The Derivative and Optimalization

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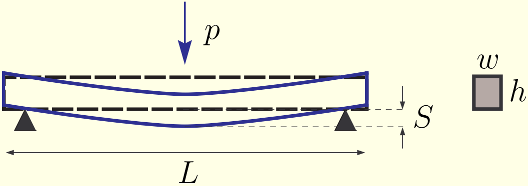
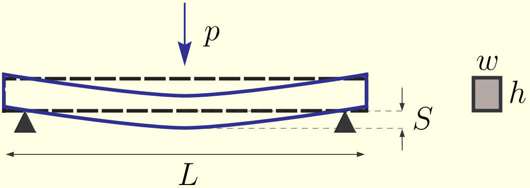
February 24, 2006



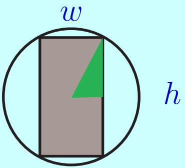
$$S = C\frac{pL^4}{wh^3}$$

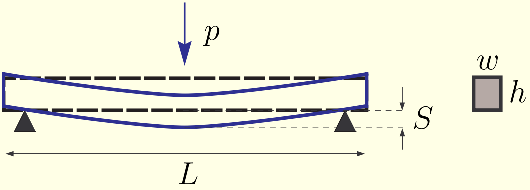
- Cut from the cylindrical log of radius r and length L a beam of the rectangular cross section $w \times h$ which is as stiff as possible.
- The formula for the sag S caused by the force p is known.



$$S = C\frac{pL^4}{wh^3}$$

$$\left(\frac{w}{2}\right)^2 + \left(\frac{h}{2}\right)^2 = r^2$$





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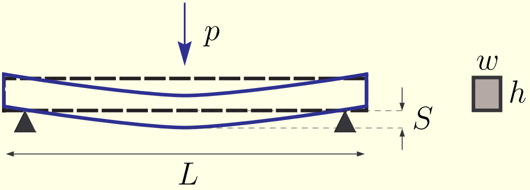
$$\left(\frac{w}{2}\right)^2 + \left(\frac{h}{2}\right)^2 = r^2$$

$$S = C\frac{pL^4}{wh^3} \to \text{minimum}$$

$$wh^3 \to \text{maximum}$$

- We have to find the values of w and h which yield minimal S.
- The quotient with constant numerator is minimal for maximal denominator.



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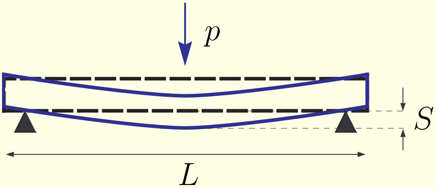
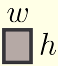
$$w = \sqrt{4r^2 - h^2}$$

There is a relationship between w and h.





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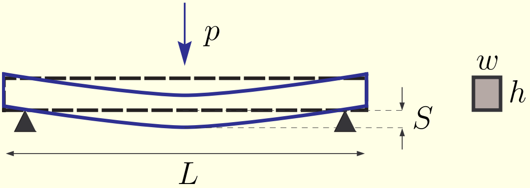
$$\sqrt{4r^2 - h^2}\, h^3 \to \text{maximum}$$

We substitute for w. Now we have a function of one variable: h.



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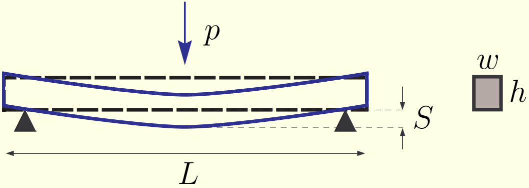
$$\sqrt{4r^2 - h^2}\, h^3 \to \text{maximum}$$

$$(4r^2 - h^2)h^6 \to \text{maximum}$$

The expression is maximal if the expression under the radical sign is maximal. This gives simpler problem (square root is removed).



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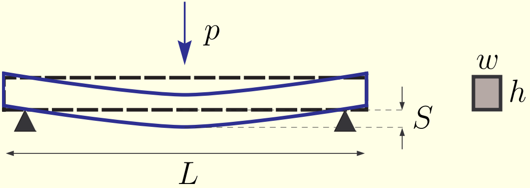
$$\begin{aligned}((4r^2 - h^2)h^6)' &= (4r^2h^6 - h^8)' \\ &= 24r^2h^5 - 8h^7 \\ &= 8h^5(3r^2 - h^2)\end{aligned}$$

We differentiate and find the stationary point.



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$$\begin{aligned} ((4r^2 - h^2)h^6)' &= (4r^2h^6 - h^8)' \\ &= 24r^2h^5 - 8h^7 \\ &= 8h^5(3r^2 - h^2) \end{aligned}$$

Derivative vanish for $h = 0$ and for $h = \sqrt{3}\,r \approx 1.73r$.

From the nature of the problem it follows that $h = \sqrt{3}\,r$ is a local maximum.

The stifness is maximal (S is minimal) if the height of the beam is equal to its width multiplied by 1.73, i.e. equal to the radius multiplied by 0.866.