The partial derivative, local extrema in two variables

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Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

z'_x

z'_y

We look for partial derivatives of the function.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$$z'_x = y^2(x^2)'_x - (x^2)'_x$$

$$z'_y$$

We differentiate with respect to x. We use the sum rule and the constant multiple rule.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$$z'_x = y^2(x^2)'_x - (x^2)'_x = y^2 2x - 2x$$

$$z'_y$$

We evaluate the derivatives.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$$\begin{aligned} z'_x &= y^2(x^2)'_x - (x^2)'_x = y^2 2x - 2x \\ &= 2x(y^2 - 1) \end{aligned}$$

$$z'_y$$

We simplify.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$$
\begin{aligned}
z'_x &= y^2(x^2)'_x - (x^2)'_x = y^2 2x - 2x \\
&= 2x(y^2 - 1) \\
z'_y &= x^2(y^2)'_y - (y^2)'_y
\end{aligned}
$$

In the same way we differentiate with respect to y.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$$\begin{aligned} z'_x &= y^2(x^2)'_x - (x^2)'_x = y^2 2x - 2x \\ &= 2x(y^2 - 1) \\ z'_y &= x^2(y^2)'_y - (y^2)'_y = x^2 2y - 2y \end{aligned}$$

We evaluate the derivatives.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$$
\begin{aligned}
z'_x &= y^2(x^2)'_x - (x^2)'_x = y^2 2x - 2x \\
&= 2x(y^2 - 1) \\
z'_y &= x^2(y^2)'_y - (y^2)'_y = x^2 2y - 2y \\
&= 2y(x^2 - 1)
\end{aligned}
$$

We simplify.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$$z'_x = 2x(y^2 - 1) \qquad ; \qquad z'_y = 2y(x^2 - 1)$$

We have the first derivatives. We look for stationary points.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1) = 0$; $\qquad z'_y = 2y(x^2 - 1) = 0$

We put the derivatives equal to zero.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1) = 0$; $\qquad z'_y = 2y(x^2 - 1) = 0$

- We solve the system of two nonlinear equations.
- We start with one of the equations.
- It is in the form "product equals zero".

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1) = 0$; $z'_y = 2y(x^2 - 1) = 0$

Case 1: $\boxed{x = 0}$

Case 2: $\boxed{y = 1}$

Case 3: $\boxed{y = -1}$

- One of the factors in the product must be zero.
- We deal independently with the cases when $x = 0$ and $(y^2 - 1) = 0$, i.e., $y = \pm 1$.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1) = 0$; $z'_y = 2y(x^2 - 1) = 0$

Case 1: $\boxed{x = 0}$

$2y(0 - 1) = 0$

Case 2: $\boxed{y = 1}$

Case 3: $\boxed{y = -1}$

- We start with the Case 1.
- We substitute $x = 0$ into the second equation.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1) = 0;$ $z'_y = 2y(x^2 - 1) = 0$

Case 1: $\boxed{x = 0}$	Case 2: $\boxed{y = 1}$	Case 3: $\boxed{y = -1}$
$2y(0 - 1) = 0$ $y = 0$		

$S_1 = [0, 0];$

We find y. We have the stationary point S_1.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2-1) = 0;$ $\quad z'_y = 2y(x^2-1) = 0$

Case 1: $\boxed{x = 0}$

$$2y(0-1) = 0$$
$$y = 0$$

Case 2: $\boxed{y = 1}$

$$2(x^2-1) = 0$$

Case 3: $\boxed{y = -1}$

$S_1 = [0, 0];$

- Similarly, we work with the Case 2.
- We substitute $y = 1$ into the second equation.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2-1) = 0$; $z'_y = 2y(x^2-1) = 0$

Case 1: $\boxed{x=0}$

$$2y(0-1) = 0$$
$$y = 0$$

Case 2: $\boxed{y=1}$

$$2(x^2-1) = 0$$
$$x^2 = \pm 1$$

Case 3: $\boxed{y=-1}$

$S_1 = [0,0]$; $S_2 = [1,1]$; $S_3 = [-1,1]$;

- We solve the quadratic equation for x.
- We have two solutions and two new stationary points.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2-1) = 0$; $z'_y = 2y(x^2-1) = 0$

Case 1: $\boxed{x = 0}$

$$2y(0-1) = 0$$
$$y = 0$$

Case 2: $\boxed{y = 1}$

$$2(x^2-1) = 0$$
$$x^2 = \pm 1$$

Case 3: $\boxed{y = -1}$

$$-2(x^2-1) = 0$$

$S_1 = [0, 0]$; $S_2 = [1, 1]$; $S_3 = [-1, 1]$;

- Similarly, we work with the Case 3.
- We substitute $y = -1$ into the second equation.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1) = 0$; $z'_y = 2y(x^2 - 1) = 0$

Case 1: $\boxed{x = 0}$	Case 2: $\boxed{y = 1}$	Case 3: $\boxed{y = -1}$
$2y(0 - 1) = 0$	$2(x^2 - 1) = 0$	$-2(x^2 - 1) = 0$
$y = 0$	$x^2 = \pm 1$	$x^2 = \pm 1$

$S_1 = [0, 0]$; $S_2 = [1, 1]$; $S_3 = [-1, 1]$; $S_4 = [1, -1]$; $S_5 = [-1, -1]$

- We solve the quadratic equation for x.
- We have two solutions and two new stationary points.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0, 0]$; $S_2 = [1, 1]$; $S_3 = [-1, 1]$; $S_4 = [1, -1]$; $S_5 = [-1, -1]$

z''_{xx}

z''_{xy}

z''_{yy}

There are five stationary points. Now we look for the second derivatives.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0, 0]$; $S_2 = [1, 1]$; $S_3 = [-1, 1]$; $S_4 = [1, -1]$; $S_5 = [-1, -1]$

$$z''_{xx} = 2(y^2 - 1)(x)'_x = 2(y^2 - 1) \cdot 1$$

$$z''_{xy}$$

$$z''_{yy}$$

We differentiate z'_x with respect to x and simplify.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0, 0]$; $S_2 = [1, 1]$; $S_3 = [-1, 1]$; $S_4 = [1, -1]$; $S_5 = [-1, -1]$

$$z''_{xx} = 2(y^2 - 1)(x)'_x = 2(y^2 - 1) \cdot 1$$
$$z''_{xy} = 2x(y^2 - 1)'_y = 2x \cdot (2y + 0) = 4xy$$
$$z''_{yy}$$

We differentiate z'_x with respect to y and simplify.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0, 0]$; $S_2 = [1, 1]$; $S_3 = [-1, 1]$; $S_4 = [1, -1]$; $S_5 = [-1, -1]$

$$z''_{xx} = 2(y^2 - 1)(x)'_x = 2(y^2 - 1) \cdot 1$$
$$z''_{xy} = 2x(y^2 - 1)'_y = 2x \cdot (2y + 0) = 4xy$$
$$z''_{yy} = 2(x^2 - 1)(y)'_y = 2(x^2 - 1) \cdot 1$$

We differentiate z'_y with respect to y and simplify.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0, 0]$; $S_2 = [1, 1]$; $S_3 = [-1, 1]$; $S_4 = [1, -1]$; $S_5 = [-1, -1]$

$z''_{xx} = 2(y^2 - 1)$; $z''_{xy} = 4xy$; $z''_{yy} = 2(x^2 - 1)$

We employ the second derivative test for each stationary point separately. We start with S_1 and evaluate Hessian at this point.

$$H(S_1) = \begin{vmatrix} z''_{xx} & z''_{xy} \\ z''_{xy} & z''_{yy} \end{vmatrix}_{[x,y]=[0,0]} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$$

A local maximum appears at S_1.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0,0]$; $S_2 = [1,1]$; $S_3 = [-1,1]$; $S_4 = [1,-1]$; $S_5 = [-1,-1]$

$z''_{xx} = 2(y^2 - 1)$; $z''_{xy} = 4xy$; $z''_{yy} = 2(x^2 - 1)$

$$H(S_2) = \begin{vmatrix} z''_{xx} & z''_{xy} \\ z''_{xy} & z''_{yy} \end{vmatrix}_{[x,y]=[1,1]} = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0$$

There is no local extremum at S_2.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0, 0]$; $S_2 = [1, 1]$; $S_3 = [-1, 1]$; $S_4 = [1, -1]$; $S_5 = [-1, -1]$

$z''_{xx} = 2(y^2 - 1)$; $z''_{xy} = 4xy$; $z''_{yy} = 2(x^2 - 1)$

$$H(S_3) = \begin{vmatrix} z''_{xx} & z''_{xy} \\ z''_{xy} & z''_{yy} \end{vmatrix}_{[x,y]=[-1,1]} = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0$$

There is no local extremum at S_3.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0,0]$; $S_2 = [1,1]$; $S_3 = [-1,1]$; $S_4 = [1,-1]$; $S_5 = [-1,-1]$

$z''_{xx} = 2(y^2 - 1)$; $z''_{xy} = 4xy$; $z''_{yy} = 2(x^2 - 1)$

$$H(S_4) = \begin{vmatrix} z''_{xx} & z''_{xy} \\ z''_{xy} & z''_{yy} \end{vmatrix}_{[x,y]=[1,-1]} = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0$$

There is no local extremum at S_4.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0,0]$; $S_2 = [1,1]$; $S_3 = [-1,1]$; $S_4 = [1,-1]$; $S_5 = [-1,-1]$

$z''_{xx} = 2(y^2 - 1)$; $z''_{xy} = 4xy$; $z''_{yy} = 2(x^2 - 1)$

$$H(S_5) = \begin{vmatrix} z''_{xx} & z''_{xy} \\ z''_{xy} & z''_{yy} \end{vmatrix}_{[x,y]=[-1,-1]} = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0$$

There is no local extremum at S_5.

Find local extrema of the function $z = x^2y^2 - x^2 - y^2$

$z'_x = 2x(y^2 - 1)$; $z'_y = 2y(x^2 - 1)$

$S_1 = [0, 0]$; $S_2 = [1, 1]$; $S_3 = [-1, 1]$; $S_4 = [1, -1]$; $S_5 = [-1, -1]$

$z''_{xx} = 2(y^2 - 1)$; $z''_{xy} = 4xy$; $z''_{yy} = 2(x^2 - 1)$

- The only local extremum is at $S_1 = [0, 0]$. It is a local maximum.
- The other stationary points are saddle points.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y$, $z'_y = 4y^3 - 4x$,

- We evaluate the partial derivatives.
- Differentiating with respect to x we consider y to be constant and vice versa.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y = 0, \qquad z'_y = 4y^3 - 4x = 0,$

We look for the stationary points. We put derivatives equal to zero and solve the system of two equations.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y = 0, \qquad z'_y = 4y^3 - 4x = 0,$

$$\boxed{\begin{aligned} 4x^3 - 4y &= 0, \\ 4y^3 - 4x &= 0. \end{aligned}}$$

This is the system to be solved.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y = 0, \qquad z'_y = 4y^3 - 4x = 0,$

$$\boxed{\begin{aligned} &4x^3 - 4y = 0,\\ &4y^3 - 4x = 0.\end{aligned}}$$

$$y = x^3$$

We isolate y from the first equation.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y = 0, \qquad z'_y = 4y^3 - 4x = 0,$

$$\boxed{\begin{aligned} 4x^3 - 4y &= 0, \\ 4y^3 - 4x &= 0. \end{aligned}}$$

$y = x^3$

$4(x^3)^3 - 4x = 0,$

We substitute for y into the second equation.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y = 0,$ $\quad z'_y = 4y^3 - 4x = 0,$

$$\boxed{\begin{aligned} 4x^3 - 4y &= 0, \\ 4y^3 - 4x &= 0. \end{aligned}}$$

$y = x^3$

$4(x^3)^3 - 4x = 0,$

$4x^9 - 4x = 0,$

We simplify.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y = 0, \qquad z'_y = 4y^3 - 4x = 0,$

$$\boxed{\begin{aligned} 4x^3 - 4y &= 0, \\ 4y^3 - 4x &= 0. \end{aligned}}$$

$$y = x^3$$

$$\begin{aligned} 4(x^3)^3 - 4x &= 0, \\ x^9 - \ \ x &= 0, \\ x(x^8 - 1) &= 0. \end{aligned}$$

We factor.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y = 0, \qquad z'_y = 4y^3 - 4x = 0,$

$$\boxed{\begin{aligned} 4x^3 - 4y &= 0,\\ 4y^3 - 4x &= 0.\end{aligned}}$$

$$y = x^3$$

$$\begin{aligned} 4(x^3)^3 - 4x &= 0,\\ x^9 - x &= 0,\\ x(x^8 - 1) &= 0.\end{aligned}$$

CASE 1:
$x = 0,$

CASE 2:
$x = 1,$

CASE 3:
$x = -1,$

- Either $x = 0$ or $(x^8 - 1) = 0$.
- The latter case yields $x^8 = 1$ and $x = \pm 1$.
- We consider three different cases. In each case the second equation is satisfied, provided $y = x^3$ remains true.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y = 0,\qquad z'_y = 4y^3 - 4x = 0,$

$$\boxed{\begin{aligned} 4x^3 - 4y &= 0,\\ 4y^3 - 4x &= 0. \end{aligned}}$$

$y = x^3$

$$\begin{aligned} 4(x^3)^3 - 4x &= 0,\\ x^9 - x &= 0,\\ x(x^8 - 1) &= 0. \end{aligned}$$

CASE 1:

$x = 0$, $y = 0$

$S_1 = [0, 0]$,

CASE 2:

$x = 1$, $y = 1$

$S_2 = [1, 1]$,

CASE 3:

$x = -1$, $y = -1$

$S_3 = [-1, -1]$.

We find the corresponding y to each x. We get three stationary points.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y$, $z'_y = 4y^3 - 4x$,

$S_1 = [0, 0]$, $S_2 = [1, 1]$, $S_3 = [-1, -1]$.

$z''_{xx} = 12x^2$, $z''_{xy} = -4$, $z''_{yy} = 12y^2$.

- We have three stationary points.
- To employ the second derivative test we have to find the second derivatives.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y$, $z'_y = 4y^3 - 4x$,

$S_1 = [0, 0]$, $S_2 = [1, 1]$, $S_3 = [-1, -1]$.

$z''_{xx} = 12x^2$, $z''_{xy} = -4$, $z''_{yy} = 12y^2$.

$H(S_1) = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0$, saddle point at $[0, 0]$

- We evaluate the hessian at S_1.
- The hessian is negative and no local extremum occurs at S_1.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y$, $z'_y = 4y^3 - 4x$,

$S_1 = [0,0]$, $S_2 = [1,1]$, $S_3 = [-1,-1]$.

$z''_{xx} = 12x^2$, $z''_{xy} = -4$, $z''_{yy} = 12y^2$.

$H(S_1) = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0$, saddle point at $[0,0]$

$H(S_2) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 12^2 - 16 > 0$, local minimum at $[1,1]$

- The hessian is positive at S_2 and the function possesses a local extremum at S_2.
- Since $z''_{xx} = 16 > 0$, the point S_2 is a local minimum.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y$, $z'_y = 4y^3 - 4x$,

$S_1 = [0,0]$, $S_2 = [1,1]$, $S_3 = [-1,-1]$.

$z''_{xx} = 12x^2$, $z''_{xy} = -4$, $z''_{yy} = 12y^2$.

$H(S_1) = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0$, saddle point at $[0,0]$

$H(S_2) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 12^2 - 16 > 0$, local minimum at $[1,1]$

$H(S_3) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 12^2 - 16 > 0$, local minimum at $[-1,-1]$

- The hessian is positive at S_3 and the function possesses a local extremum at S_3.
- Since $z''_{xx} = 16 > 0$, the point S_3 is a local minimum.

Find local extrema of the function $z = x^4 + y^4 - 4xy + 30$

$z'_x = 4x^3 - 4y$, $z'_y = 4y^3 - 4x$,

$S_1 = [0, 0]$, $S_2 = [1, 1]$, $S_3 = [-1, -1]$.

$z''_{xx} = 12x^2$, $z''_{xy} = -4$, $z''_{yy} = 12y^2$.

$H(S_1) = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0$, saddle point at $[0, 0]$

$H(S_2) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 12^2 - 16 > 0$, local minimum at $[1, 1]$

$H(S_3) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 12^2 - 16 > 0$, local minimum at $[-1, -1]$

The problem is solved.

Find local extrema of the function $z = y\ln(x^2 + y)$.

Find local extrema of the function $z = y\ln(x^2 + y)$.

$Dom(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y > 0\}$

The $\ln(\cdot)$ function yields restrictions to the domain of the function.

Find local extrema of the function $z = y\ln(x^2 + y)$.

$Dom(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y > 0\}$

$z'_x = \dfrac{2xy}{x^2 + y}$,

We find the partial derivatives. Differentiating with respect to x we use the constant multiple rule, since in the product $y\ln(x^2 + y)$ the factor y is treated as a constant. The chain rule follows, since the function $\ln(x^2 + y)$ is a composite function with inside function $(x^2 + y)$.

$$(y\ln(x^2 + y))'_x = y(\ln(x^2 + y))'_x = y\frac{1}{x^2 + y}(2x + 0)$$

Find local extrema of the function $z = y\ln(x^2+y)$.

$Dom(f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y > 0\}$

$$z'_x = \frac{2xy}{x^2+y} \quad , \quad z'_y = \ln(x^2+y) + \frac{y}{x^2+y}$$

Differentiating with respect to y we use the product rule, since both factors y and $\ln(x^2+y)$ are functions (x is treated as a constant and y as a variable).

Find local extrema of the function $z = y\ln(x^2 + y)$.

$$Dom(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y > 0\}$$

$$z'_x = \frac{2xy}{x^2 + y} = 0, \quad z'_y = \ln(x^2 + y) + \frac{y}{x^2 + y} = 0$$

To find stationary points we put the derivatives equal to zero.

Find local extrema of the function $z = y\ln(x^2+y)$.

$Dom(f) = \{(x,y) \in \mathbb{R}^2 : x^2+y>0\}$

$z'_x = \dfrac{2xy}{x^2+y} = 0, \quad z'_y = \ln(x^2+y) + \dfrac{y}{x^2+y} = 0$

$2xy = 0$

- We start with the first (simpler) equation.
- The fraction equals zero iff the numerator is zero.

Find local extrema of the function $z = y\ln(x^2 + y)$.

$Dom(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y > 0\}$

$z'_x = \dfrac{2xy}{x^2 + y} = 0, \quad z'_y = \ln(x^2 + y) + \dfrac{y}{x^2 + y} = 0$

$2xy = 0$

CASE 1: $x = 0$

CASE 2: $y = 0$

- To ensure that a product is zero, (at least) one of the factors has to be zero.
- We distinguish two possible cases.

Find local extrema of the function $z = y\ln(x^2+y)$.

$Dom(f) = \{(x,y) \in \mathbb{R}^2 : x^2+y>0\}$

$z'_x = \dfrac{2xy}{x^2+y} = 0$, $\quad z'_y = \ln(x^2+y) + \dfrac{y}{x^2+y} = 0$

$2xy = 0$

CASE 1: $x = 0$

$$\ln y + \frac{y}{y} = 0,$$

$$\ln y = -1,$$

CASE 2: $y = 0$

We substitute $x = 0$ into the second equation and simplify.

Find local extrema of the function $z = y\ln(x^2+y)$.

$Dom(f) = \{(x,y) \in \mathbb{R}^2 : x^2+y>0\}$

$z'_x = \dfrac{2xy}{x^2+y} = 0, \qquad z'_y = \ln(x^2+y) + \dfrac{y}{x^2+y} = 0$

$2xy = 0$

CASE 1: $x = 0$

$$\ln y + \frac{y}{y} = 0,$$
$$\ln y = -1,$$
$$y = e^{-1}$$

CASE 2: $y = 0$

The inverse function to $\ln$ function is an exponential function.

Find local extrema of the function $z = y\ln(x^2+y)$.

$Dom(f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y > 0\}$

$z'_x = \dfrac{2xy}{x^2+y} = 0, \quad z'_y = \ln(x^2+y) + \dfrac{y}{x^2+y} = 0$

$2xy = 0$

CASE 1: $x = 0$

$$\ln y + \frac{y}{y} = 0,$$
$$\ln y = -1,$$
$$y = e^{-1}$$

$\boxed{S_1 = [0, e^{-1}]}$

CASE 2: $y = 0$

- We have the stationary point $S_1 = [0, e^{-1}]$. We check that $S_1 \in Dom(f)$.

Find local extrema of the function $z = y\ln(x^2+y)$.

$Dom(f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y > 0\}$

$$z'_x = \frac{2xy}{x^2+y} = 0, \quad z'_y = \ln(x^2+y) + \frac{y}{x^2+y} = 0$$

$2xy = 0$

CASE 1: $x = 0$

$$\ln y + \frac{y}{y} = 0,$$
$$\ln y = -1,$$
$$y = e^{-1}$$

$\boxed{S_1 = [0, e^{-1}]}$

CASE 2: $y = 0$

$$\ln(x^2) = 0$$

- We return to the Case 2.
- We put $y = 0$ into the red equation.

Find local extrema of the function $z = y\ln(x^2+y)$.

$$Dom(f) = \{(x,y) \in \mathbb{R}^2 : x^2+y>0\}$$

$$z'_x = \frac{2xy}{x^2+y} = 0, \qquad z'_y = \ln(x^2+y) + \frac{y}{x^2+y} = 0$$

$$2xy = 0$$

CASE 1: $x = 0$

$$\ln y + \frac{y}{y} = 0,$$

$$\ln y = -1,$$

$$y = e^{-1}$$

$$\boxed{S_1 = [0, e^{-1}]}$$

CASE 2: $y = 0$

$$\ln(x^2) = 0$$

$$x^2 = e^0 = 1$$

$$x = \pm 1$$

We isolate x^2 and solve for x.

$\boxed{\text{Find local extrema of the function } z = y\ln(x^2+y).}$

$Dom(f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y > 0\}$

$z'_x = \dfrac{2xy}{x^2+y}$, $z'_y = \ln(x^2+y) + \dfrac{y}{x^2+y} = 0$

$2xy = 0$

CASE 1: $x = 0$

$$\ln y + \frac{y}{y} = 0,$$
$$\ln y = -1,$$
$$y = e^{-1}$$

$\boxed{S_1 = [0, e^{-1}]}$

CASE 2: $y = 0$

$$\ln(x^2) = 0$$
$$x^2 = e^0 = 1$$
$$x = \pm 1$$

$\boxed{S_2 = [1,0]}$ and $\boxed{S_3 = [-1,0]}$.

We have two stationary points. We check that both belong to $Dom(f)$.

Find local extrema of the function $z = y\ln(x^2 + y)$.

$$z'_x = \frac{2xy}{x^2+y} \quad , \quad z'_y = \ln(x^2+y) + \frac{y}{x^2+y}$$

$S_1 = [0, e^{-1}]$, $S_2 = [1, 0]$, $S_3 = [-1, 0]$

Up to now we have this.

Find local extrema of the function $z = y\ln(x^2+y)$.

$$z'_x = \frac{2xy}{x^2+y} \quad , \quad z'_y = \ln(x^2+y) + \frac{y}{x^2+y}$$

$S_1 = [0, e^{-1}]$, $S_2 = [1, 0]$, $S_3 = [-1, 0]$

z''_{xx}

z''_{xy}

z''_{yy}

We will use the second derivative test to recognize, whether a local extremum appears at the stationary points.

Find local extrema of the function $z = y\ln(x^2+y)$.

$z'_x = \dfrac{2xy}{x^2+y}$, $z'_y = \ln(x^2+y) + \dfrac{y}{x^2+y}$

$S_1 = [0, e^{-1}]$, $S_2 = [1, 0]$, $S_3 = [-1, 0]$

$$z''_{xx} = \frac{2y(x^2+y) - 2xy2x}{(x^2+y)^2},$$

z''_{xy}

z''_{yy}

We differentiate z'_x with respect to x. This gives z''_{xx}.

Find local extrema of the function $z = y\ln(x^2 + y)$.

$$z'_x = \frac{2xy}{x^2 + y}, \quad z'_y = \ln(x^2 + y) + \frac{y}{x^2 + y}$$

$S_1 = [0, e^{-1}]$, $S_2 = [1, 0]$, $S_3 = [-1, 0]$

$$z''_{xx} = \frac{2y(x^2 + y) - 2xy2x}{(x^2 + y)^2},$$

$$z''_{xy} = \frac{2x(x^2 + y) - 2xy}{(x^2 + y)^2},$$

$$z''_{yy}$$

We differentiate z'_x with respect to y. This gives z''_{xy}.

Find local extrema of the function $z = y\ln(x^2+y)$.

$$z'_x = \frac{2xy}{x^2+y} \quad , \quad z'_y = \ln(x^2+y) + \frac{y}{x^2+y}$$

$S_1 = [0, e^{-1}]$, $S_2 = [1, 0]$, $S_3 = [-1, 0]$

$$z''_{xx} = \frac{2y(x^2+y) - 2xy2x}{(x^2+y)^2},$$

$$z''_{xy} = \frac{2x(x^2+y) - 2xy}{(x^2+y)^2},$$

$$z''_{yy} = \frac{1}{x^2+y} + \frac{x^2+y-y}{(x^2+y)^2}.$$

We differentiate z''_{yy} with respect to y. This gives z''_{yy}.

Find local extrema of the function $z = y\ln(x^2 + y)$.

$$z'_x = \frac{2xy}{x^2 + y}, \quad z'_y = \ln(x^2 + y) + \frac{y}{x^2 + y}$$

$S_1 = [0, e^{-1}]$, $S_2 = [1, 0]$, $S_3 = [-1, 0]$

$$z''_{xx} = \frac{2y(x^2 + y) - 2xy2x}{(x^2 + y)^2},$$

$$z''_{xy} = \frac{2x(x^2 + y) - 2xy}{(x^2 + y)^2},$$

$$z''_{yy} = \frac{1}{x^2 + y} + \frac{x^2 + y - y}{(x^2 + y)^2}.$$

$$z''_{xx} = \frac{2y^2 - 2yx^2}{(x^2 + y)^2},$$

$$z''_{xy} = \frac{2x^3}{(x^2 + y)^2},$$

$$z''_{yy} = \frac{1}{x^2 + y} + \frac{x^2}{(x^2 + y)^2}.$$

We simplify.

Find local extrema of the function $z = y\ln(x^2 + y)$.

$$z'_x = \frac{2xy}{x^2+y} \quad , \quad z'_y = \ln(x^2+y) + \frac{y}{x^2+y}$$

$S_1 = [0, e^{-1}]$, $S_2 = [1, 0]$, $S_3 = [-1, 0]$

$$z''_{xx} = \frac{2y^2 - 2yx^2}{(x^2+y)^2},$$

$$z''_{xy} = \frac{2x^3}{(x^2+y)^2},$$

$$z''_{yy} = \frac{1}{x^2+y} + \frac{x^2}{(x^2+y)^2}.$$

$$H(S_1) = \begin{vmatrix} 2 & 0 \\ 0 & e \end{vmatrix} > 0,$$

$$H(S_2) = \begin{vmatrix} 0 & 2 \\ 2 & 2 \end{vmatrix} = -4 < 0,$$

$$H(S_3) = \begin{vmatrix} 0 & -2 \\ -2 & 2 \end{vmatrix} = -4 < 0.$$

We evaluate the hessian at each of the stationary points.

Find local extrema of the function $z = y\ln(x^2+y)$.

$$z'_x = \frac{2xy}{x^2+y} \quad , \quad z'_y = \ln(x^2+y) + \frac{y}{x^2+y}$$

$S_1 = [0, e^{-1}]$, $S_2 = [1, 0]$, $S_3 = [-1, 0]$

$$z''_{xx} = \frac{2y^2 - 2yx^2}{(x^2+y)^2},$$

$$z''_{xy} = \frac{2x^3}{(x^2+y)^2},$$

$$z''_{yy} = \frac{1}{x^2+y} + \frac{x^2}{(x^2+y)^2}.$$

$$H(S_1) = \begin{vmatrix} 2 & 0 \\ 0 & e \end{vmatrix} > 0,$$

$$H(S_2) = \begin{vmatrix} 0 & 2 \\ 2 & 2 \end{vmatrix} = -4 < 0,$$

$$H(S_3) = \begin{vmatrix} 0 & -2 \\ -2 & 2 \end{vmatrix} = -4 < 0.$$

Local minimum at $[0, e^{-1}]$. No other local extremum.

According to the second derivative test we obtain the following conclusion.

That's all . . .