The first derivative, local extrema and monotonicity

Robert Mařík

May 9, 2006

Contents

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$ and establish the intervals of monotonicity.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$

- We find the domain of the function.
- There is no restriction on x and the domain $\mathbb{R}$.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$

$$y' = (x^3)' - 2(x^2)' + (x)' + (1)'$$

We differentiate. We use the sum rule and the constant multiple rule.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$

$$y' = (x^3)' - 2(x^2)' + (x)' + (1)'$$
$$= 3x^2 - 4x + 1 + 0$$

We find derivatives by the formula $\boxed{(x^n)' = nx^{n-1}}$.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$

$$\begin{aligned} y' &= (x^3)' - 2(x^2)' + (x)' + (1)' \\ &= 3x^2 - 4x + 1 + 0 \\ &= 3x^2 - 4x + 1 \end{aligned}$$

We simplify.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$; $\quad y' = 3x^2 - 4x + 1$

$$3x^2 - 4x + 1 = 0$$

- We have to find the intervals of monotonicity first.
- We have to find the sign of the derivative.
- We have to find points where the derivative may change its sign. There are no points of discontinuity and we have to find stationary points.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$; $\quad y' = 3x^2 - 4x + 1$

$$3x^2 - 4x + 1 = 0$$

$$x_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3}$$

We solve the quadratic equation by the formula. The solutions of $ax^2 + bx + c = 0$ are

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$; $\quad y' = 3x^2 - 4x + 1$

$$3x^2 - 4x + 1 = 0$$

$$x_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3}$$

$$= \frac{4 \pm 2}{6}$$

We simplify.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$; $y' = 3x^2 - 4x + 1$; Stac. points: $x_1 = 1$, $x_2 = \dfrac{1}{3}$

$$3x^2 - 4x + 1 = 0$$

$$x_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} = \frac{4 \pm 2}{6}$$

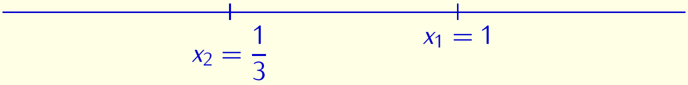
$$x_1 = 1$$

$$x_2 = \frac{1}{3}$$

We find the solution. There are two real zeros of the derivative.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

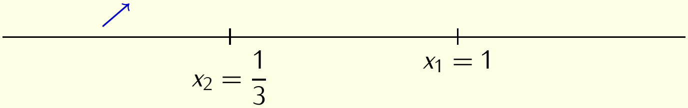
$Dom(f) = \mathbb{R}$; $y' = 3x^2 - 4x + 1$; Stac. points: $x_1 = 1$, $x_2 = \frac{1}{3}$



- We draw stationary points on the real axis.
- There are no points of discontinuity and we have three subintervals.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

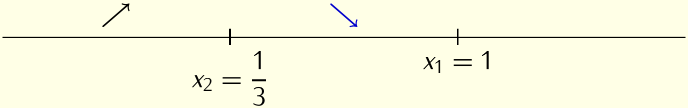
$Dom(f) = \mathbb{R}$; $y' = 3x^2 - 4x + 1$; Stac. points: $x_1 = 1$, $x_2 = \frac{1}{3}$



- We consider an arbitrary number from the first interval $(-\infty, \frac{1}{3})$. Let us consider the test number $\xi_1 = 0$.
- We find $y'(0) = 3 \cdot 0^2 - 4 \cdot 0 + 1 = 1 > 0$. The function is increasing on $(-\infty, \frac{1}{3})$.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$; $y' = 3x^2 - 4x + 1$; Stac. points: $x_1 = 1$, $x_2 = \frac{1}{3}$

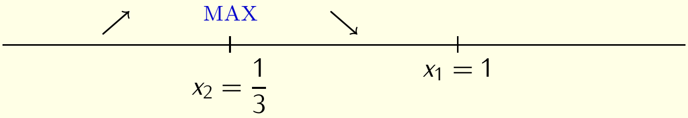


$y'(0) > 0$ $\quad$ $y'(\frac{1}{2}) < 0$

In a similar way, we find $y'(\frac{1}{2}) = 3\frac{1}{4} - 4\frac{1}{2} + 1 = -\frac{1}{4} < 0$ and the function is decreasing on $(\frac{1}{3}, 1)$.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$; $y' = 3x^2 - 4x + 1$; Stac. points: $x_1 = 1$, $x_2 = \frac{1}{3}$

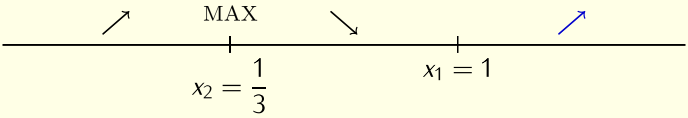


$y'(0) > 0$ $\qquad$ $y'(\frac{1}{2}) < 0$

There is a change in monotonicity at x_2. There is a local maximum at this point.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$; $y' = 3x^2 - 4x + 1$; Stac. points: $x_1 = 1$, $x_2 = \frac{1}{3}$

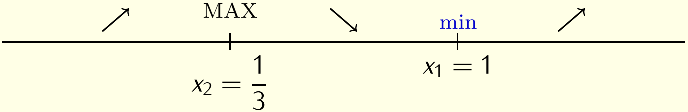


$y'(0) > 0$ $y'(\frac{1}{2}) < 0$ $y'(2) > 0$

We find $y'(2) = 3 \cdot 2^2 - 4 \cdot 2 + 1 = 5$

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$; $y' = 3x^2 - 4x + 1$; Stac. points: $x_1 = 1$, $x_2 = \frac{1}{3}$

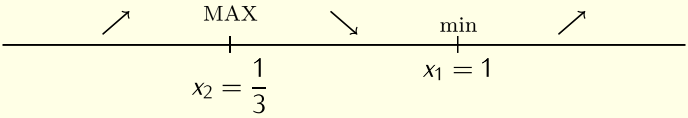


$y'(0) > 0$ $y'(\frac{1}{2}) < 0$ $y'(2) > 0$

The type of monotonicity changes at $x_1 = 1$ and there is a local minimum at this point.

Find local extrema of the function $y = x^3 - 2x^2 + x + 1$.

$Dom(f) = \mathbb{R}$; $y' = 3x^2 - 4x + 1$; Stac. points: $x_1 = 1$, $x_2 = \frac{1}{3}$



Finished!

Find local extrema of the function $y = \dfrac{x^3}{x-1}$ and establish the intervals of monotonicity.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$;

We find the natural domain. The expression in the denominator of the fraction must be nonzero.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$;

$$y' = \frac{(x^3)'(x-1) - x^3(x-1)'}{(x-1)^2}$$

We use the quotient rule

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}.$$

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$;

$$y' = \frac{(x^3)'(x-1) - x^3(x-1)'}{(x-1)^2}$$
$$= \frac{3x^2(x-1) - x^3(1-0)}{(x-1)^2}$$

We differentiate.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$;

$$y' = \frac{(x^3)'(x-1) - x^3(x-1)'}{(x-1)^2} = \frac{3x^2(x-1) - x^3(1-0)}{(x-1)^2} = \frac{2x^3 - 3x^2}{(x-1)^2}$$

We simplify.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = \dfrac{x^2(2x-3)}{(x-1)^2}$;

$$\begin{aligned} y' &= \frac{(x^3)'(x-1) - x^3(x-1)'}{(x-1)^2} \\ &= \frac{3x^2(x-1) - x^3(1-0)}{(x-1)^2} \\ &= \frac{2x^3 - 3x^2}{(x-1)^2} \\ &= \frac{x^2(2x-3)}{(x-1)^2} \end{aligned}$$

We find a factorization of the numerator.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = \dfrac{x^2(2x-3)}{(x-1)^2}$;

$$\frac{x^2(2x-3)}{(x-1)^2} = 0$$

- We investigate zero points, discontinuities and sign of the first derivative.
- The derivative has a discontinuity at $x = 1$.
- We have to solve the equation $y' = 0$.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$$Dom(f) = \mathbb{R} \setminus \{1\}; \quad y' = \frac{x^2(2x-3)}{(x-1)^2};$$

$$\frac{x^2(2x-3)}{(x-1)^2} = 0$$

$$x^2(2x-3) = 0$$

The quotient is zero iff the numerator is zero.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = \dfrac{x^2(2x-3)}{(x-1)^2}$; $\quad x_{1,2} = 0,\ x_3 = \dfrac{3}{2}$

$$\frac{x^2(2x-3)}{(x-1)^2} = 0$$

$$x^2(2x-3) = 0$$

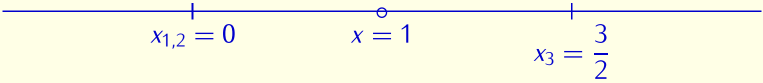
$$x_{1,2} = 0$$

$$x_3 = \frac{3}{2}$$

The product is zero iff at least one of its factors is zero. We continue with two equations $x^2 = 0$ and $2x - 3 = 0$.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

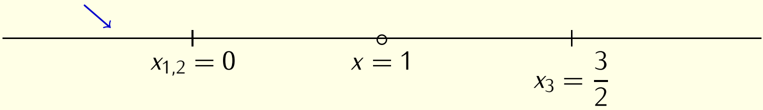
$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = \dfrac{x^2(2x-3)}{(x-1)^2}$; $\quad x_{1,2} = 0,\ x_3 = \dfrac{3}{2}$



- We have found discontinuities and stationary points.
- We draw these points on real axis.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\};$ $\quad y' = \dfrac{x^2(2x-3)}{(x-1)^2};$ $\quad x_{1,2} = 0,\ x_3 = \dfrac{3}{2}$



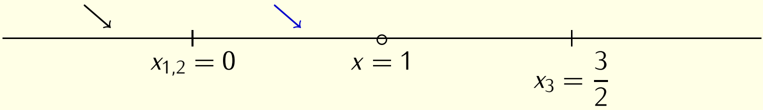
$y'(-1) < 0$

We evaluate the derivative at test points from subintervals on real axis..

$$y'(-1) = \frac{(-1)^2(-2-3)}{\text{positive}} = \frac{-5}{\text{positive}} < 0$$

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = \dfrac{x^2(2x-3)}{(x-1)^2}$; $\quad x_{1,2} = 0,\ x_3 = \dfrac{3}{2}$

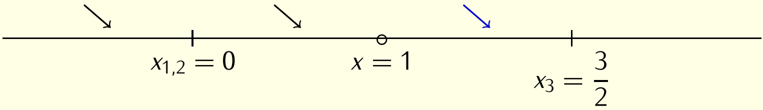


$y'(-1) < 0 \qquad y'(\frac{1}{2}) < 0$

$$y'(\frac{1}{2}) = \frac{\frac{1}{4}(1-3)}{\text{positive}} < 0$$

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = \dfrac{x^2(2x-3)}{(x-1)^2}$; $\quad x_{1,2} = 0,\ x_3 = \dfrac{3}{2}$

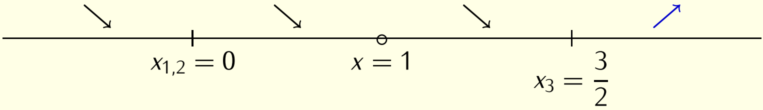


$y'(-1) < 0 \qquad y'(\frac{1}{2}) < 0 \qquad y'(1,2) < 0$

$$y'(1,2) = \frac{(1,2)^2(2,4-3)}{\text{positive}} < 0$$

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\};$ $\quad y' = \dfrac{x^2(2x-3)}{(x-1)^2};$ $\quad x_{1,2} = 0,\ x_3 = \dfrac{3}{2}$

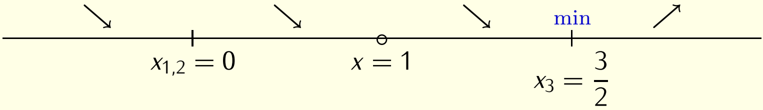


$y'(-1) < 0$ $\quad y'(\frac{1}{2}) < 0$ $\quad y'(1,2) < 0$ $\quad y'(2) > 0$

$$y'(2) = \frac{(2)^2(4-3)}{\text{positive}} > 0$$

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$$Dom(f) = \mathbb{R} \setminus \{1\}; \quad y' = \frac{x^2(2x-3)}{(x-1)^2}; \quad x_{1,2} = 0,\ x_3 = \frac{3}{2}$$

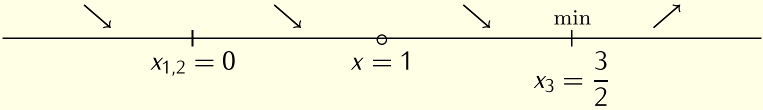


$$y'(-1) < 0 \qquad y'(\tfrac{1}{2}) < 0 \qquad y'(1,2) < 0 \qquad y'(2) > 0$$

The type of monotonicity changes at $x = \frac{3}{2}$. The function is continuous in a neighborhood of this point and hence a local extremum (minimum) appears here.

Find local extrema of the function $y = \dfrac{x^3}{x-1}$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = \dfrac{x^2(2x-3)}{(x-1)^2}$; $\quad x_{1,2} = 0,\ x_3 = \dfrac{3}{2}$



$y'(-1) < 0 \qquad y'(\frac{1}{2}) < 0 \qquad y'(1,2) < 0 \qquad y'(2) > 0$

Finished.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$ and establish the intervals of monotonicity.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$;

We establish the domain of the function. The only restriction follows from the denominator of the fraction:

$$1 - x \neq 0,$$

i.e.

$$x \neq 1.$$

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$;

$$y' = 4\left(\frac{1+x}{1-x}\right)^3 \frac{1(1-x)-(1+x)(-1)}{(1-x)^2}$$

- We differentiate the function. The outside function is differentiated by the power rule $(x^4)' = 4x^3$.
- The inside function is a fraction and it is differentiated by the quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$;

$$y' = 4\left(\frac{1+x}{1-x}\right)^3 \frac{1(1-x)-(1+x)(-1)}{(1-x)^2}$$
$$= 4\frac{(1+x)^3}{(1-x)^3}\frac{1-x+1+x}{(1-x)^2}$$

We simplify the numerator of the second fraction.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$;

$$\begin{aligned} y' &= 4\left(\frac{1+x}{1-x}\right)^3 \frac{1(1-x)-(1+x)(-1)}{(1-x)^2} \\ &= 4\frac{(1+x)^3}{(1-x)^3}\frac{1-x+1+x}{(1-x)^2} \\ &= 8\frac{(1+x)^3}{(1-x)^5} \end{aligned}$$

And simplify even more.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $y' = 8\frac{(1+x)^3}{(1-x)^5}$;

We have the derivative y'. The restriction on x are the same as for the original function and hence the domain of the derivative is $\mathbb{R} \setminus \{1\}$.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $y' = 8\frac{(1+x)^3}{(1-x)^5}$;

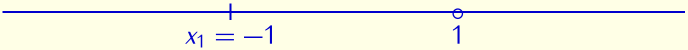
Stationary point: $x_1 = -1$

- We look for the point where $y' = 0$ first.
- The fraction equals zero iff the numerator equals zero. Hence the unique stationary point is a solution of

$$(1+x)^3 = 0.$$

Find local extrema of the function $y = \left(\dfrac{1+x}{1-x}\right)^4$.

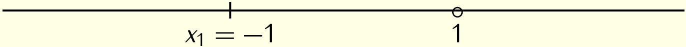
$Dom(f) = \mathbb{R} \setminus \{1\}$; $y' = 8\dfrac{(1+x)^3}{(1-x)^5}$; $x_1 = -1$



- We mark the stationary point and the point of discontinuity on the real axis.
- The real axis is divided into three subintervals. The function has the same type of monotonicity for all x belonging to the same subinterval.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

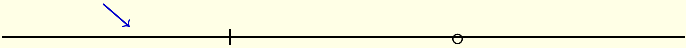
$Dom(f) = \mathbb{R} \setminus \{1\}$; $y' = 8\frac{(1+x)^3}{(1-x)^5}$; $x_1 = -1$



- We investigate the monotonicity on the interval $(-\infty, -1)$
- We choose a test number from this interval
- Let $\xi_1 = -2$ be the test number.
- We evaluate the derivative at the test point.

Find local extrema of the function $y = \left(\dfrac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = 8\dfrac{(1+x)^3}{(1-x)^5}$; $\quad x_1 = -1$



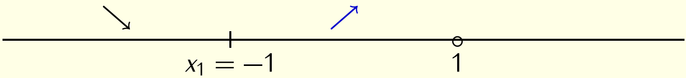
$y'(-2) < 0$

$$y'(-2) = 8\frac{(1-2)^3}{(1-(-2))^5} = 8\frac{-1}{3^5} < 0.$$

The derivative is negative and the function is decreasing at $\xi_2 = -2$ and on $(-\infty, -1)$.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = 8\frac{(1+x)^3}{(1-x)^5}$; $\quad x_1 = -1$



$y'(-2) < 0 \qquad y'(0) > 0$

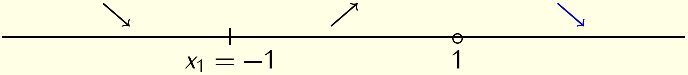
Similarly, the test point $\xi_2 = 0$ belongs to $(-1, 1)$ and

$$y'(0) = 8\frac{1}{1^5} > 0.$$

The function is increasing at $\xi_2 = 0$ and on $(-1, 1)$.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $y' = 8\frac{(1+x)^3}{(1-x)^5}$; $x_1 = -1$



$y'(-2) < 0$ $y'(0) > 0$ $y'(2) < 0$

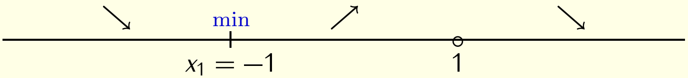
Finally, the test point $\xi_3 = 2$ belongs to $(1, \infty)$ and

$$y'(2) = 8\frac{(1+2)^3}{(1-2)^5} < 0.$$

The function is decreasing at $\xi_3 = 2$ and on $(1, \infty)$.

Find local extrema of the function $y = \left(\frac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $\quad y' = 8\frac{(1+x)^3}{(1-x)^5}$; $\quad x_1 = -1$

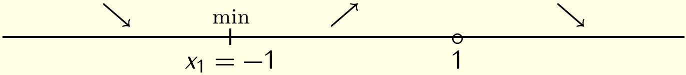


$y'(-2) < 0 \qquad y'(0) > 0 \qquad y'(2) < 0$

- The function has a local minimum at $x = -1$.
- The function has no other local extremum. Particularly, there is no local extremum at $x = 1$, since $1 \notin Dom(f)$.

Find local extrema of the function $y = \left(\dfrac{1+x}{1-x}\right)^4$.

$Dom(f) = \mathbb{R} \setminus \{1\}$; $y' = 8\dfrac{(1+x)^3}{(1-x)^5}$; $x_1 = -1$



Finished!

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$ and establish the intervals of monotonicity.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$;

We establish the domain of the function. The only restriction follows from the denominator of the fraction:

$$1 + x \neq 0,$$

i.e.

$$x \neq -1.$$

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$;

$$y' = \frac{1\,(1+x)^3 - x\,3(1+x)^2}{((1+x)^3)^2}$$

- We differentiate the function. We use the quotient rule
- When differentiating the denominator $(1+x)^3$ we use the chain rule $((1+x)^3)' = 3(1+x)^2(1+x)' = 3(1+x)^2$. this allows a factorization of the numerator in the forthcoming steps.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$;

$$y' = \frac{1\,(1+x)^3 - x\,3(1+x)^2}{((1+x)^3)^2}$$
$$= \frac{(1+x)^2(1+x-3x)}{(1+x)^6}$$

We simplify the numerator of the second fraction. We take the common factor $(1+x)^2$ from the parenthesis.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$;

$$y' = \frac{1\,(1+x)^3 - x\,3(1+x)^2}{((1+x)^3)^2} = \frac{(1+x)^2(1+x-3x)}{(1+x)^6} = \frac{1-2x}{(1+x)^4}$$

We cancel $(1+x)^2$ and simplify the remaining part of the numerator.

Find local extrema of the function $y = \frac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$; $\qquad y' = \frac{1-2x}{(1+x)^4}$;

- We have the derivative y'.
- The restriction on x are the same as for the original function and hence the domain of the derivative is $\mathbb{R} \setminus \{-1\}$.
- We will investigate the sign of the derivative.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$; $y' = \dfrac{1-2x}{(1+x)^4}$;

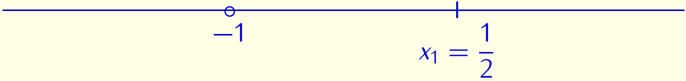
Stationary point: $x_1 = \dfrac{1}{2}$

- We look for the point where $y' = 0$ first.
- The fraction equals zero iff the numerator equals zero. Hence the unique stationary point is a solution of

$$1 - 2x = 0.$$

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

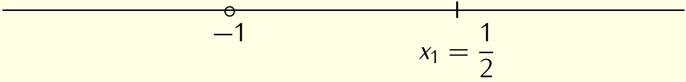
$Dom(f) = \mathbb{R} \setminus \{-1\}$; $\quad y' = \dfrac{1-2x}{(1+x)^4}$; $\quad x_1 = \dfrac{1}{2}$



- We mark the stationary point and the point of discontinuity on the real axis.
- The real axis is divided into three subintervals. The function has the same type of monotonicity for all x belonging to the same subinterval.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

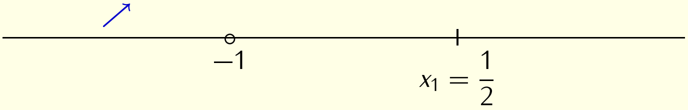
$Dom(f) = \mathbb{R} \setminus \{-1\}$; $\quad y' = \dfrac{1-2x}{(1+x)^4}$; $\quad x_1 = \dfrac{1}{2}$



- We investigate the monotonicity on the interval $(-\infty, -1)$
- We choose a test number from this interval
- Let $\xi_1 = -2$ be the test number.
- We evaluate the derivative at the test point.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$; $y' = \dfrac{1-2x}{(1+x)^4}$; $x_1 = \dfrac{1}{2}$



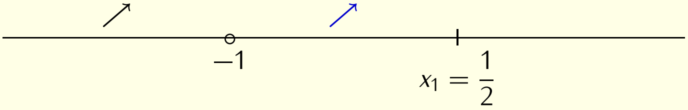
$y'(-2) > 0$

$$y'(-2) = \frac{1-2(-2)}{(1-2)^6} = \frac{5}{1} > 0.$$

The derivative is positive and the function is increasing at $\xi_2 = -2$ and on $(-\infty, -1)$.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$; $y' = \dfrac{1-2x}{(1+x)^4}$; $x_1 = \dfrac{1}{2}$

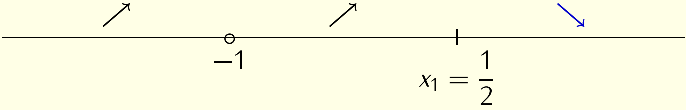


$y'(-2) > 0$ $y'(0) > 0$

Similarly, the test point $\xi_2 = 0$ belongs to $(-1, \frac{1}{2})$ and $y'(0) = \frac{1}{1} > 0$. The function is increasing at $\xi_2 = 0$ and on $(-1, \frac{1}{2})$.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$; $\quad y' = \dfrac{1-2x}{(1+x)^4}$; $\quad x_1 = \dfrac{1}{2}$

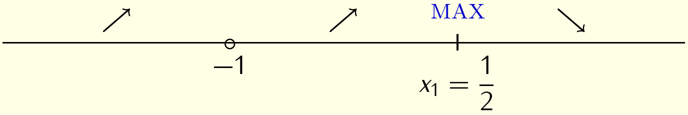


$y'(-2) > 0 \qquad y'(0) > 0 \qquad y'(2) < 0$

Finally $y'(2) = \dfrac{1-4}{3^4} < 0$. The function is decreasing at $\xi_3 = 2$ and on $(\frac{1}{2}, \infty)$.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$; $\quad y' = \dfrac{1-2x}{(1+x)^4}$; $\quad x_1 = \dfrac{1}{2}$

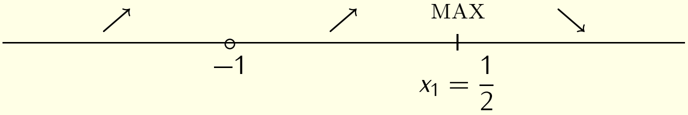


$y'(-2) > 0 \qquad y'(0) > 0 \qquad y'(2) < 0$

- The function has a local maximum at $x = \dfrac{1}{2}$.
- The function has no other local extremum.

Find local extrema of the function $y = \dfrac{x}{(1+x)^3}$.

$Dom(f) = \mathbb{R} \setminus \{-1\}$; $\quad y' = \dfrac{1-2x}{(1+x)^4}$; $\quad x_1 = \dfrac{1}{2}$



Finished!

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$ and establish the intervals of monotonicity.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$;

We establish the domain of the function. The only restriction on x arises from the denominator of the fraction. Hence $x \neq 0$.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$;

$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2}$$

We use the quotient rule

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

with $u = 3x + 1$ and $v = x^3$.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$;

$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

- We will look for the points where $y' = 0$.
- From this reason it is useful to simplify and to factor the derivative as much as possible.
- We take out the common factor x^2 in the numerator.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$;

$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

$$= 3\frac{x - 3x - 1}{x^4}$$

- We cancel the factor x^2 which is in both numerator and denominator.
- We take the constant factor 3 from the fraction.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$;

$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

$$= 3\frac{x - 3x - 1}{x^4} = 3\frac{-2x-1}{x^4}$$

We simplify.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$;

$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

$$= 3\frac{x - 3x - 1}{x^4} = 3\frac{-2x-1}{x^4} = -3\frac{2x+1}{x^4}$$

We take the minus sign from the numerator.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$; $y'(x) = -3\dfrac{2x+1}{x^4}$;

- The domain of the derivative is the same as the domain of the function f (the same restriction $x \neq 0$).
- In order to find the intervals where the derivative is positive or negative, we have to find the points where $y'(x) = 0$.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

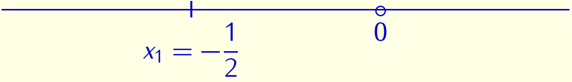
$Dom(f) = \mathbb{R} \setminus \{0\}$; $\qquad y'(x) = -3\dfrac{2x+1}{x^4}$;

Stationary point: $x_1 = -\dfrac{1}{2}$.

- The fraction equals zero iff the numerator equals zero.
- $2x+1 = 0$ for $x = -\dfrac{1}{2}$. Hence $x_1 = -\dfrac{1}{2}$ is the unique stationary point of the function.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

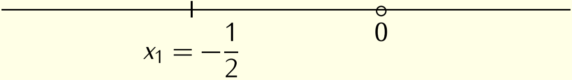
$Dom(f) = \mathbb{R} \setminus \{0\}$; $y'(x) = -3\dfrac{2x+1}{x^4}$; $x_1 = -\dfrac{1}{2}$



- We mark the point of discontinuity of the derivative and the stationary points to the real axis.
- The axis is divided into three subintervals. In each of these subintervals the type of the monotonicity is preserved for all x.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

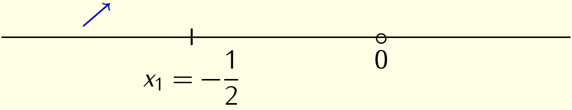
$Dom(f) = \mathbb{R} \setminus \{0\}$; $y'(x) = -3\dfrac{2x+1}{x^4}$; $x_1 = -\dfrac{1}{2}$



We choose an arbitrary test number from the first interval $(-\infty, -\frac{1}{2})$. Let $\xi_1 = -1$ be such a number. We evaluate the derivative at ξ_1.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$; $y'(x) = -3\dfrac{2x+1}{x^4}$; $x_1 = -\dfrac{1}{2}$

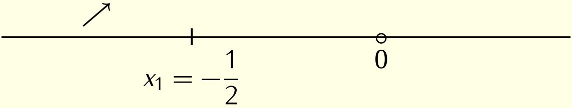


$$y'(-1) = -3\frac{-2+1}{(-1)^4} > 0$$

Hence the function is increasing at $\xi_1 = -1$ and the same is true for the interval $(-\infty, -\frac{1}{2})$.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

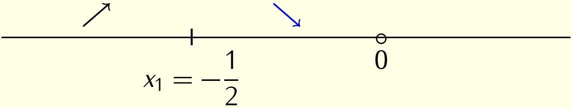
$Dom(f) = \mathbb{R} \setminus \{0\}$; $\quad y'(x) = -3\dfrac{2x+1}{x^4}$; $\quad x_1 = -\dfrac{1}{2}$



We choose the test number $\xi_2 = -\dfrac{1}{4}$ from the second interval $(-\dfrac{1}{2}, 0)$. We evaluate the derivative at this point.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$; $\quad y'(x) = -3\dfrac{2x+1}{x^4}$; $\quad x_1 = -\dfrac{1}{2}$

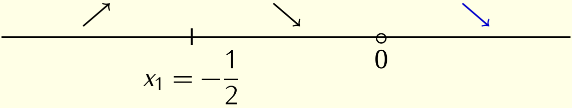


$$y'(-1/4) = -3\frac{-\frac{1}{2}+1}{\text{positive}} < 0$$

and hence the function is decreasing at $\xi_2 = -1/4$ and also on the interval $(-\frac{1}{2}, 0)$.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$; $\qquad y'(x) = -3\dfrac{2x+1}{x^4}$; $\qquad x_1 = -\dfrac{1}{2}$



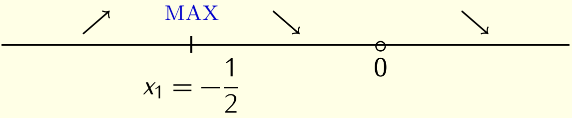
Similarly, for $\xi_3 = 1$ we have

$$y'(1) = -3\frac{2+1}{1^4}$$

and hence the function is increasing at $\xi_3 = 1$ and also on the interval $(0, \infty)$.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

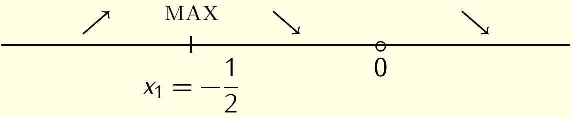
$Dom(f) = \mathbb{R} \setminus \{0\}$; $y'(x) = -3\dfrac{2x+1}{x^4}$; $x_1 = -\dfrac{1}{2}$



- The function is continuous on $\mathbb{R} \setminus \{0\}$ (why? explain!).
- From the scheme of monotonicity it follows that the function possesses a local maximum at $x = -\dfrac{1}{2}$ and no other local extremum.

Find local extrema of the function $y = \dfrac{3x+1}{x^3}$.

$Dom(f) = \mathbb{R} \setminus \{0\}$; $\quad y'(x) = -3\dfrac{2x+1}{x^4}$; $\quad x_1 = -\dfrac{1}{2}$



- The problem is solved!
- Everything concerning monotonicity and local extrema is clear from the picture.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

$Dom(f) = \mathbb{R}$;

We establish the domain of the function. There is no restriction for x and hence the domain is $\mathbb{R}$.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

$Dom(f) = \mathbb{R}$;

$$y' = (x^2)' e^{-x} + x^2 (e^{-x})'$$

We use the chain rule

$$(uv)' = u'v + uv'$$

with $u = x^2$ and $v = e^{-x}$.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

$Dom(f) = \mathbb{R}$;

$$y' = (x^2)' e^{-x} + x^2 (e^{-x})' = 2x e^{-x} + x^2(-1)e^{-x}$$

We use the power rule for derivative of x^2 and the formula and the chain rule for derivative of e^{-x}.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

$Dom(f) = \mathbb{R}$;

$$y' = (x^2)' e^{-x} + x^2 (e^{-x})' = 2x e^{-x} + x^2 (-1) e^{-x}$$
$$= e^{-x}(2x - x^2)$$

- We will look for the points where $y' = 0$.
- From this reason it is useful to factor the derivative.
- We take out the common factor e^{-x}.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

$Dom(f) = \mathbb{R}$;

$$y' = (x^2)' e^{-x} + x^2 (e^{-x})' = 2x e^{-x} + x^2(-1)e^{-x}$$
$$= e^{-x}(2x - x^2) = e^{-x} x(2 - x)$$

The quadratic expression in the parentheses can be factored by taking out the factor x.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

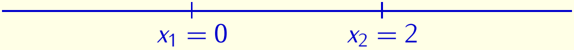
$Dom(f) = \mathbb{R}$;　　　　$y'(x) = e^{-x} x (2 - x)$;

Stationary points: $x_1 = 0$, $x_2 = 2$.

- Now it is easy to find the stationary points.
- The derivative equals zero iff one of its factors equals to zero.
- The factor e^{-x} is never equal to zero.
- The factor $(x - 2)$ equals zero iff $x = 2$.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

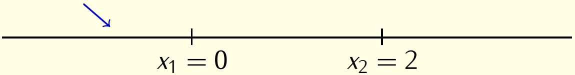
$Dom(f) = \mathbb{R}$; $y'(x) = e^{-x}x(2 - x)$; $x_1 = 0$, $x_2 = 2$.



- We mark the domain of the derivative (no restriction) and the stationary points to the real axis.
- The axis is divided into three subintervals.
- In each of these subintervals the type of the monotonicity is preserved for all x.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

$Dom(f) = \mathbb{R}$; $y'(x) = e^{-x}x(2-x)$; $x_1 = 0$, $x_2 = 2$.



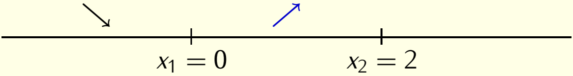
We choose an arbitrary test number from the first interval $(-\infty, 0)$. Let $\xi_1 = -1$ be such a number. We evaluate the derivative at ξ_1:

$$y'(-1) = e^{-(-1)}(-1)(2-(-1)) = e^1(-1)3 < 0$$

Hence the function is decreasing at ξ_1 and the same is true for the interval $(-\infty, 0)$.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

$Dom(f) = \mathbb{R}$; $y'(x) = e^{-x}x(2-x)$; $x_1 = 0$, $x_2 = 2$.



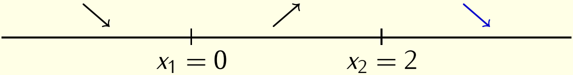
We choose the test number $\xi_2 = 1$ from the second interval $(0, 2)$. The derivative evaluated at this point is

$$y'(1) = e^{-1}1(2-1) = e^{-1} > 0$$

and hence the function is increasing at $\xi_2 = 1$ and also on the interval $(0, 2)$.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

$Dom(f) = \mathbb{R}$; $y'(x) = e^{-x}x(2-x)$; $x_1 = 0$, $x_2 = 2$.



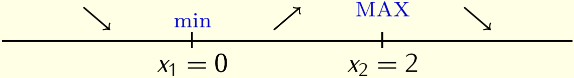
We choose the test number $\xi_3 = 3$ from the last interval $(2, \infty)$. The derivative evaluated at this point is

$$y'(3) = e^{-3}3(2-3) = -3e^{-3} < 0$$

and hence the function is decreasing at $\xi_3 = 3$ and also on the interval $(2, \infty)$.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

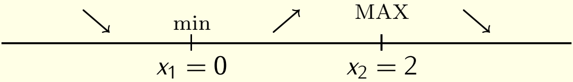
$Dom(f) = \mathbb{R}$; $y'(x) = e^{-x}x(2-x)$; $x_1 = 0$, $x_2 = 2$.



- The function is continuous on $\mathbb{R}$ (why? explain!).
- From the scheme of monotonicity it follows that the function possesses a local minimum at $x = 0$ and a local maximum at $x = 2$.

Find local extrema of the function $y = x^2 e^{-x}$ and establish the intervals of monotonicity.

$Dom(f) = \mathbb{R}$; $y'(x) = e^{-x}x(2-x)$; $x_1 = 0$, $x_2 = 2$.



- The problem is solved!
- Everything concerning monotonicity and local extrema is clear from the picture.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$. Establish the intervals of monotonicity.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = \mathbb{R}^+ \setminus \{1\} = (0,1) \cup (1,\infty)$.

- We establish the domain of the function.
- There is a restriction $x > 0$ from the $\ln(\cdot)$ function.
- There is a restriction $\ln x \neq 0$ from the denominator of the fraction. Since $\ln x = 0$ for $x = e^0 = 1$, this is equivalent to the restriction $x \neq 1$.
- The domain is $Dom(f) = \mathbb{R}^+ \setminus \{1\} = (0,1) \cup (1,\infty)$.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = \mathbb{R}^+ \setminus \{1\} = (0,1) \cup (1,\infty)$.

$$y' = \frac{2x\ln x - x^2\frac{1}{x}}{\ln^2 x}$$

We differentiate by the quotient rule

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

with $u = x^2$ and $v = \ln x$.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = \mathbb{R}^+ \setminus \{1\} = (0,1) \cup (1,\infty)$.

$$y' = \frac{2x\ln x - x^2\frac{1}{x}}{\ln^2 x} = \frac{2x\ln x - x}{\ln^2 x}$$

We simplify the numerator.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = \mathbb{R}^+ \setminus \{1\} = (0,1) \cup (1,\infty)$.

$$y' = \frac{2x\ln x - x^2\frac{1}{x}}{\ln^2 x} = \frac{2x\ln x - x}{\ln^2 x} = \frac{x(2\ln x - 1)}{\ln^2 x}$$

- We will look for the points where $y' = 0$.
- The fraction equals zero iff the numerator equals zero.
- From this reason it is useful to factor the numerator.
- We take out the common factor x in the numerator.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = \mathbb{R}^+ \setminus \{1\} = (0,1) \cup (1,\infty)$.

$$y' = \frac{2x\ln x - x^2\frac{1}{x}}{\ln^2 x} = \frac{2x\ln x - x}{\ln^2 x} = \frac{x(2\ln x - 1)}{\ln^2 x}$$

- Now it is easy to find the stationary points.
- The fraction equals zero iff one of the factors in the numerator equals to zero.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = \mathbb{R}^+ \setminus \{1\} = (0,1) \cup (1,\infty)$.

$$y' = \frac{2x\ln x - x^2\frac{1}{x}}{\ln^2 x} = \frac{2x\ln x - x}{\ln^2 x} = \frac{x(2\ln x - 1)}{\ln^2 x}$$

Stationary point: $x_1 = e^{1/2}$.

- The factor $(2\ln x - 1)$ equals zero for $\ln x = \dfrac{1}{2}$, i.e. for $x = e^{1/2}$

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = \mathbb{R}^+ \setminus \{1\} = (0, 1) \cup (1, \infty)$.

$$y' = \frac{2x \ln x - x^2 \frac{1}{x}}{\ln^2 x} = \frac{2x \ln x - x}{\ln^2 x} = \frac{x(2 \ln x - 1)}{\ln^2 x}$$

Stationary point: $x_1 = e^{1/2}$.

- The factor x never equals zero due to the restriction on the domain.
- There is no other stationary point

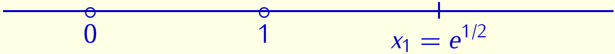
Find local extrema of the function $y = \frac{x^2}{\ln x}$.

$Dom(f) = (0,1) \cup (1,\infty)$; $y' = \frac{x(2\ln x - 1)}{\ln^2 x}$; $x_1 = e^{1/2}$.

- We will work with the derivative and the stationary point.
- We have to find the domain of the derivative. Since the restrictions are the same as for the original function, the domain of f' is the same as the domain of f.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

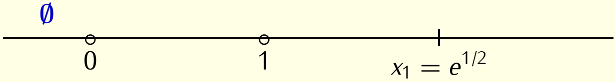
$Dom(f) = (0,1) \cup (1,\infty)$; $y' = \dfrac{x(2\ln x - 1)}{\ln^2 x}$; $x_1 = e^{1/2}$.



- We mark the domain of the derivative (including the point of discontinuity) and the stationary point to the real axis.
- Since $1 = e^0$ and $0 < 1/2$, then $1 < e^{1/2}$. (The exponential function is increasing)

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

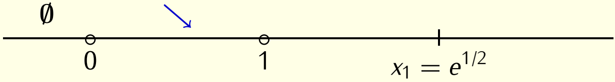
$Dom(f) = (0,1) \cup (1,\infty)$; $y' = \dfrac{x(2\ln x - 1)}{\ln^2 x}$; $x_1 = e^{1/2}$.



- The axis is divided into four subintervals. One of these subintervals does not belong to the domain.
- In each of the remaining subintervals the type of the monotonicity is preserved for all x.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

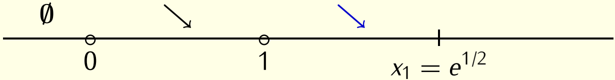
$Dom(f) = (0,1) \cup (1,\infty)$; $y' = \dfrac{x(2\ln x - 1)}{\ln^2 x}$; $x_1 = e^{1/2}$.



Let $\xi_1 = e^{-1}$ is a test number from the first subinterval. The derivative at ξ_1 is negative, since $y'(-1) = \dfrac{e^{-1}(-2-1)}{(-1)^2} < 0$, where we used $\ln(e^{-1}) = -1$. Hence the function is decreasing at ξ_1 and the same is true for the interval $(0,1)$.

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = (0,1) \cup (1,\infty)$; $\quad y' = \dfrac{x(2\ln x - 1)}{\ln^2 x}$; $\quad x_1 = e^{1/2}$.

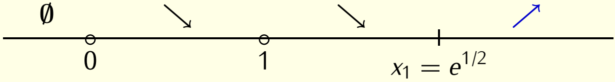


$\xi_2 = e^{1/4}$ satisfies $1 < e^{1/4} < e^{1/2}$ and $\ln(e^{1/2}) = \dfrac{1}{2}$. Hence

$$y'(e^{1/4}) = \frac{e^{1/4}(\frac{1}{2} - 1)}{\left(\frac{1}{2}\right)^2} < 0.$$

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = (0,1) \cup (1,\infty)$; $\quad y' = \dfrac{x(2\ln x - 1)}{\ln^2 x}$; $\quad x_1 = e^{1/2}$.

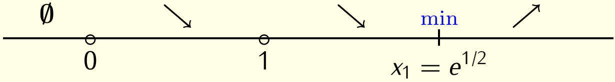


$\xi_3 = e$ satisfies $1 < e$ and $\ln(e) = 1$. Hence

$$y'(e) = \frac{e(2-1)}{1^2} > 0.$$

Find local extrema of the function $y = \dfrac{x^2}{\ln x}$.

$Dom(f) = (0,1) \cup (1,\infty)$; $\quad y' = \dfrac{x(2\ln x - 1)}{\ln^2 x}$; $\quad x_1 = e^{1/2}$.



Finished. The function possesses unique local minimum at $x = e^{\frac{1}{2}}$ and no local maximum.

THAT'S ALL