infty$



We draw vertical asymptotes and the graph near these asymptotes.

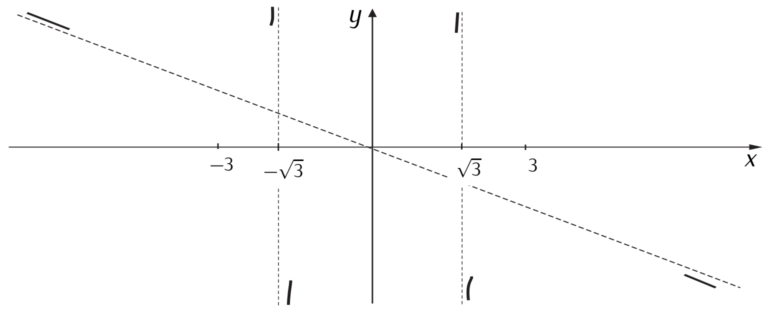




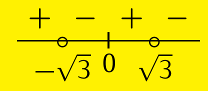


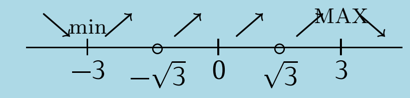
$f(0) = 0;$ $\qquad f(\pm 3) = \mp\frac{9}{2}$

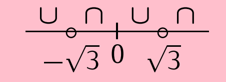
$f(\pm\infty) = \mp\infty;$ $\qquad f(-\sqrt{3}\pm) = \mp\infty;$ $\qquad f(\sqrt{3}\pm) = \mp\infty$



We draw the inclined asymptote and the graph for large $|x|$. We are aware of the type of concavity.

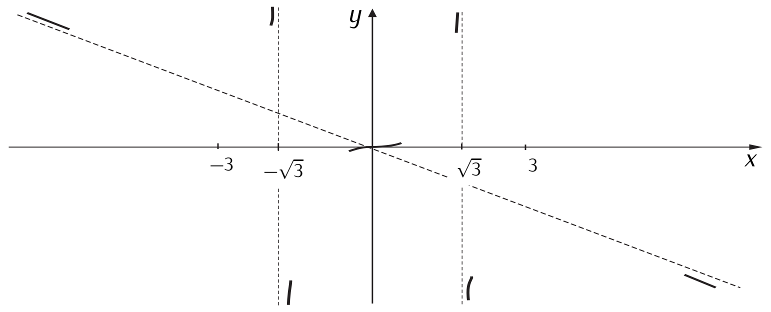




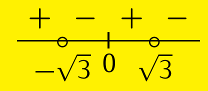


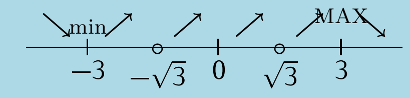
$f(0)=0;$ $\qquad f(\pm 3)=\mp\dfrac{9}{2}$

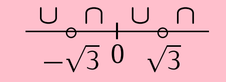
$f(\pm\infty)=\mp\infty;$ $\qquad f(-\sqrt{3}\pm)=\mp\infty;$ $\qquad f(\sqrt{3}\pm)=\mp\infty$



We draw the function near the stationary point.

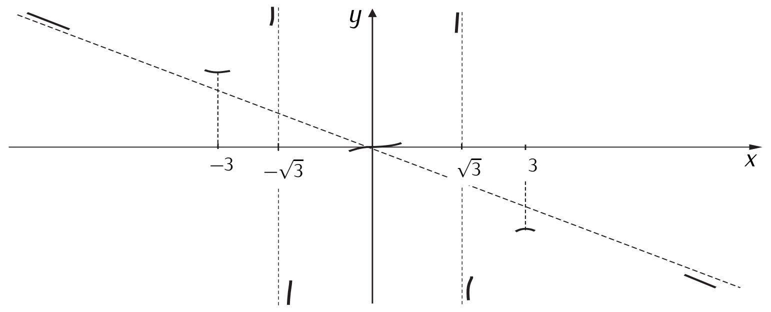




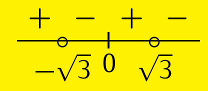


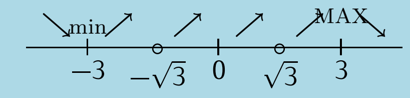
$f(0)=0;$ $\qquad f(\pm 3)=\mp\frac{9}{2}$

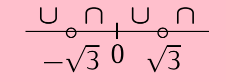
$f(\pm\infty)=\mp\infty;$ $\qquad f(-\sqrt{3}\pm)=\mp\infty;$ $\qquad f(\sqrt{3}\pm)=\mp\infty$



We draw the function at local extrema.

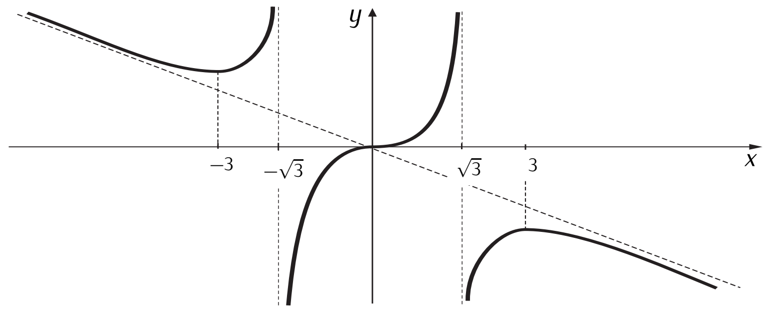






$f(0) = 0;$ $\qquad f(\pm 3) = \mp\dfrac{9}{2}$

$f(\pm\infty) = \mp\infty;$ $\qquad f(-\sqrt{3}\pm) = \mp\infty;$ $\qquad f(\sqrt{3}\pm) = \mp\infty$



We complete our drawing.

$$\boxed{y = (x+1)e^x}$$

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$;

There is no restriction on the domain. Hence the domain is $\mathbb{R}$.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$,

Substituting $x = 0$ into $f(x)$ we get the y-intercept.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$,

$$(x+1)e^x = 0$$

Solving $y = 0$ we get the x-intercept.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$,

$$\begin{aligned} (x+1)e^x &= 0 \\ x+1 &= 0 \end{aligned}$$

- Product equals zero iff one of the factors is zero.
- The factor e^x never equals zero.

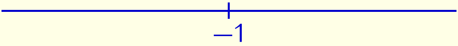
$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$$\begin{aligned} (x+1)e^x &= 0 \\ x+1 &= 0 \\ x &= -1 \end{aligned}$$

The x-intercept is $x = -1$.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

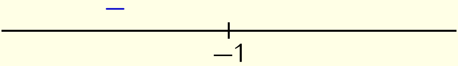
$$\begin{aligned}(x+1)e^x &= 0\\ x+1 &= 0\\ x &= -1\end{aligned}$$



- We mark the x-intercept on the real axis.
- There is no point of discontinuity.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$$\begin{aligned} (x+1)e^x &= 0 \\ x+1 &= 0 \\ x &= -1 \end{aligned}$$

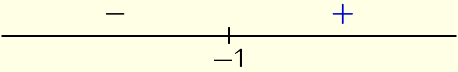


$f(-2) = (-2+1) \cdot e^{-2} = -e^{-2} < 0$

Evaluating $f(-2)$ we see that the function is negative at -2. The same is true for all x in interval $(-\infty, -1)$.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$$
\begin{aligned}
(x+1)e^x &= 0\\
x+1 &= 0\\
x &= -1
\end{aligned}
$$



$f(-2) = (-2+1)\cdot e^{-2} = -e^{-2} < 0$

$f(0) = 1 > 0$ as we have seen above

The function is positive at $x = 0$ and also on $(-1, \infty)$.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$$\lim_{x\to\infty}(x+1)e^x = \infty \cdot \infty = \infty$$

- We evaluate limits at $\pm\infty$. We start with the limit at $+\infty$.
- We use $\infty + 1 = \infty$ and $\lim\limits_{x\to\infty} e^x = \infty$.

$\boxed{y=(x+1)e^x}$ $Dom(f)=\mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$$\lim_{x\to\infty}(x+1)e^x=\infty\cdot\infty=\infty$$

$$\lim_{x\to-\infty}(x+1)e^x=(-\infty)\cdot e^{-\infty}=(-\infty)\cdot 0$$

- We evaluate limit at $-\infty$.
- We substitute (in the sense of limits) $x=-\infty$ and use $-\infty+1=-\infty$ and $\lim\limits_{x\to-\infty}e^x=0$.
- The expression $0\times\infty$ is an indeterminate form.
- We have to look for an alternative method, than the algebra of limits.

$\boxed{y=(x+1)e^x}$ $Dom(f)=\mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$$\lim_{x\to\infty}(x+1)e^x=\infty\cdot\infty=\infty$$

$$\lim_{x\to-\infty}(x+1)e^x=(-\infty)\cdot e^{-\infty}=(-\infty)\cdot 0$$

$$=\lim_{x\to-\infty}\frac{x+1}{e^{-x}}=\frac{-\infty}{\infty}$$

- We convert into fraction by using $\boxed{e^x=\dfrac{1}{e^{-x}}}$. The limit becomes to be expression which is convenient for l'Hospital's rule.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$$\lim_{x\to\infty} (x+1)e^x = \infty \cdot \infty = \infty$$

$$\lim_{x\to-\infty} (x+1)e^x = (-\infty) \cdot e^{-\infty} = (-\infty) \cdot 0$$

$$= \lim_{x\to-\infty} \frac{x+1}{e^{-x}} = \frac{-\infty}{\infty}$$

$$= \lim_{x\to-\infty} \frac{1}{-e^{-x}}$$

We use the l'Hospital's rule (we differentiate separately both the numerator and hte denominator). The function e^{-x} is differentiated by the chain rule as follows.

$$(e^{-x})' = e^{-x}(-x)' = e^{-x} \cdot (-1)$$

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$$\lim_{x\to\infty}(x+1)e^x = \infty\cdot\infty = \infty$$

$$\lim_{x\to-\infty}(x+1)e^x = (-\infty)\cdot e^{-\infty} = (-\infty)\cdot 0$$

$$= \lim_{x\to-\infty}\frac{x+1}{e^{-x}} = \frac{-\infty}{\infty}$$

$$= \lim_{x\to-\infty}\frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^x$$

We simplify.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$$\lim_{x\to\infty} (x+1)e^x = \infty \cdot \infty = \infty$$

$$\lim_{x\to-\infty} (x+1)e^x = (-\infty) \cdot e^{-\infty} = (-\infty) \cdot 0$$

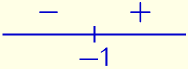
$$= \lim_{x\to-\infty} \frac{x+1}{e^{-x}} = \frac{-\infty}{\infty}$$

$$= \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^x = -e^{-\infty} = 0$$

We substitute. An examination of the graph shows that $\lim_{x\to-\infty} e^x = e^{-\infty} = 0$.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

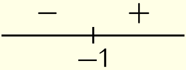
$f(+\infty) = \infty$, $f(-\infty) = 0$;



This has been established. Now we will continue by exploring the derivative.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;



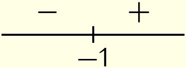
$$y' = (x+1)' \cdot e^x + (x+1) \cdot (e^x)'$$

The function $y = (x+1) \cdot e^x$ is differentiated by the produt rule

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$\boxed{y=(x+1)e^x}$ $Dom(f)=\mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$f(+\infty)=\infty$, $f(-\infty)=0$;

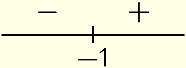


$$\begin{aligned} y' &= (x+1)'\cdot e^x+(x+1)\cdot (e^x)' \\ &= 1\cdot e^x+(x+1)\cdot e^x \end{aligned}$$

We evaluate the derivatives.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;

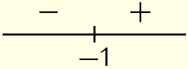


$$\begin{aligned} y' &= (x+1)' \cdot e^x + (x+1) \cdot (e^x)' \\ &= 1 \cdot e^x + (x+1) \cdot e^x \\ &= e^x(1 + x + 1) \end{aligned}$$

We take out the common factor e^x.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;

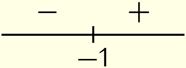


$$\begin{aligned} y' &= (x+1)' \cdot e^x + (x+1) \cdot (e^x)' \\ &= 1 \cdot e^x + (x+1) \cdot e^x \\ &= e^x(1 + x + 1) \\ &= e^x(x+2) \end{aligned}$$

We simplify.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;

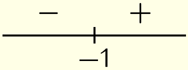


$y' = e^x(x+2)$;

The derivative has been evaluated.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;

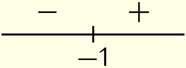


$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$

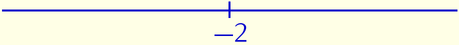
- The derivative is zero iff $\boxed{(x+2) = 0}$, since e^x is never equal zero. This yields stationary point $x = -2$.
- A qiuick evaluateion shows $f(-2) = (-2+1)e^{-2} = -e^{-2}$ and using the calculator we get $f(-2) \approx -0.14$.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;



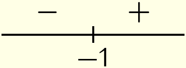
$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$



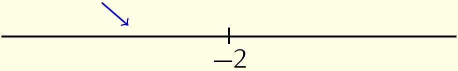
We draw the real axis with the stationary point. There is no point of discontinuity.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;



$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$



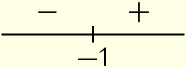
We choose $x = -3$ and evaluate $f'(-3)$:

$$y'(-3) = e^{-3}(-3+2) = -e^{-3} < 0.$$

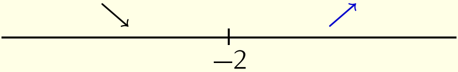
The function is decreasing at $x = -3$ and the same is true on the whole interval $(-\infty, -2)$.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;



$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$



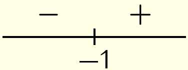
We choose $x = 0$ and evaluate $f'(0)$:

$$y'(0) = e^0(0+2) = 2 > 0.$$

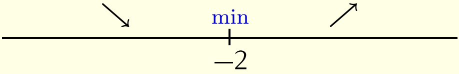
The function is increasing at $x = 0$ and the same is true on the whole interval $(-2, \infty)$.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;



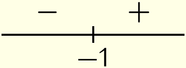
$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$



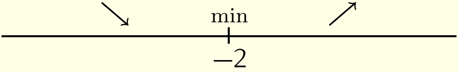
The local minimum appears at $x = -2$.

$\boxed{y=(x+1)e^x}$ $Dom(f)=\mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$f(+\infty)=\infty$, $f(-\infty)=0$;



$y'=e^x(x+2)$; stac. point is $x=-2$; $f(-2)=-e^{-2}\approx-0.14$



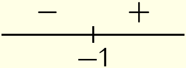
$$y''=e^x\cdot(x+2)+e^x\cdot 1$$

We find y''. We differentate the product $y'=e^x\cdot(x+2)$ by the product rule

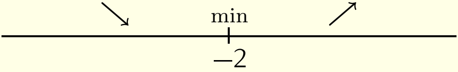
$$(u\cdot v)'=u'\cdot v+u\cdot v'$$

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;



$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$



$$\begin{aligned} y'' &= e^x \cdot (x+2) + e^x \cdot 1 \\ &= e^x(x+2+1) \\ &= e^x(x+3) \end{aligned}$$

We simplify.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;

− +

−1

$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$

↘ min ↗

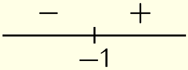
−2

$y'' = e^x(x+3)$;

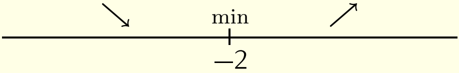
The second derivative is known.

$\boxed{y=(x+1)e^x}$ $Dom(f)=\mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$f(+\infty)=\infty$, $f(-\infty)=0$;



$y'=e^x(x+2)$; stac. point is $x=-2$; $f(-2)=-e^{-2}\approx -0.14$



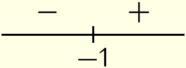
$y''=e^x(x+3)$; $y''=0$ for $x=-3$, $f(-3)=-2e^{-3}\approx -0.01$

- We look for the points where $y''=0$. Since e^x never equals zero, the only possibility is $(x+3)=0$ and hence $x=-3$.
- Evaluating the function at $x=-3$ we get

$$f(-3)=(-3+1)e^{-3}=-2e^{-3}\approx -0.01$$

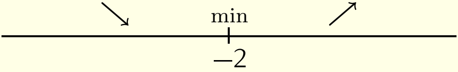
$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

$f(+\infty) = \infty$, $f(-\infty) = 0$;


− +

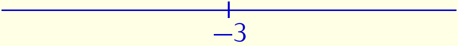
−1

$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$


↘ min ↗

−2

$y'' = e^x(x+3)$; $y'' = 0$ for $x = -3$, $f(-3) = -2e^{-3} \approx -0.01$


−3

We draw the real axis with the point where the second derivative equals zero. There is no point of discontinuity and the second derivative may change its sign only by passing through zero.

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

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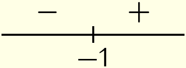
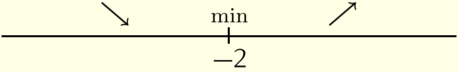
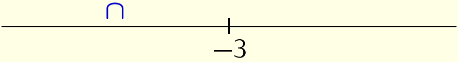


$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$



$y'' = e^x(x+3)$; $y'' = 0$ for $x = -3$, $f(-3) = -2e^{-3} \approx -0.01$

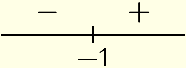


The function is concave down on $(-\infty, -3)$, since $-4 \in (-\infty, 3)$ and

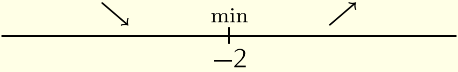
$$y''(-4) = e^{-4}(-4+3) = -e^{-4} < 0.$$

$\boxed{y = (x+1)e^x}$ $Dom(f) = \mathbb{R}$; y-int. is $[0, 1]$, x-int. is $[-1, 0]$,

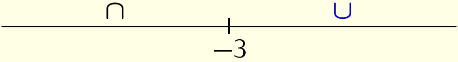
$f(+\infty) = \infty$, $f(-\infty) = 0$;



$y' = e^x(x+2)$; stac. point is $x = -2$; $f(-2) = -e^{-2} \approx -0.14$



$y'' = e^x(x+3)$; $y'' = 0$ for $x = -3$, $f(-3) = -2e^{-3} \approx -0.01$

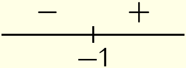


The function is concave up on $(-3, \infty)$, since $-2 \in (-3, \infty)$ and a local minimum appears at $x = -2$. Among others,

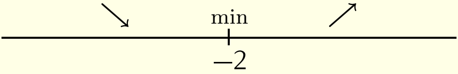
$$y''(-2) = e^{-2}(-2+3) = e^{-2} < 0.$$

$\boxed{y=(x+1)e^x}$ $Dom(f)=\mathbb{R}$; y-int. is $[0,1]$, x-int. is $[-1,0]$,

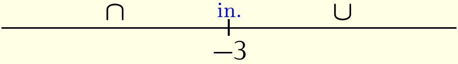
$f(+\infty)=\infty$, $f(-\infty)=0$;



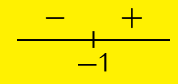
$y'=e^x(x+2)$; stac. point is $x=-2$; $f(-2)=-e^{-2}\approx -0.14$

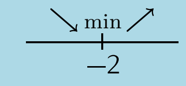


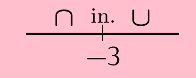
$y''=e^x(x+3)$; $y''=0$ for $x=-3$, $f(-3)=-2e^{-3}\approx -0.01$



The inflection appears at $x=-3$.





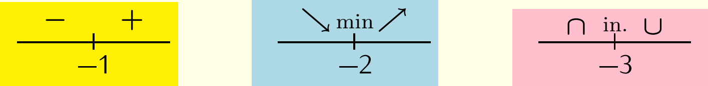


$f(0) = 1$
$f(-1) = 0$

$f(-2) \approx -0.14$
$f(-3) \approx -0.01$

$f(+\infty) = \infty$
$f(-\infty) = 0$

We summarize all important computations.



$f(0) = 1$
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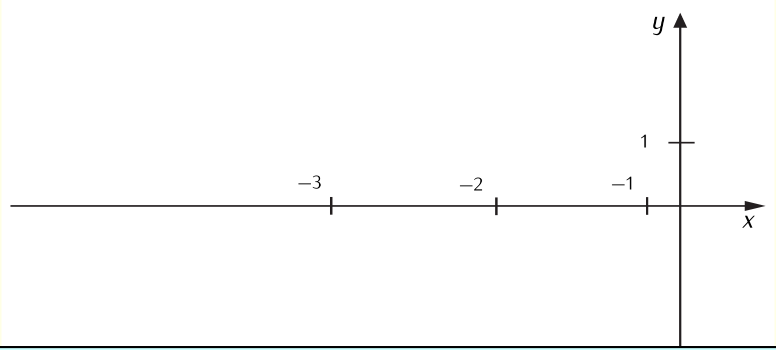
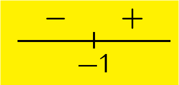
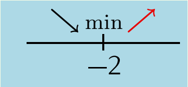
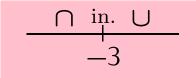


We draw the coordinate system.







$f(0) = 1$
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$f(-2) \approx -0.14$
$f(-3) \approx -0.01$

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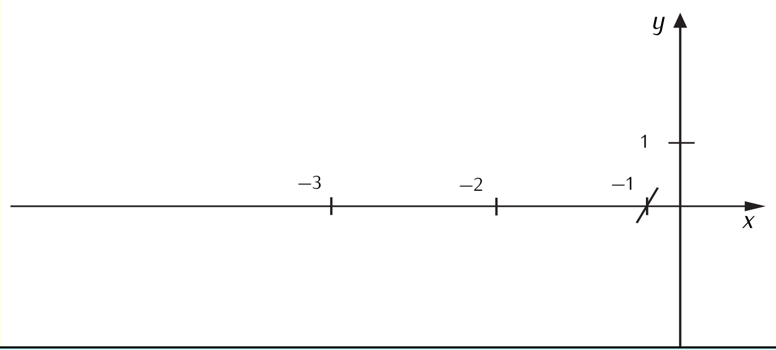
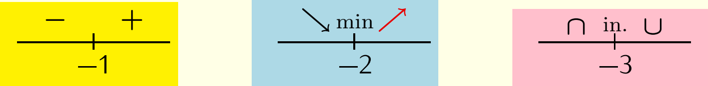


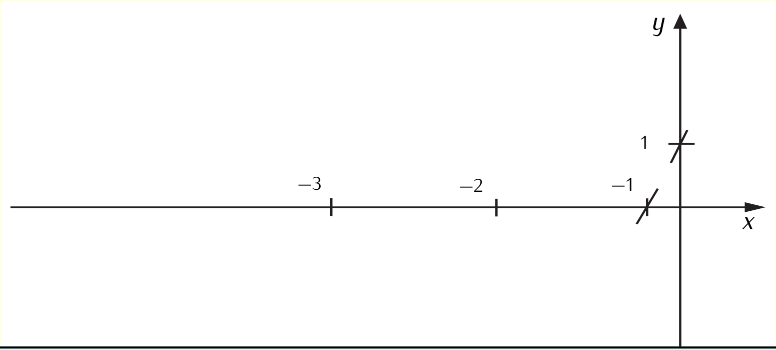
We mark the x-intercept $x = -1$. The function is increasing at this point.



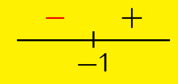
$f(0) = 1$
$f(-1) = 0$

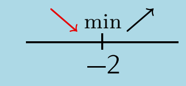
$f(-2) \approx -0.14$
$f(-3) \approx -0.01$

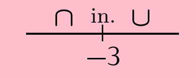
$f(+\infty) = \infty$
$f(-\infty) = 0$



We mark the y-intercept $y = 1$. The function is increasing at this point.







$f(0) = 1$

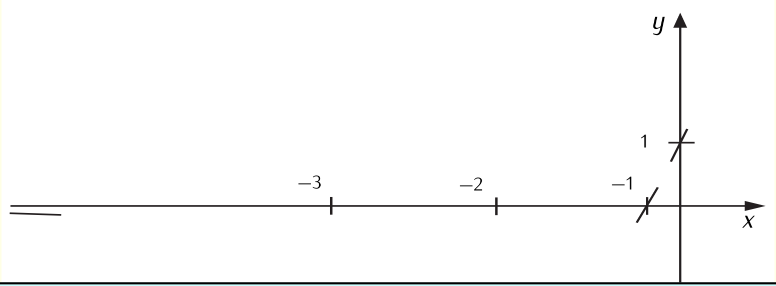
$f(-1) = 0$

$f(-2) \approx -0.14$

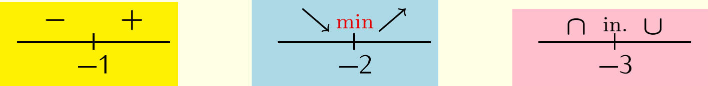
$f(-3) \approx -0.01$

$f(+\infty) = \infty$

$f(-\infty) = 0$



We draw a short mark near the asymptote at $-\infty$. We are aware of the fact that the function is decreasing and negative near $-\infty$ and hence the graph is below the asymptote.



$f(0) = 1$
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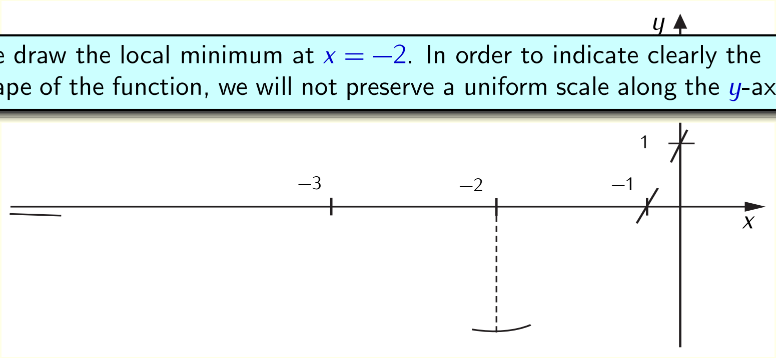
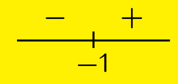
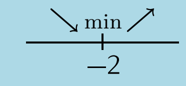
$f(-2) \approx -0.14$
$f(-3) \approx -0.01$

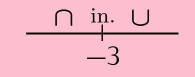
$f(+\infty) = \infty$
$f(-\infty) = 0$



We draw the local minimum at $x = -2$. In order to indicate clearly the shape of the function, we will not preserve a uniform scale along the y-axis.



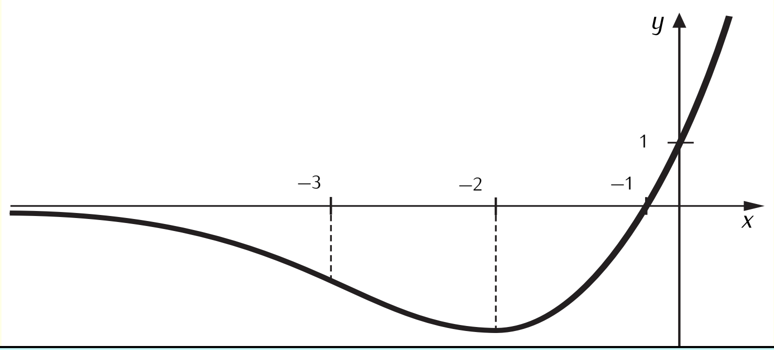




$f(0) = 1$
$f(-1) = 0$

$f(-2) \approx -0.14$
$f(-3) \approx -0.01$

$f(+\infty) = \infty$
$f(-\infty) = 0$



We join the pieces of the graph. Finished.

That’s all.