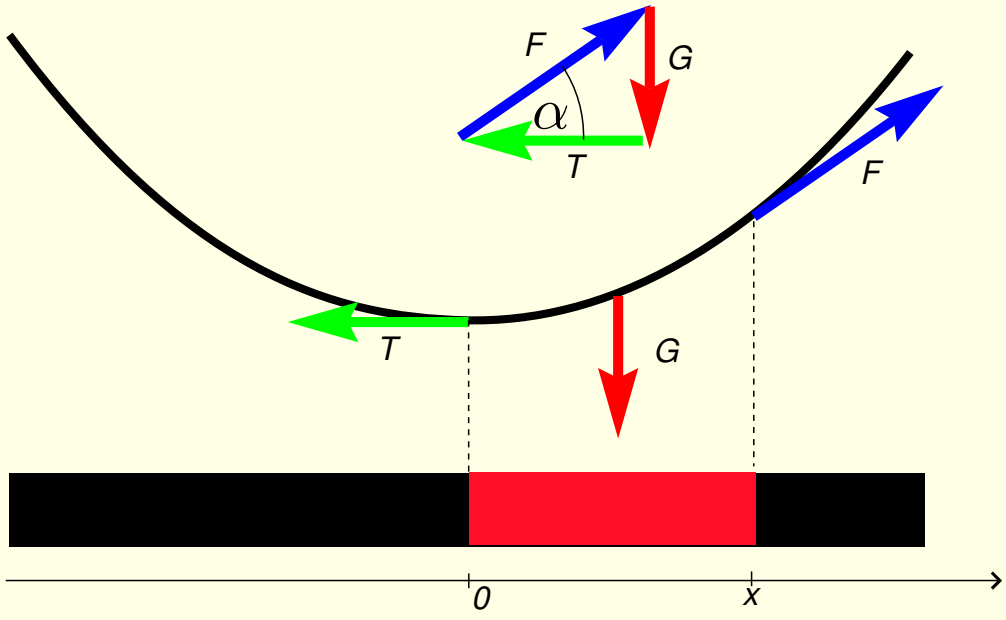
Differential calculus, suspension brigdes and hanging chains.

Robert Mařík

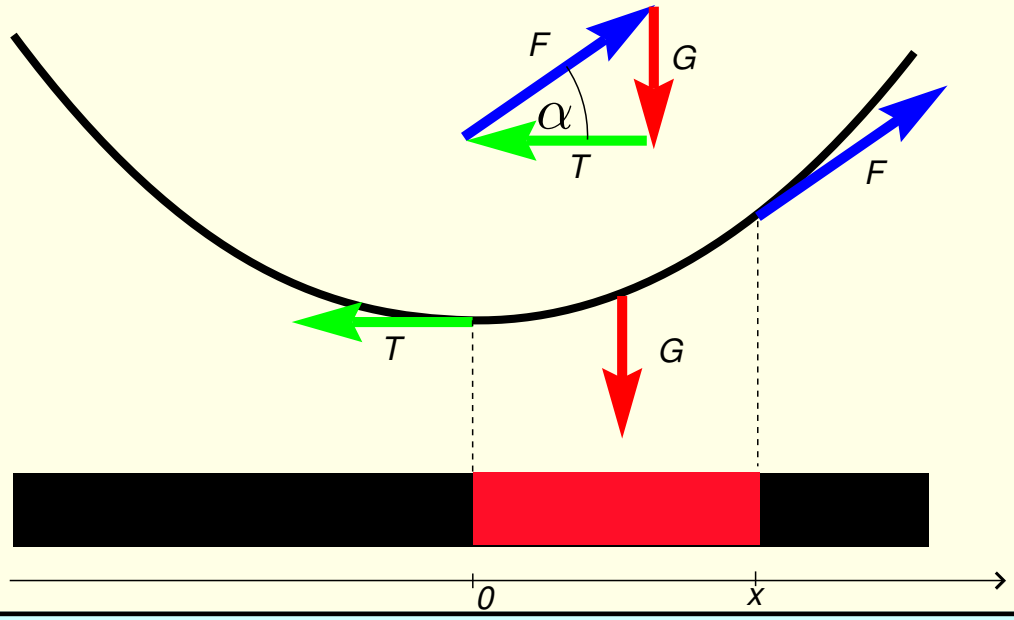
27. ledna 2006

Problem 1: Consider the bridge on the picture. The road deck hangs on vertical cables suspended from the main cables. The cables themselves may be large, but their mass is insignificant when compared with that of the deck, so disregard the mass of the cable. Suppose that the deck is horizontal, and suppose that its cross section is uniform throughout the length of the bridge. Also, the vertical cables are so close together that we can assume that the weight of any part of the deck is transferred to the part of the main cable directly above it. These conditions are reasonably close to the real thing. We have to find the optimal shape of the main cable.

Rather than compute the height of the cable at a given point, let us try to determine the slope of the cable at that point. Naturally, the cable will align itself with the direction of the forces pulling on it. There is one point on this cable where that direction is easy to see. At the lowest point, the vertex, the cable is horizontal. Label that point O and let that be the origin of a set of coordinate axes. Now pick an arbitrary point, P, on the cable, somewhere to the right of O. What is the direction of the tension at point P? How do we determine that?

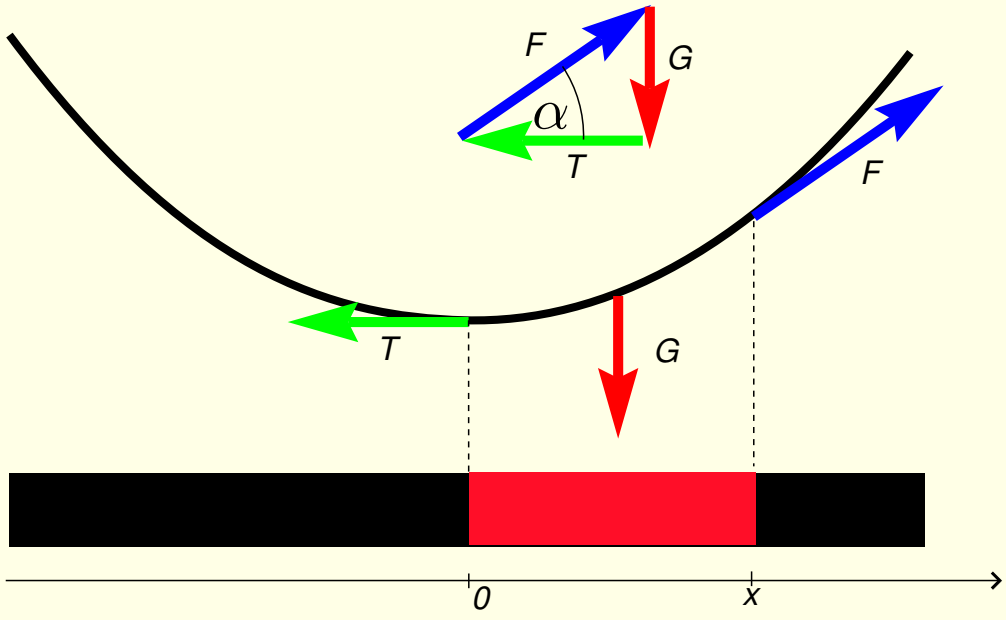


Consider the coordinate system shown on the picture. Consider only the red part of the bridge and the cable above this part. This cable is motionless and hence the sum of all forces action on this part od the cable is zero. Each force is identified with a vector arrow.

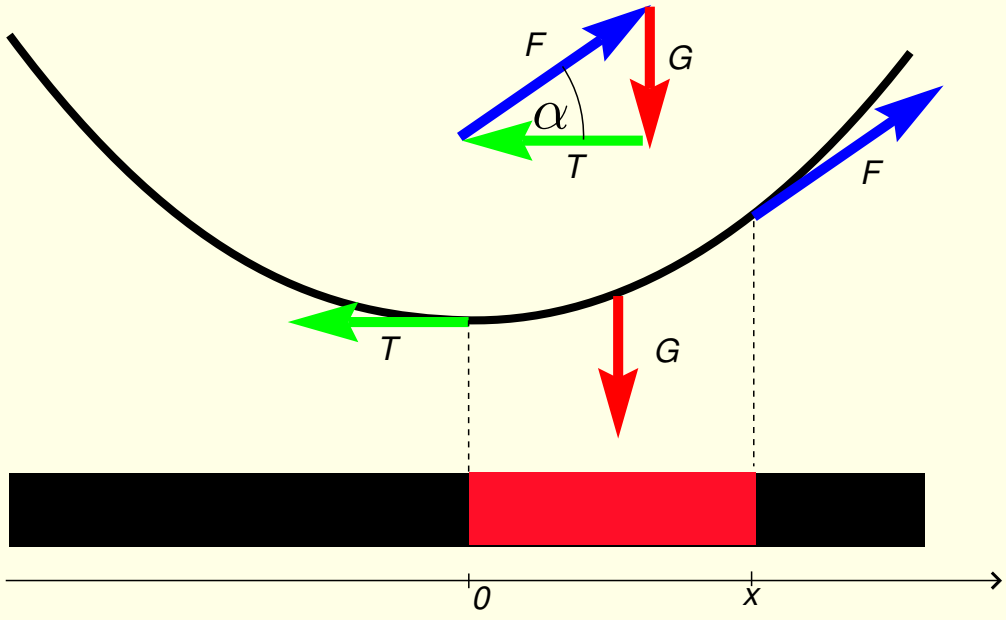


The forces acting in this problem are:

- Tension T at the point $x = 0$. This force is horizontal.
- Tension F at the general point x. This force is tangent to the main cable. The slope of this force is equal to the slope of the curve, we are looking for.
- The vertical gravitational force $G = mg$. Since the linear density of the deck is constant along the length of the bridge, the weight of the red part of the bridge is the product of its length x and linear density τ.



The sum of these forces is a zero vector. Hence these forces form a (right) triangle. From this triangle we get $\operatorname{tg}\alpha = \dfrac{G}{T} = \dfrac{\tau x g}{T} = \mu x$, where $\mu = \dfrac{\tau g}{T}$ is a real constant.

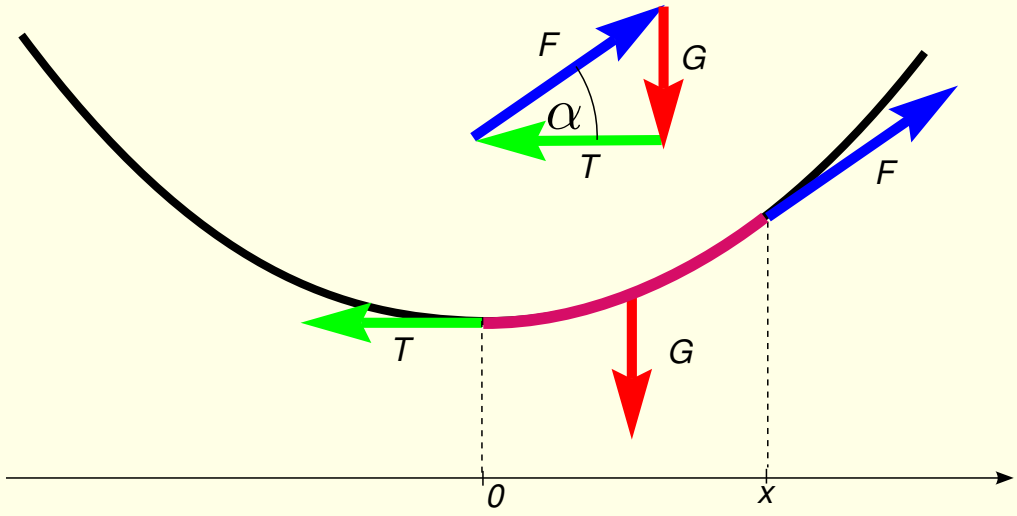


Matematical description: Find the curve $y = y(x)$ such that the derivative of this funciton satisfies $y' = \mu x$.

Solution: This is the basic problem of integral calculus and we solve the problem by integration. $y(x) = \int y'(x)\,\mathrm{d}x = \int \mu x\,\mathrm{d}x = \mu\frac{1}{2}x^2 + C.$

the main cable has a parabolical shape.

Problém 2: Consider a similar problem, but the load is uniform along the length of the cable (or chain).



The only different thing now is the formula for the mass. The mass of the red part is a product of the length of the cable and the linear density α. Hence

$$y' = \alpha \underbrace{\int_0^x \sqrt{1 + [y'(t)]^2}\,\mathrm{d}t}_{\text{length}}.$$

Matematical problem: Find the function $y = y(x)$ which satisfies

$$y' = \alpha \int_0^x \sqrt{1 + [y'(t)]^2}\, \mathrm{d}t. \tag{1}$$

Solution: Differentiating (1) we get

$$y'' = \alpha \sqrt{1 + [y'(x)]^2}.$$

Really: if the funtion $\mathcal{F}(x)$ is a primitive function to $\sqrt{1 + y'^2(x)}$, then the definite integral can be evaluated by Newton–Leibniz formula as $\mathcal{F}(x) - \mathcal{F}(0)$. Differentiating with respect to x we get $F'(x)$, which is equal to $\sqrt{1 + y'^2(x)}$ (remember that we assumed that $\mathcal{F}$ is a primitive function). The problem is reducet to the problem to find the function $y = y(x)$ which satisfies

$$y'' = \alpha \sqrt{1 + y'^2}.$$

The substitution $z(x) = y'(x)$, $z'(x) = y''(x)$ transforms this equation into the equation

$$z' = \alpha \sqrt{1 + z^2}.$$

Separating variables we get

$$\frac{dz}{\sqrt{1+z^2}} = \alpha dx$$

and integrating

$$\ln\left(z + \sqrt{1+z^2}\right) = \alpha x + C.$$

From here

$$\begin{aligned}
z + \sqrt{1+z^2} &= e^{\alpha x + C} \\
\sqrt{1+z^2} &= e^{\alpha x + C} - z \\
1 + z^2 &= e^{2(\alpha x + C)} - 2ze^{\alpha x + C} + z^2 \\
2ze^{\alpha x + C} &= e^{2(\alpha x + C)} - 1 \\
z &= \frac{1}{2}\left[e^{\alpha x + C} - e^{-(\alpha x + C)}\right]
\end{aligned}$$

Hence

$$y' = \frac{1}{2}\left[e^{\alpha x + C} - e^{-(\alpha x + C)}\right]$$

and integration gives

$$y' = \frac{1}{2\alpha}\left[e^{\alpha x + C} + e^{-(\alpha x + C)}\right] = \frac{1}{\alpha}\cosh(\alpha x + C).$$

The shape of the cable is the hyperbolic cosine function.

Further reading

- http://whistleralley.com/hanging/hanging.htm
- http://www.du.edu/ jcalvert/math/parabola.htm
- http://www.du.edu/ jcalvert/math/catenary.htm
- http://www.math.rutgers.edu/courses/321/F04/essay1.pdf
- http://mathworld.wolfram.com/Catenary.html
- http://en.wikipedia.org/wiki/Suspension_bridge
- http://en.wikipedia.org/wiki/Catenary