Linear algebra

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Contents

1 Algebraic linear space

Definition (algebraic linear space). The set $\mathbb{R}^n$ of ordered n-tuples of real numbers $(a_1, a_2, \dots, a_n)$ with the operations addition of vectors and multiplication of a vector by a real number defined for every $c \in \mathbb{R}$ and $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ by the relations

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \tag{1}$$

$$c \cdot (a_1, a_2, \dots, a_n) = (c \cdot a_1, c \cdot a_2, \dots, c \cdot a_n) \tag{2}$$

is called an *algebraic linear space*, or an *algebraic vector space*, shortly a *vector space*.

Definition (vectors). Consider the vector space $\mathbb{R}^n$.

Elements of this space are called (*algebraic*) *vectors*. The fact that a variable is a vector will be denoted by an arrow symbol over this variable: $\vec{a}$.

The numbers $a_1, \dots, a_n$ are called *components of the vector* $(a_1, a_2, \dots, a_n)$.

The number n is called a *dimension* of the space $\mathbb{R}^n$. A vector from $\mathbb{R}^n$ is called an *n-vector*.

Remark 1 (column vector). The components of the vector can be also rearranged into columns. In such a case we speak about a *column vector*, e.g.,

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

is a 3-dimensional column vector (or shortly, a column 3-vector).

Remark 2. The operations onvectors are defined as operations on the corresponding components. From this reason these operations preserve their basic properties known from the algebra of real numbers. Among others, the addition of vectors is associative, commutative and distributive with respect to the multiplication by a real number.

Remark 3 (zero vector). The vector $\vec{o} := (0, 0, \ldots, 0)$ is called a *zero vector*. From the definition of vector operations it follows that $t\vec{o} = \vec{o}$, $\vec{o} + \vec{u} = \vec{u}$ and $0\vec{u} = \vec{o}$ for an arbitrary vector $\vec{u}$ and an arbitrary real number t.

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Vector algebra.

$\vec{a} = (1, 2, 1),\qquad \vec{b} = (3, 0, -1),\qquad \vec{c} = (2, 1, 0),\qquad \vec{o} = (0, 0, 0)$

$$\vec{a} + 2 \cdot \vec{b} - \vec{c}$$

$$\vec{a} + \vec{o}$$

$$0 \cdot \vec{a} + 0 \cdot \vec{b} + 0 \cdot \vec{c}$$

$$\vec{a} + \vec{b} - 2 \cdot \vec{c}$$

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$$\begin{aligned}\vec{a} + 2\cdot\vec{b} - \vec{c} &= (1, 2, 1) + 2\cdot(3, 0, -1) - (2, 1, 0)\\ &= (1, 2, 1) + (6, 0, -2) - (2, 1, 0)\end{aligned}$$

$$\vec{a} + \vec{o}$$

$$0\cdot\vec{a} + 0\cdot\vec{b} + 0\cdot\vec{c}$$

$$\vec{a} + \vec{b} - 2\cdot\vec{c}$$

We substitute for vectors and multiply the vector $\vec{b}$ by two.

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$$\vec{a} + \vec{o}$$

$$0 \cdot \vec{a} + 0 \cdot \vec{b} + 0 \cdot \vec{c}$$

$$\vec{a} + \vec{b} - 2 \cdot \vec{c}$$

We add the corresponding components.

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$$\vec{a} + \vec{o}$$

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$$\vec{a} + \vec{b} - 2 \cdot \vec{c}$$

We simplify.

Vector algebra.

$\vec{a} = (1, 2, 1)$, $\vec{b} = (3, 0, -1)$, $\vec{c} = (2, 1, 0)$, $\vec{o} = (0, 0, 0)$

$$\vec{a} + \vec{o} = (1, 2, 1) + (0, 0, 0) = (1, 2, 1) = \vec{a}$$

$$0 \cdot \vec{a} + 0 \cdot \vec{b} + 0 \cdot \vec{c}$$

$$\vec{a} + \vec{b} - 2 \cdot \vec{c}$$

Teh zero vector is a neutral e,ement with respect to addition.

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The trivial linear combination gives zero vector.

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$$\begin{aligned} \vec{a} + \vec{b} - 2 \cdot \vec{c} &= (1, 2, 1) + (3, 0, -1) - (4, 2, 0) \\ &= (0, 0, 0) \end{aligned}$$

Sometimes it is possible to get the zero vector as nontrivial linear combination. In this case we say that vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linear dependent.

Definition (linear combination). Let $\vec{u}_1$, $\vec{u}_2$, ..., $\vec{u}_k$ be a finite sequence of vectors from $\mathbb{R}^n$. The vector $\vec{u}$ which satisfies

$$\vec{u} = t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_k\vec{u}_k, \qquad (3)$$

for some real numbers t_1, t_2, ..., t_k is said to be a *linear combination of vectors* $\vec{u}_1$, $\vec{u}_2$, ..., $\vec{u}_k$. The numbers t_1, t_2, ..., t_k are said to be *coefficients of this linear combination*.

Remark 4 (trivial linear combination). If all coefficients in a linear combination equal zero (*trivial linear combination*), the right-hand side of (3) gives the zero vector. Hence the zero vector can be always written as a linear combination of given vectors. Now we state an important question:

> *Is the trivial linear combination the unique linear combination with this property? This means: Given a set U of vectors, is there a possibility how to obtain the zero vector as a nontrivial linear combination of vectors from U?*

The answer is: For some vectors *yes* and for some *no*. It turns out to be important to distinguish these cases. This is a motivation for the following definition.

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Definition (linear (in-)dependence of vectors). Vectors $\vec{u}_1$, $\vec{u}_2$, ..., $\vec{u}_k$ are said to be *linearly dependent* iff there exists at least one nontrivial linear combination of all these vectors which yields the zero vector. More precisely, the vectors are linearly dependent if there exist real numbers t_1, t_2, ..., t_k such that at least one of these numbers is nonzero and

$$\vec{o} = t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_k\vec{u}_k \tag{4}$$

holds. The vectors are said to be *linearly independent* if they are not linearly dependent.

2 Matrix

Definition (matrix). A rectangular array

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

where $a_{ij} \in \mathbb{R}$ for $i = 1..m$ and $j = 1..n$ is called an $m \times n$ *matrix* or shortly a *matrix*.

The set of all $m \times n$ matrices will be denoted by $\mathbb{R}^{m\times n}$. Shortly we write $A = (a_{ij})_{i=1}^{m}{}_{j=1}^{n}$ or $\boldsymbol{A = (a_{ij})}$.

An $m \times n$ matrix is called a *square matrix* if $m = n$ and a *rectangular matrix* otherwise.

The elements a_{ii} are called elements of the *main diagonal*.

Definition (transposed matrix). Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ be an $m \times n$ matrix. The $n \times m$ matrix $\boldsymbol{A^T}$ which is obtained from the matrix A by interchanging the rows and the columns is called a *transpose matrix to the matrix* A, i.e., $A^T \in \mathbb{R}^{n\times m}$ and

$$A^T = (a_{ji}),$$

where a_{ij} are the elements of the matrix A.

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \qquad A^T = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$$

Definition (basic matrix operations). Let $A = (a_{ij})$, $B = (b_{ij})$ be $m \times n$ matrices. Under a *sum of the matrices A and B* we understand the $m \times n$ matrix $C = (c_{ij})$ with entries $c_{ij} = a_{ij} + b_{ij}$ for all i, j. We write $\boldsymbol{C = A + B}$.

Let $A = (a_{ij})$ be an $m \times n$ matrix and $t \in \mathbb{R}$ be a real number. Under a *product of the number t and the matrix A* we understand the $m \times n$ matrix $D = (d_{ij})$ with entries $d_{ij} = t.a_{ij}$ for all i, j. We write $\boldsymbol{D = tA}$.

$$\begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 3 \\ 3 & 2 & 1 \\ 4 & 4 & 2 \end{pmatrix}$$

$$3 \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -3 & 6 \\ 9 & 3 & -6 \\ 6 & 0 & 3 \end{pmatrix}$$

Definition (matrix multiplication). Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times p$ matrix. Under the *product of the matrices A and B (in this order!)* we understand the $m \times p$ matrix $G = (g_{ij})$ defined

$$g_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

for every $i = 1..m$, $j = 1..p$. In other words, a scalar product of the vector from the i-th row of the matrix A and the j-th column of the matrix B is at the position ij in the matrix G is. We write $\boldsymbol{G = AB}$ (in this order!).

Scalar product in $\mathbb{R}^3$

From the high school you know that the scalar product of two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is the number

$$\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3 = \sum_{i=1}^{3} u_i v_i.$$

From the definition of the matrix multiplication it follws that

$$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3 \end{pmatrix}$$

is simply another notation for the same operation.

Multiply matrices

$$\begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ -1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ -1 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix}$$

$$A \cdot B = C, \quad c_{ij} = \sum_k a_{ik} b_{kj}$$

Multiply matrices

$$\begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ -1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 2 + (-1)\cdot(-1) + 2\cdot 3 & 2\cdot 4 - 1\cdot 2 + 2\cdot 1 \\ 3\cdot 2 + 1\cdot(-1) - 2\cdot 3 & 3\cdot 4 + 1\cdot 2 - 2\cdot 1 \\ 2\cdot 2 + 0\cdot(-1) + 1\cdot 3 & 2\cdot 4 + 0\cdot 2 + 1\cdot 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ -1 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix}$$

The element c_{ij} of the matrix product C si scalar product of the i-th row of the matrix A and j-th column of the matix B. As a summary, the matrix product AB consists of six scalar product.

Multiply matrices

$$\begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ -1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 2 + (-1)\cdot(-1) + 2\cdot 3 & 2\cdot 4 - 1\cdot 2 + 2\cdot 1 \\ 3\cdot 2 + 1\cdot(-1) - 2\cdot 3 & 3\cdot 4 + 1\cdot 2 - 2\cdot 1 \\ 2\cdot 2 + 0\cdot(-1) + 1\cdot 3 & 2\cdot 4 + 0\cdot 2 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 11 & 8 \\ -1 & 12 \\ 7 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ -1 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix}$$

Multiply matrices

$$\begin{pmatrix}2 & -1 & 2\\ 3 & 1 & -2\\ 2 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}2 & 4\\ -1 & 2\\ 3 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2\cdot 2+(-1)\cdot(-1)+2\cdot 3 & 2\cdot 4-1\cdot 2+2\cdot 1\\ 3\cdot 2+1\cdot(-1)-2\cdot 3 & 3\cdot 4+1\cdot 2-2\cdot 1\\ 2\cdot 2+0\cdot(-1)+1\cdot 3 & 2\cdot 4+0\cdot 2+1\cdot 1\end{pmatrix}=\begin{pmatrix}11 & 8\\ -1 & 12\\ 7 & 9\end{pmatrix}$$

$$\begin{pmatrix}2 & 4\\ -1 & 2\\ 3 & 1\end{pmatrix}\cdot\begin{pmatrix}2 & -1 & 2\\ 3 & 1 & -2\\ 2 & 0 & 1\end{pmatrix}=\text{undefined}$$

Matrix product and linear combinations.

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & -1 \\ 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} -2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 6 \end{pmatrix}$$

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Theorem 1 (properties of the matrix multiplication). The matrix multiplication is associative and distributive from both left and right, i.e., the following relations hold whenever they have sense.

$$A(BC) = (AB)C \qquad \text{(the associative law)}$$
$$A(B + C) = AB + AC \qquad \text{(the left distributive law)}$$
$$(B + C)A = BA + CA \qquad \text{(the right distributive law)}$$

Definition (identity matrix). Under an $n \times n$ *identity matrix* we understand the $n \times n$ matrix with the numbers 1 in the main diagonal and the numbers 0 outside this diagonal. The $n \times n$ identity matrix is denoted by $\boldsymbol{I_n}$.

Example 1. The 3×3 identity matrix is the matrix

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 2 (properties of identity matrix). Let A be a matrix and I the identity matrix. Then $IA = A$ and $AI = A$ whenever this product is defined.

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3 Rank

Definition (rank of a matrix)**.** Let A be a matrix. Under the *rank of the matrix* A we understand the maximal number of the linearly independent rows of the matrix A. The rank of the matrix A will be denoted by $\mathbf{rank}\,(A)$.

Definition (pivot, row echelon form)**.** Let A be an $m \times n$ matrix. The first nonzero element of each row of the matrix A is said to be a *pivot* of this row. The matrix A is said to be in the *row echelon form* if

- all zero rows (if exists any) are at the bottom of the matrix,
- if two successive rows are non-zero, then the second row starts with more zeros than the first one, i.e. the pivot of each row appears after the pivot of the preceding row.

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Theorem 3 (rank of a matrix in the row echelon form)**.** The rank of a matrix in the row echelon form equals to the number of the nonzero rows of this matrix.

Example 2. The matrix $A = \begin{pmatrix} 2 & 2 & 2 & 3 & -1 & 5 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is in the row echelon form and $\operatorname{rank}(A) = 3$. The matrix $B = \begin{pmatrix} 2 & 2 & 2 & 3 & -1 & 5 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 3 & -1 & 2 & 1 \end{pmatrix}$ is not in the row echelon form.

Remark 5 (linear (in–)dependence of vectors)**.** Let A be an $m \times n$ matrix. **The rows of the matrix A are linearly independent n-vectors if and only if rank $(A) = m$.** Knowing the rank of the matrix, we can resolve the problem concerning a linear dependence or independence of the rows of the matrix.

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Theorem 3 (rank of a matrix in the row echelon form)**.** The rank of a matrix in the row echelon form equals to the number of the nonzero rows of this matrix.

Example 2. The matrix $A = \begin{pmatrix} 2 & 2 & 2 & 3 & -1 & 5 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is in the row echelon form and $\operatorname{rank}(A) = 3$. The matrix $B = \begin{pmatrix} 2 & 2 & 2 & 3 & -1 & 5 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 3 & -1 & 2 & 1 \end{pmatrix}$ is not in the row echelon form.

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Theorem 4 (row operations preserving the rank)**.** The following row operations preserve the rank of matrices.

1. Omitting a row which satisfies one of the following condition:
   - it contains only zeros, or
   - it equals to another row, or
   - it equals to a constant multiple of another row.
2. Multiplying any row by a nonzero real number.
3. Interchanging the order of the rows in an arbitrary way.
4. Keeping one row without any change and adding arbitrary multiples of this row to arbitrary nonzero multiples of another rows.

Theorem 5. Any matrix can be after application of a finite number of row operations from Theorem 4 transformed into its row echelon form.

Arrangement: The process of conversion a matrix by row operations preserving rank into its row-echelon form will be referred as *row-reduction*.

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Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix}$$

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

- We choose the red row to be the pivot row.
- This row remains and comes as first.

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \begin{matrix} 2 \\ (-3) \end{matrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ & & & & \\ & & & & \end{pmatrix}$$

$2R_1 - 3R_2 = \dots$

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \begin{matrix} (-3) \\ 2 \end{matrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ & & & & \end{pmatrix}$$

$2R_3 - 3R_2 = \dots$

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ \color{red}{2} & \color{red}{1} & \color{red}{-1} & \color{red}{2} & \color{red}{-3} \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \overset{(-1)}{\sim} \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ \color{blue}{0} & \color{blue}{-6} & \color{blue}{2} & \color{blue}{-4} & \color{blue}{5} \end{pmatrix}$$

$R_4 - R_2 = \dots$

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ 0 & -6 & 2 & -4 & 5 \end{pmatrix} \sim$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ & & & & \\ & & & & \end{pmatrix}$$

The first row remains.

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ 0 & -6 & 2 & -4 & 5 \end{pmatrix} \sim$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ & & & & \\ & & & & \end{pmatrix}$$

- The next pivot row will be one of the red rows.
- However, it would be difficult to pivot directly on one of these rows and produce zeros at a_{32} and a_{42}.

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ 0 & -6 & 2 & -4 & 5 \end{pmatrix} (-1) \sim$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ & & & & \end{pmatrix}$$

- We choose the red row as a pivot row. This row remains.
- As the first step, we will decrease the numbers in the remaining rows. Particularly, we perform $R_2 - R_3 = \dots$

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ 0 & -6 & 2 & -4 & 5 \end{pmatrix} (-1) \sim$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$R_2 - R_4 = \dots$

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ 0 & -6 & 2 & -4 & 5 \end{pmatrix} \sim$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The row R_3 is a multiple of row R_4 and one of these rows can be deleted.

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ 0 & -6 & 2 & -4 & 5 \end{pmatrix} \sim$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ & & & & \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ & & & & \end{pmatrix}$$

- The first row remains.
- The last row will be the next pivot row and comes as the second.

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ 0 & -6 & 2 & -4 & 5 \end{pmatrix} \sim$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ & & & & \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}_{5} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 8 & -4 & 5 \end{pmatrix}$$

We pivot on the red row. $5R_3 + R_2 = \dots$

Find rank of the matrix A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 & -2 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ 0 & -7 & 1 & -4 & 5 \\ 0 & -6 & 2 & -4 & 5 \end{pmatrix} \sim$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & -5 & 3 & -4 & 5 \\ & & & & \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 8 & -4 & 5 \end{pmatrix}$$

$\operatorname{rank}(A) = 3$

- The matrix is in the row echelon form.
- The row echelon form has three rows, hence $\operatorname{rank}(A) = 3$.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix}$$

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

- The row R_1 will be the pivot row.
- This row remains and comes as the first.
- We pivot on $a_{11} = 1$.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix}^{-3} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ & & & & \\ & & & & \end{pmatrix}$$

We clean the element a_{21}. We use the operation $-3R_1 + R_2$.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ 0 & 4 & -6 & 2 & 4 \\ & & & & \end{pmatrix}$$

We clean the element a_{31}. We use the operation $R_1 + R_3$.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ 0 & 4 & -6 & 2 & 4 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix}$$

The element a_{41} is zero and hence the last row remains.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ 0 & 4 & -6 & 2 & 4 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ & & & & \\ & & & & \end{pmatrix}$$

- The first row remains.
- The new pivot row will be the last row (red), since the number $a_{42} = 1$ is more convenient for pivoting than the numbers $a_{22} = -5$ and $a_{23} = 4$.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ 0 & 4 & -6 & 2 & 4 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} 5$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ & & & & \end{pmatrix}$$

We clean a_{22}. We use $5R_4 + R_2$.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ 0 & 4 & -6 & 2 & 4 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} {\scriptstyle -4}$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ 0 & 0 & -18 & 18 & 0 \end{pmatrix}$$

We clean a_{32}. We use $-4R_4 + R_2$.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ 0 & 4 & -6 & 2 & 4 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ 0 & 0 & -18 & 18 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

The last row can be divided by the number 18. All other rows remain.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ 0 & 4 & -6 & 2 & 4 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ 0 & 0 & -18 & 18 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ & & & & \end{pmatrix}$$

- The first two row remain.
- The number $a_{34} = -1$ is more convenient for pivoting than the number $a_{33} = 26$. From this reason we use R_4 as the new pivot row.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ 0 & 4 & -6 & 2 & 4 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ 0 & 0 & -18 & 18 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix} 26$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 9 & 9 \end{pmatrix}$$

It remains to clean a_{33}. We use $26R_4 + R_3$.

Find rank of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 3 & 1 & -4 & 6 & -2 \\ -1 & 2 & -1 & 1 & 6 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & -5 & 11 & 3 & 4 \\ 0 & 4 & -6 & 2 & 4 \\ 0 & 1 & 3 & -4 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ 0 & 0 & -18 & 18 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 26 & -17 & 9 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -5 & 1 & -2 \\ 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 9 & 9 \end{pmatrix}$$

$\operatorname{rank}(B) = 4$

The matrix is in a row echlon form. The row echelon form consists from four nonzero rows. Hence the rank is four.

4 Inverse Matrix

Definition (invertible matrix, inverse matrix). Let A be an $n \times n$ square matrix. If there exists an $n \times n$ matrix A^{-1} which satisfies the relations

$$A^{-1}A = I = AA^{-1}, \tag{5}$$

then the matrix A is said to be *invertible*. The matrix A^{-1} is said to be the *inverse matrix* to A.

Definition (reduced row echelon form). An $m \times n$ matrix A is said to be in the *reduced row echelon form* if

- A is in the row echelon form,
- pivots in all rows equal 1,
- each of the pivots is the only nonzero number in its column.

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Definition (reduced row echelon form)**.** An $m \times n$ matrix A is said to be in the *reduced row echelon form* if

- A is in the row echelon form,
- pivots in all rows equal 1,
- each of the pivots is the only nonzero number in its column.

Calculation of the inverse matrix. Given a square $n \times n$ matrix A, the inverse matrix A^{-1} can be calculated in the following steps.

1. We write the matrix A and the $n \times n$ identity matrix I_n together.
2. We convert the matrix A into its reduced row-echelon form by row operations from Theorem 4.
3. We distinguish two mutually different cases.
   - If the reduced row echelon form of the matrix A is not the $n \times n$ identity matrix, then A is not invertible.
   - If the reduced row echelon form is the $n \times n$ identity matrix, then the application of all of the steps which convert A into its reduced row echelon form onto the identity matrix yields the inverse A^{-1}.
4. Remark that we *cannot* use any of the column operations.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{rrr|rrr} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right)$$

We write the matrix A and the 3×3 identity matrix.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} -1 & 1 & 3 & 0 & 1 & 0 \\ & & & & & \\ & & & & & \end{array}\right)$$

- We choose the second row as a pivot row. The reason is that the number -1 is more convenient for pivoting than the numbers 6 or 2.
- The pivot row comes as the first.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) 6 \sim \left(\begin{array}{ccc|ccc} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 6 & 0 \\ & & & & & \end{array}\right)$$

We adjust the element $a_{11} = 6$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{rrr|rrr} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) 2 \sim \left(\begin{array}{rrr|rrr} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{array}\right)$$

We adjust the element $a_{31} = 2$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{rrr|rrr} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|rrr} & & & & & \\ 0 & 1 & 0 & 0 & 2 & 1 \\ & & & & & \end{array}\right)$$

- The new pivot can be either the second or the third row.
- We choose the last row. This row has to be written as the second.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{rrr|rrr} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{array}\right) (-1)$$

$$\sim \left(\begin{array}{rrr|rrr} -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ & & & & & \end{array}\right)$$

We adjust the element $a_{12} = 1$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{array}\right) (-2)$$

$$\sim \left(\begin{array}{ccc|ccc} -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right)$$

We adjust the element $a_{22} = 2$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right)$$

- The last pivot will be the last row.
- This row has to remain as the last row.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{rrr|rrr} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|rrr} -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ \color{red}{0} & \color{red}{0} & \color{red}{1} & \color{red}{1} & \color{red}{2} & \color{red}{-2} \end{array}\right) \sim \left(\begin{array}{rrr|rrr} & & & & & \\ \color{blue}{0} & \color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{2} & \color{blue}{1} \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right)$$

The second row is good. This row remains.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right) \begin{matrix} (-1) \\ \\ 3 \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 7 & -5 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right)$$

We adjust the element $a_{13} = 3$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 6 & -4 & -17 \\ -1 & 1 & 3 \\ 2 & -1 & -6 \end{pmatrix}$$

$$\left(\begin{array}{rrr|rrr} 6 & -4 & -17 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|rrr} -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 7 & -5 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & 7 & -5 \\ 0 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}$$

The matrix on the left is the identity matrix and hence the second matrix is the inverse.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix}$$

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

We start with the matrix and the 3×3 identity matrix.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ & & & & & \\ & & & & & \end{array}\right)$$

- We choose the second row to be a pivot row (contains the smallest numbers).
- This pivot row will be the first row.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right) (-1)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ & & & & & \end{array}\right)$$

We adjust the element $a_{11} = 1$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right) (-1)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 3 & 5 & 0 & -1 & 1 \end{array}\right)$$

We adjust the element $a_{31} = 1$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 3 & 5 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} & & & & & \\ 0 & 1 & 3 & 1 & -1 & 0 \\ & & & & & \end{array}\right)$$

- We choose the second row as the next pivot row.
- This row remains as the second.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 3 & 5 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ & & & & & \end{array}\right)$$

We adjust the element $a_{12} = -1$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 3 & 5 & 0 & -1 & 1 \end{array}\right) (-3) \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 0 & -4 & -3 & 2 & 1 \end{array}\right)$$

We adjust the element $a_{32} = 3$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 3 & 5 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 0 & -4 & -3 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ 0 & 0 & 4 & 3 & -1 & -1 \end{array}\right)$$

The last row remains. It will be the new pivot row.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 3 & 5 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 0 & -4 & -3 & 2 & 1 \end{array}\right) \begin{array}{c} \\ 4 \\ 3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} & & & & & \\ 0 & 4 & 0 & -5 & 2 & 3 \\ 0 & 0 & 4 & 3 & -1 & -1 \end{array}\right)$$

We adjust the element $a_{23} = 3$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 3 & 5 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 0 & -4 & -3 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & 4 & 0 & -5 & 2 & 3 \\ 0 & 0 & 4 & 3 & -1 & -1 \end{array}\right)$$

We adjust the element $a_{13} = 4$ to zero.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 3 & 5 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 0 & -4 & -3 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & 4 & 0 & -5 & 2 & 3 \\ 0 & 0 & 4 & 3 & -1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & 0 & -5/4 & 2/4 & 3/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & -1/4 \end{array}\right)$$

We divide each row by the leftmost nonzero number.

Given a matrix A, find the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 6 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 3 & 5 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 0 & -4 & -3 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & 4 & 0 & -5 & 2 & 3 \\ 0 & 0 & 4 & 3 & -1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & 0 & -5/4 & 2/4 & 3/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & -1/4 \end{array}\right) \ ; A^{-1} = \frac{1}{4}\begin{pmatrix} -8 & 8 & 4 \\ -5 & 2 & 3 \\ 3 & -2 & -1 \end{pmatrix}$$

The identity matrix is on the left. The inverse matrix is on the right. The common denominator $\frac{1}{4}$ can be taken out.

5 Systems of linear equations

Consider the following problems: Find real numbers x_1, x_2, which satisfy:

Problem 1 :

$$\begin{aligned} 4x_1 + 5x_2 &= 7 \\ x_1 - 2x_2 &= 4 \end{aligned}$$

Problem 2 :

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 5 \\ -2 \end{pmatrix} x_2 = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

Problem 3

$$\begin{pmatrix} 4 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

All three problems are equivalent. The difference is in notation only.

Definition (system of linear equations). Under a *system of m linear equations in n unknowns* we understand the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\ &\vdots\\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{6}$$

Variables x_1, x_2, ..., x_n are said to be *unknowns*. Real numbers a_{ij} are said to be *coefficients of the left-hand sides of equations*, real numbers b_j *coefficients of the right-hand sides* or *constant terms of equations*.

Under a *solution of the system* (6) we understand the n-tuple of real numbers which, substituted for the unknowns, convert the equations into identities.

Definition (matrix of the system). Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \tag{7}$$

is said to be a *matrix of the system* (6) (or a *coefficients matrix*). Matrix

$$A^* = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right) \tag{8}$$

is said to be an *augmented matrix of the system* (6).

Remark 6 (vector notation of the system of linear equations)**.** The system (6) can written in an equivalent form of a vector equation. Really, denote

$$\vec{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ \vec{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ \vec{a}_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{pmatrix}, \ \dots, \vec{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}, \ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}. \tag{9}$$

Clearly

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \cdots + x_n\vec{a}_n = \vec{b} \tag{10}$$

is equivalent to (6). Now we see that the problem to find a solution of linear system (6) possesses the following equivalent formulation (we use the notation introduced in (9)).

Write the vector $\vec{b}$ as a linear combination of vectors $\vec{a}_1$, $\vec{a}_2$, . . . , $\vec{a}_n$.

Definition (homogeneous system)**.** If

$$b_1 = b_2 = \cdots = b_m = 0$$

holds, then system (6) is said to be *homogeneous*.

Remark 7 (trivial solution)**.** Every homogeneous system possesses a solution. Really, it is clear that the n-tuple $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$ is a solution of an arbitrary homogeneous system. This solution is called a *trivial solution*.

Remark 8 (matrix notation)**.** The linear combination in (10) can be written as a matrix product. This leads to the matrix equation

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}. \tag{11}$$

Denote by A the matrix (7) of the system, by $\vec{b}$ the column vector of the right-hand sides and by $\vec{x}$ the vector of unknowns , i.e.

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{and} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Using this notation, the linear system can be written as the matrix equation

$$A\vec{x} = \vec{b}. \tag{12}$$

This form is used frequently in the engineering computations for simplicity and brevity.

Theorem 6 (Frobenius). System (6) has a solution if and only if the augmented matrix of this system has the same rank as the matrix of this system, i.e. $\operatorname{rank}(A) = \operatorname{rank}(A^*)$.

Solution of linear system – Gauss elimination

1. We write the augmented matrix of the system. The i-th column contains the coefficients at x_i and the last column contains the right-hand sides. The order of the rows is arbitrary.

2. We convert the augmented matrix into its row echelon form. We use row operations[1] from Theorem 4.

3. We rewrite back the augmented matrix in the row echelon form into a system of linear equations (in the original unknowns). The set of all solutions of this new system is the same as the set of all solutions of the original system.

4. We start with the last equation. Three mutually different cases are possible

[1] no operations on columns!

(a) The last equation does not contain any unknown, i.e. it has the form $0 = a$, where a is a nonzero number. In this case the system possesses no solution.

(b) The last equation contains exactly one unknown. In this case we solve the equation for this unknown and continue with the next step.

(c) There are k unknowns in the last equation $(k > 1)$. In this case we solve one arbitrary of these unknowns through the other $(k-1)$ ones. These $(k-1)$ unknowns are called *free unknowns*. The free unknowns can be considered as parameters and can take any real values.

5. We continue with the last but one equation. The unknowns which appeared in the preceding steps are considered to be known already. Two cases can occur.

   (a) The equation contains one "new" unknown (i.e. all of the unknowns, with exception of one unknown, are free or known already). We solve the equation for this unknown. The formula for this unknown may contain also the free unknowns.

   (b) The equation contains at least two "new" unknowns. If there is l, $l > 1$, new unknowns, then we isolate one of them and the other

$(l-1)$ will be free.

6. We repeat the last step until we reach the first equation. At this stage the system is solved. We either calculate all of the unknowns (the system has unique solution) or we calculate the non-free unknowns in terms of the free ones. These free variables can be considered as parameters and can take any real value. Hence, if at least one of the free variables is present, then the system has infinitely many solutions.

Remark that the choice of the free unknowns is not unique and two equivalent sets of solutions can be written in several, very different, forms.

Remark 9. The following three mutually different cases may occur:

1. The system has no solution if and only if $\operatorname{rank}(A) \neq \operatorname{rank}(A^*)$. This occurs if the last line of the augmented matrix in the row echelon form corresponds to the equation $0 = a$ where a is a real nonzero number. This equation clearly has no solution and hence the whole system has no solution.

2. The system has exactly one solutions if and only if $\operatorname{rank}(A) = \operatorname{rank}(A^*) = n$.

3. The system has infinitely many solutions if and only if $\operatorname{rank}(A) = \operatorname{rank}(A^*) < n$. In this case the unknowns can be computed in terms of $(n - \operatorname{rank}(A))$ independent parameters, or, in other words, in $(n - \operatorname{rank}(A))$ free variables.

Solve the system

$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad\ \ +\ x_3 \qquad\quad &= 3 \end{aligned}$$

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad +\ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 6 & 2 & -1 & 7 & 0\\ 4 & 2 & -3 & 5 & -4\\ 1 & 1 & -1 & -1 & 0\\ 1 & 0 & 1 & 0 & 3 \end{array}\right)$$

We write the augmented matrix A^* of the system.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1\qquad+\ x_3\qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 6 & 2 & -1 & 7 & 0\\ 4 & 2 & -3 & 5 & -4\\ 1 & 1 & -1 & -1 & 0\\ 1 & 0 & 1 & 0 & 3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ & & & & \\ & & & & \\ & & & & \end{array}\right)$$

We choose the fourth row to be the pivot row. This row remains and comes as first.

Solve the system
$$\begin{aligned}6x_1+2x_2-\ x_3+7x_4&=0\\4x_1+2x_2-3x_3+5x_4&=-4\\x_1+\ x_2-\ x_3-\ x_4&=0\\x_1\qquad+\ x_3\qquad&=3\end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 6 & 2 & -1 & 7 & 0\\ 4 & 2 & -3 & 5 & -4\\ 1 & 1 & -1 & -1 & 0\\ 1 & 0 & 1 & 0 & 3\end{array}\right)(-1) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ & & & & \\ & & & & \end{array}\right)$$

$R_3 - R_4 = \dots$

Solve the system
$$\begin{aligned}6x_1+2x_2-\ x_3+7x_4&=0\\4x_1+2x_2-3x_3+5x_4&=-4\\x_1+\ x_2-\ x_3-\ x_4&=0\\x_1\qquad+\ x_3\qquad&=3\end{aligned}$$

$$A^*\sim\left(\begin{array}{cccc|c}6&2&-1&7&0\\4&2&-3&5&-4\\1&1&-1&-1&0\\1&0&1&0&3\end{array}\right)(-4)\sim\left(\begin{array}{cccc|c}1&0&1&0&3\\0&1&-2&-1&-3\\0&2&-7&5&-16\\&&&&\end{array}\right)$$

$R_2-4R_4=\dots$

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1\qquad +\ x_3\qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 6 & 2 & -1 & 7 & 0\\ 4 & 2 & -3 & 5 & -4\\ 1 & 1 & -1 & -1 & 0\\ 1 & 0 & 1 & 0 & 3 \end{array}\right) (-6) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 2 & -7 & 5 & -16\\ 0 & 2 & -7 & 7 & -18 \end{array}\right)$$

$R_1 - 6R_4 = \dots$

Solve the system

$$\begin{aligned} 6x_1+2x_2-x_3+7x_4&=0\\ 4x_1+2x_2-3x_3+5x_4&=-4\\ x_1+x_2-x_3-x_4&=0\\ x_1+x_3&=3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 6 & 2 & -1 & 7 & 0\\ 4 & 2 & -3 & 5 & -4\\ 1 & 1 & -1 & -1 & 0\\ 1 & 0 & 1 & 0 & 3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 2 & -7 & 5 & -16\\ 0 & 2 & -7 & 7 & -18 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ & & & & \\ & & & & \end{array}\right)$$

The first row remains and the second row will be the new pivot row.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad +\ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 6 & 2 & -1 & 7 & 0\\ 4 & 2 & -3 & 5 & -4\\ 1 & 1 & -1 & -1 & 0\\ 1 & 0 & 1 & 0 & 3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 2 & -7 & 5 & -16\\ 0 & 2 & -7 & 7 & -18 \end{array}\right) \begin{matrix} \\ (-2) \\ \\ \\ \end{matrix} \sim$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ & & & & \end{array}\right)$$

$-2R_2 + R_3 = \dots$

Solve the system
$$\begin{aligned}6x_1+2x_2-\ x_3+7x_4&=0\\4x_1+2x_2-3x_3+5x_4&=-4\\x_1+\ x_2-\ x_3-\ x_4&=0\\x_1\qquad+\ x_3\qquad&=3\end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrr|r} 6 & 2 & -1 & 7 & 0\\ 4 & 2 & -3 & 5 & -4\\ 1 & 1 & -1 & -1 & 0\\ 1 & 0 & 1 & 0 & 3\end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 2 & -7 & 5 & -16\\ 0 & 2 & -7 & 7 & -18\end{array}\right) (-2) \sim$$

$$\left(\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & -3 & 9 & -12\end{array}\right)$$

$-2R_2 + R_4 = \dots$

Solve the system
$$\begin{aligned} 6x_1+2x_2-x_3+7x_4&=0\\ 4x_1+2x_2-3x_3+5x_4&=-4\\ x_1+x_2-x_3-x_4&=0\\ x_1+x_3&=3 \end{aligned}$$

$$A^*\sim\left(\begin{array}{cccc|c} 6&2&-1&7&0\\ 4&2&-3&5&-4\\ 1&1&-1&-1&0\\ 1&0&1&0&3 \end{array}\right)\sim\left(\begin{array}{cccc|c} 1&0&1&0&3\\ 0&1&-2&-1&-3\\ 0&2&-7&5&-16\\ 0&2&-7&7&-18 \end{array}\right)\sim$$

$$\left(\begin{array}{cccc|c} 1&0&1&0&3\\ 0&1&-2&-1&-3\\ 0&0&-3&7&-10\\ 0&0&-3&9&-12 \end{array}\right)\sim\left(\begin{array}{cccc|c} 1&0&1&0&3\\ 0&1&-2&-1&-3\\ 0&0&-3&7&-10 \end{array}\right)$$

The first two rows remain. The third row will be the new pivot row and remains as well.

Solve the system
$$\begin{aligned}6x_1+2x_2-\ x_3+7x_4&=0\\4x_1+2x_2-3x_3+5x_4&=-4\\x_1+\ x_2-\ x_3-\ x_4&=0\\x_1\qquad +\ x_3\qquad &=3\end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 6 & 2 & -1 & 7 & 0\\ 4 & 2 & -3 & 5 & -4\\ 1 & 1 & -1 & -1 & 0\\ 1 & 0 & 1 & 0 & 3\end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 2 & -7 & 5 & -16\\ 0 & 2 & -7 & 7 & -18\end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & -3 & 9 & -12\end{array}\right) (-1) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2\end{array}\right)$$

$-R_3 + R_4 = \dots$

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1\qquad +\ x_3\qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

The augmented matrix is row-equivalent to this (blue) matrix in the row-echelon form.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad + \ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$2x_4 = -2$

The system has a solution, since $\operatorname{rank}(A) = \operatorname{rank}(A^*) = 4$. Moreover $n = 4$ (the number of unknowns) and hence the system possesses a unique solution. We start from the last row in the row-echelon form. We write the corresponding equation . . .

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad +\ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$2x_4 = -2 \qquad \Rightarrow \qquad x_4 = -1$

and solve for x_4.

Solve the system
$$\begin{aligned}6x_1+2x_2-x_3+7x_4&=0\\4x_1+2x_2-3x_3+5x_4&=-4\\x_1+x_2-x_3-x_4&=0\\x_1+x_3&=3\end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c}1&0&1&0&3\\0&1&-2&-1&-3\\0&0&-3&7&-10\\0&0&0&2&-2\end{array}\right)$$

$2x_4=-2 \quad \Rightarrow \quad x_4=-1$

$-3x_3+7x_4=-10$

We write the equation corresponding to the third row in the row-echelon form.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad + \ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$$2x_4 = -2 \quad \Rightarrow \quad x_4 = -1$$

$$-3x_3 + 7x_4 = -10 \quad \Rightarrow \quad -3x_3 - 7 = -10$$

We substitute $x_4 = -1$ ...

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad +\ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$$2x_4 = -2 \quad \Rightarrow \quad x_4 = -1$$

$$-3x_3 + 7x_4 = -10 \quad \Rightarrow \quad -3x_3 - 7 = -10 \quad \Rightarrow \quad x_3 = 1$$

and solve for x_3.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1\qquad +\ x_3\qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$$2x_4 = -2 \quad \Rightarrow \quad x_4 = -1$$

$$-3x_3 + 7x_4 = -10 \quad \Rightarrow \quad -3x_3 - 7 = -10 \quad \Rightarrow \quad x_3 = 1$$

$$x_2 - 2x_3 - x_4 = -3$$

We convert the second row of the row-echelon matrix to the terms of unknowns.

Solve the system

$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1\qquad +\ x_3\qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$2x_4 = -2 \quad \Rightarrow \quad x_4 = -1$

$-3x_3 + 7x_4 = -10 \quad \Rightarrow \quad -3x_3 - 7 = -10 \quad \Rightarrow \quad x_3 = 1$

$x_2 - 2x_3 - x_4 = -3 \quad \Rightarrow \quad x_2 - 2 + 1 = -3$

We substitute $x_4 = -1$ and $x_3 = 1$ ...

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad + \ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$2x_4 = -2 \qquad \Rightarrow \qquad x_4 = -1$

$-3x_3 + 7x_4 = -10 \qquad \Rightarrow \qquad -3x_3 - 7 = -10 \qquad \Rightarrow \qquad x_3 = 1$

$x_2 - 2x_3 - x_4 = -3 \qquad \Rightarrow \qquad x_2 - 2 + 1 = -3 \qquad \Rightarrow \qquad x_2 = -2$

and solve for x_2.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ \ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ \ x_2-\ \ x_3-\ \ x_4 &= 0\\ x_1 \qquad\ \ +\ \ x_3 \qquad\ \ &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$$2x_4 = -2 \quad \Rightarrow \quad x_4 = -1$$

$$-3x_3 + 7x_4 = -10 \quad \Rightarrow \quad -3x_3 - 7 = -10 \quad \Rightarrow \quad x_3 = 1$$

$$x_2 - 2x_3 - x_4 = -3 \quad \Rightarrow \quad x_2 - 2 + 1 = -3 \quad \Rightarrow \quad x_2 = -2$$

$$x_1 + x_3 = 3$$

The equation corresponding to the first row.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad + \ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$2x_4 = -2 \quad \Rightarrow \quad x_4 = -1$

$-3x_3 + 7x_4 = -10 \quad \Rightarrow \quad -3x_3 - 7 = -10 \quad \Rightarrow \quad x_3 = 1$

$x_2 - 2x_3 - x_4 = -3 \quad \Rightarrow \quad x_2 - 2 + 1 = -3 \quad \Rightarrow \quad x_2 = -2$

$x_1 + x_3 = 3 \quad \Rightarrow \quad x_1 + 1 = 3$

Substitution $x_3 = 1$.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad + \ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$2x_4 = -2 \quad \Rightarrow \quad x_4 = -1$

$-3x_3 + 7x_4 = -10 \quad \Rightarrow \quad -3x_3 - 7 = -10 \quad \Rightarrow \quad x_3 = 1$

$x_2 - 2x_3 - x_4 = -3 \quad \Rightarrow \quad x_2 - 2 + 1 = -3 \quad \Rightarrow \quad x_2 = -2$

$x_1 + x_3 = 3 \quad \Rightarrow \quad x_1 + 1 = 3 \quad \Rightarrow \quad x_1 = 2$

We find $x_1 = 2$.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1 \qquad +\ x_3 \qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$2x_4 = -2 \quad \Rightarrow \quad x_4 = -1$

$-3x_3 + 7x_4 = -10 \quad \Rightarrow \quad -3x_3 - 7 = -10 \quad \Rightarrow \quad x_3 = 1$

$x_2 - 2x_3 - x_4 = -3 \quad \Rightarrow \quad x_2 - 2 + 1 = -3 \quad \Rightarrow \quad x_2 = -2$

$x_1 + x_3 = 3 \quad \Rightarrow \quad x_1 + 1 = 3 \quad \Rightarrow \quad x_1 = 2$

The unique solution is $[x_1 = 2, x_2 = -2, x_3 = 1, x_4 = -1]$.

Now we have all unknowns.

Solve the system
$$\begin{aligned} 6x_1+2x_2-\ x_3+7x_4 &= 0\\ 4x_1+2x_2-3x_3+5x_4 &= -4\\ x_1+\ x_2-\ x_3-\ x_4 &= 0\\ x_1\qquad +\ x_3\qquad &= 3 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3\\ 0 & 1 & -2 & -1 & -3\\ 0 & 0 & -3 & 7 & -10\\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$$2x_4 = -2 \quad\Rightarrow\quad x_4 = -1$$

$$-3x_3 + 7x_4 = -10 \quad\Rightarrow\quad -3x_3 - 7 = -10 \quad\Rightarrow\quad x_3 = 1$$

$$x_2 - 2x_3 - x_4 = -3 \quad\Rightarrow\quad x_2 - 2 + 1 = -3 \quad\Rightarrow\quad x_2 = -2$$

$$x_1 + x_3 = 3 \quad\Rightarrow\quad x_1 + 1 = 3 \quad\Rightarrow\quad x_1 = 2$$

The unique solution is $[x_1 = 2, x_2 = -2, x_3 = 1, x_4 = -1]$.

The problem is solved.

Solve the system

$$\begin{aligned}
3x_1 - 2x_2 + 6x_3 + 2x_4 - 4x_5 &= 5\\
x_1 \qquad\quad + 2x_3 - x_4 + 2x_5 &= 3\\
x_1 + 2x_2 + 2x_3 \qquad\qquad\quad &= 1\\
2x_1 - 6x_2 + 4x_3 + 2x_4 - 4x_5 &= 5
\end{aligned}$$

Solve the system
$$\begin{aligned} 3x_1 - 2x_2 + 6x_3 + 2x_4 - 4x_5 &= 5 \\ x_1 \qquad + 2x_3 - x_4 + 2x_5 &= 3 \\ x_1 + 2x_2 + 2x_3 \qquad &= 1 \\ 2x_1 - 6x_2 + 4x_3 + 2x_4 - 4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 3 & -2 & 6 & 2 & -4 & 5 \\ 1 & 0 & 2 & -1 & 2 & 3 \\ 1 & 2 & 2 & 0 & 0 & 1 \\ 2 & -6 & 4 & 2 & -4 & 5 \end{array}\right)$$

We write the augmented matrix.

Solve the system

$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5 &= 5\\ x_1 \qquad +2x_3- x_4+2x_5 &= 3\\ x_1+2x_2+2x_3 \qquad\qquad &= 1\\ 2x_1-6x_2+4x_3+2x_4-4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 3 & -2 & 6 & 2 & -4 & 5\\ 1 & 0 & 2 & -1 & 2 & 3\\ 1 & 2 & 2 & 0 & 0 & 1\\ 2 & -6 & 4 & 2 & -4 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right)$$

We choose the second row as a pivot row. This row will be the first new row.

Solve the system

$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5 &= 5\\ x_1+2x_3-x_4+2x_5 &= 3\\ x_1+2x_2+2x_3 &= 1\\ 2x_1-6x_2+4x_3+2x_4-4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrrr|r} 3 & -2 & 6 & 2 & -4 & 5\\ 1 & 0 & 2 & -1 & 2 & 3\\ 1 & 2 & 2 & 0 & 0 & 1\\ 2 & -6 & 4 & 2 & -4 & 5 \end{array}\right) \overset{(-3)}{\sim} \left(\begin{array}{rrrrr|r} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & -2 & 0 & 5 & -10 & -4\\ & & & & & \\ & & & & & \end{array}\right)$$

We adjust the first row of the preceding matrix.

Solve the system

$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5&=5\\ x_1\quad\;\;+2x_3-\;\,x_4+2x_5&=3\\ x_1+2x_2+2x_3\qquad\qquad\quad&=1\\ 2x_1-6x_2+4x_3+2x_4-4x_5&=5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrrr|r} 3 & -2 & 6 & 2 & -4 & 5\\ 1 & 0 & 2 & -1 & 2 & 3\\ 1 & 2 & 2 & 0 & 0 & 1\\ 2 & -6 & 4 & 2 & -4 & 5 \end{array}\right) \overset{(-1)}{\sim} \left(\begin{array}{rrrrr|r} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & -2 & 0 & 5 & -10 & -4\\ 0 & 2 & 0 & 1 & -2 & -2\\ & & & & & \end{array}\right)$$

We adjust the third row of the preceding matrix.

Solve the system

$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5 &= 5\\ x_1 \qquad +2x_3- \ x_4+2x_5 &= 3\\ x_1+2x_2+2x_3 \qquad\qquad &= 1\\ 2x_1-6x_2+4x_3+2x_4-4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrrr|r} 3 & -2 & 6 & 2 & -4 & 5\\ 1 & 0 & 2 & -1 & 2 & 3\\ 1 & 2 & 2 & 0 & 0 & 1\\ 2 & -6 & 4 & 2 & -4 & 5 \end{array}\right) \overset{(-2)}{\sim} \left(\begin{array}{rrrrr|r} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & -2 & 0 & 5 & -10 & -4\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & -6 & 0 & 4 & -8 & -1 \end{array}\right)$$

We adjust the last row of the preceding matrix.

Solve the system
$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5 &= 5\\ x_1 \qquad +2x_3-\ x_4+2x_5 &= 3\\ x_1+2x_2+2x_3 \qquad\qquad &= 1\\ 2x_1-6x_2+4x_3+2x_4-4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 3 & -2 & 6 & 2 & -4 & 5\\ 1 & 0 & 2 & -1 & 2 & 3\\ 1 & 2 & 2 & 0 & 0 & 1\\ 2 & -6 & 4 & 2 & -4 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & -2 & 0 & 5 & -10 & -4\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & -6 & 0 & 4 & -8 & -1 \end{array}\right) \sim$$
$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ & & & & & \\ & & & & & \end{array}\right)$$

The red row will be the next pivot row. The first row remains and the pivot row will be the second.

Solve the system

$$\begin{aligned} 3x_1 - 2x_2 + 6x_3 + 2x_4 - 4x_5 &= 5 \\ x_1 \phantom{{}-2x_2} + 2x_3 - x_4 + 2x_5 &= 3 \\ x_1 + 2x_2 + 2x_3 \phantom{{}+2x_4-4x_5} &= 1 \\ 2x_1 - 6x_2 + 4x_3 + 2x_4 - 4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrrr|r} 3 & -2 & 6 & 2 & -4 & 5 \\ 1 & 0 & 2 & -1 & 2 & 3 \\ 1 & 2 & 2 & 0 & 0 & 1 \\ 2 & -6 & 4 & 2 & -4 & 5 \end{array}\right) \sim \left(\begin{array}{rrrrr|r} 1 & 0 & 2 & -1 & 2 & 3 \\ 0 & -2 & 0 & 5 & -10 & -4 \\ 0 & 2 & 0 & 1 & -2 & -2 \\ 0 & -6 & 0 & 4 & -8 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{rrrrr|r} 1 & 0 & 2 & -1 & 2 & 3 \\ 0 & 2 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 6 & -12 & -6 \\ & & & & & \end{array}\right)$$

We adjust the second row of the preceding matrix.

Solve the system
$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5 &= 5\\ x_1 \quad\quad +2x_3-\ x_4+2x_5 &= 3\\ x_1+2x_2+2x_3 \quad\quad\quad\quad &= 1\\ 2x_1-6x_2+4x_3+2x_4-4x_5 &= 5\end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 3 & -2 & 6 & 2 & -4 & 5\\ 1 & 0 & 2 & -1 & 2 & 3\\ 1 & 2 & 2 & 0 & 0 & 1\\ 2 & -6 & 4 & 2 & -4 & 5\end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & -2 & 0 & 5 & -10 & -4\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & -6 & 0 & 4 & -8 & -1\end{array}\right) \begin{matrix} \\ \\ 3 \\ \end{matrix} \sim$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 6 & -12 & -6\\ 0 & 0 & 0 & 7 & -14 & -7\end{array}\right)$$

We adjust the last row of the preceding matrix.

Solve the system

$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5 &= 5\\ x_1 \qquad +2x_3- \ x_4+2x_5 &= 3\\ x_1+2x_2+2x_3 \qquad\qquad &= 1\\ 2x_1-6x_2+4x_3+2x_4-4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 3 & -2 & 6 & 2 & -4 & 5\\ 1 & 0 & 2 & -1 & 2 & 3\\ 1 & 2 & 2 & 0 & 0 & 1\\ 2 & -6 & 4 & 2 & -4 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & -2 & 0 & 5 & -10 & -4\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & -6 & 0 & 4 & -8 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 6 & -12 & -6\\ 0 & 0 & 0 & 7 & -14 & -7 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 1 & -2 & -1\\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

The green rows can be divided by the numbers 6 and 7, respectively.

Solve the system
$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5&=5\\ x_1\qquad +2x_3-\ x_4+2x_5&=3\\ x_1+2x_2+2x_3\qquad\qquad &=1\\ 2x_1-6x_2+4x_3+2x_4-4x_5&=5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 3 & -2 & 6 & 2 & -4 & 5\\ 1 & 0 & 2 & -1 & 2 & 3\\ 1 & 2 & 2 & 0 & 0 & 1\\ 2 & -6 & 4 & 2 & -4 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & -2 & 0 & 5 & -10 & -4\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & -6 & 0 & 4 & -8 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 6 & -12 & -6\\ 0 & 0 & 0 & 7 & -14 & -7 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

The last two rows are identical. We keep one of these row only.

Solve the system

$$\begin{aligned} 3x_1 - 2x_2 + 6x_3 + 2x_4 - 4x_5 &= 5 \\ x_1 \qquad + 2x_3 - \ x_4 + 2x_5 &= 3 \\ x_1 + 2x_2 + 2x_3 \qquad\qquad &= 1 \\ 2x_1 - 6x_2 + 4x_3 + 2x_4 - 4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3 \\ 0 & 2 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

The augmented matrix is in the row echelon form. The rank of the augmented matrix is 3, the rank of the coefficient matrix is also 3. Hence the system has a solution. The number of free variables is unknowns $-$ rank $= 5 - 3 = 2$.

Solve the system

$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5 &= 5\\ x_1 \qquad +2x_3- \ x_4+2x_5 &= 3\\ x_1+2x_2+2x_3 \qquad\qquad &= 1\\ 2x_1-6x_2+4x_3+2x_4-4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

$x_4 - 2x_5 = -1$

We write the equation corresponding to the last line of the matrix.

Solve the system

$$\begin{aligned} 3x_1 - 2x_2 + 6x_3 + 2x_4 - 4x_5 &= 5 \\ x_1 \quad\quad + 2x_3 - x_4 + 2x_5 &= 3 \\ x_1 + 2x_2 + 2x_3 \quad\quad &= 1 \\ 2x_1 - 6x_2 + 4x_3 + 2x_4 - 4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3 \\ 0 & 2 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

$x_4 - 2x_5 = -1 \quad \Rightarrow \quad x_4 = 2x_5 - 1, \quad x_5$ is free

There is one equation with two unknowns. We choose x_5 to be a free variable and solve for x_4.

Solve the system

$$\begin{aligned} 3x_1 - 2x_2 + 6x_3 + 2x_4 - 4x_5 &= 5 \\ x_1 \qquad + 2x_3 - x_4 + 2x_5 &= 3 \\ x_1 + 2x_2 + 2x_3 \qquad\qquad &= 1 \\ 2x_1 - 6x_2 + 4x_3 + 2x_4 - 4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrrr|r} 1 & 0 & 2 & -1 & 2 & 3 \\ 0 & 2 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

$x_4 - 2x_5 = -1 \quad \Rightarrow \quad x_4 = 2x_5 - 1, \quad x_5$ is free

$2x_2 + x_4 - 2x_5 = -2$

We write the equation corresponding to the second line of the matrix.

Solve the system
$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5&=5\\ x_1\qquad\;\; +2x_3-\;x_4+2x_5&=3\\ x_1+2x_2+2x_3\qquad\qquad\quad&=1\\ 2x_1-6x_2+4x_3+2x_4-4x_5&=5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

$x_4 - 2x_5 = -1 \quad \Rightarrow \quad x_4 = 2x_5 - 1, \quad x_5$ is free

$2x_2 + x_4 - 2x_5 = -2 \quad \Rightarrow \quad 2x_2 + (2x_5 - 1) - 2x_5 = -2$

We substitute for x_4. x_5 is free and only the variable x_2 remains.

Solve the system

$$\begin{aligned} 3x_1 - 2x_2 + 6x_3 + 2x_4 - 4x_5 &= 5 \\ x_1 \qquad + 2x_3 - x_4 + 2x_5 &= 3 \\ x_1 + 2x_2 + 2x_3 \qquad &= 1 \\ 2x_1 - 6x_2 + 4x_3 + 2x_4 - 4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrrr|r} 1 & 0 & 2 & -1 & 2 & 3 \\ 0 & 2 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

$x_4 - 2x_5 = -1 \quad \Rightarrow \quad x_4 = 2x_5 - 1, \quad x_5$ is free

$2x_2 + x_4 - 2x_5 = -2 \quad \Rightarrow \quad 2x_2 + (2x_5 - 1) - 2x_5 = -2 \quad \Rightarrow \quad x_2 = -\frac{1}{2}$

We solve the equation for x_2. We have $2x_2 = -2 - 2x_5 + 1 + 2x_5$ and from here we have x_2.

Solve the system
$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5&=5\\ x_1 \qquad +2x_3- \ x_4+2x_5&=3\\ x_1+2x_2+2x_3 \qquad\qquad &=1\\ 2x_1-6x_2+4x_3+2x_4-4x_5&=5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

$x_4 - 2x_5 = -1 \quad \Rightarrow \quad x_4 = 2x_5 - 1, \quad x_5$ is free

$2x_2 + x_4 - 2x_5 = -2 \quad \Rightarrow \quad 2x_2 + (2x_5 - 1) - 2x_5 = -2 \quad \Rightarrow \quad x_2 = -\frac{1}{2}$

$x_1 + 2x_3 - x_4 + 2x_5 = 3$

We write the equation corresponding to the first line of the matrix.

Solve the system

$$\begin{aligned} 3x_1 - 2x_2 + 6x_3 + 2x_4 - 4x_5 &= 5 \\ x_1 \qquad + 2x_3 - x_4 + 2x_5 &= 3 \\ x_1 + 2x_2 + 2x_3 \qquad &= 1 \\ 2x_1 - 6x_2 + 4x_3 + 2x_4 - 4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrrr|r} 1 & 0 & 2 & -1 & 2 & 3 \\ 0 & 2 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

$$x_4 - 2x_5 = -1 \quad \Rightarrow \quad x_4 = 2x_5 - 1, \quad x_5 \text{ is free}$$

$$2x_2 + x_4 - 2x_5 = -2 \quad \Rightarrow \quad 2x_2 + (2x_5 - 1) - 2x_5 = -2 \quad \Rightarrow \quad x_2 = -\frac{1}{2}$$

$$x_1 + 2x_3 - x_4 + 2x_5 = 3 \quad \Rightarrow \quad x_1 + 2x_3 - (2x_5 - 1) + 2x_5 = 3, \quad x_3 \text{ is free}$$

We substitute for x_4. The variable x_5 is free and two new variables x_1 and x_3 remain. We choose x_3 to be the second free variable.

Solve the system
$$\begin{aligned}3x_1-2x_2+6x_3+2x_4-4x_5&=5\\ x_1\qquad +2x_3-\ x_4+2x_5&=3\\ x_1+2x_2+2x_3\qquad\qquad &=1\\ 2x_1-6x_2+4x_3+2x_4-4x_5&=5\end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 1 & -2 & -1\end{array}\right)$$

$x_4 - 2x_5 = -1 \quad \Rightarrow \quad x_4 = 2x_5 - 1,\quad x_5$ is free

$2x_2 + x_4 - 2x_5 = -2 \quad \Rightarrow \quad 2x_2 + (2x_5 - 1) - 2x_5 = -2 \quad \Rightarrow \quad x_2 = -\frac{1}{2}$

$x_1 + 2x_3 - x_4 + 2x_5 = 3 \quad \Rightarrow \quad x_1 + 2x_3 - (2x_5 - 1) + 2x_5 = 3,\quad x_3$ is free

$x_1 = 2 - 2x_3$

We solve the equation with respect to x_1.

Solve the system
$$\begin{aligned}3x_1-2x_2+6x_3+2x_4-4x_5&=5\\ x_1+2x_3-x_4+2x_5&=3\\ x_1+2x_2+2x_3&=1\\ 2x_1-6x_2+4x_3+2x_4-4x_5&=5\end{aligned}$$

$$A^*\sim\left(\begin{array}{rrrrr|r}1&0&2&-1&2&3\\0&2&0&1&-2&-2\\0&0&0&1&-2&-1\end{array}\right)$$

$x_4-2x_5=-1 \quad\Rightarrow\quad x_4=2x_5-1,\quad x_5$ is free

$2x_2+x_4-2x_5=-2 \quad\Rightarrow\quad 2x_2+(2x_5-1)-2x_5=-2 \quad\Rightarrow\quad x_2=-\dfrac{1}{2}$

$x_1+2x_3-x_4+2x_5=3 \quad\Rightarrow\quad x_1+2x_3-(2x_5-1)+2x_5=3,\quad x_3$ is free

$x_1=2-2x_3$

The solution is $[x_1=2-2x_3, x_2=-\dfrac{1}{2}, x_3, x_4=2x_5-2, x_5]$, where x_3 and x_5 are free variables.

The system is solved.

Solve the system
$$\begin{aligned} 3x_1-2x_2+6x_3+2x_4-4x_5 &= 5\\ x_1 \qquad\quad +2x_3-\ x_4+2x_5 &= 3\\ x_1+2x_2+2x_3 \qquad\qquad\quad &= 1\\ 2x_1-6x_2+4x_3+2x_4-4x_5 &= 5 \end{aligned}$$

$$A^* \sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 3\\ 0 & 2 & 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

$x_4 - 2x_5 = -1 \quad \Rightarrow \quad x_4 = 2x_5 - 1, \quad x_5$ is free

$2x_2 + x_4 - 2x_5 = -2 \quad \Rightarrow \quad 2x_2 + (2x_5 - 1) - 2x_5 = -2 \quad \Rightarrow \quad x_2 = -\dfrac{1}{2}$

$x_1 + 2x_3 - x_4 + 2x_5 = 3 \quad \Rightarrow \quad x_1 + 2x_3 - (2x_5 - 1) + 2x_5 = 3, \quad x_3$ is free

$x_1 = 2 - 2x_3$

The solution is $[x_1 = 2 - 2x_3, x_2 = -\dfrac{1}{2}, x_3, x_4 = 2x_5 - 2, x_5]$, where x_3 and x_5 are free variables.

Solve the system

$$\begin{aligned} 2x_1+2x_2-2x_3+\ x_4 &= 1 \\ x_1+2x_2+\ x_3-2x_4 &= 1 \\ 3x_1+4x_2-\ x_3+2x_4 &= 5 \\ x_1+3x_2+3x_3-2x_4 &= 4 \end{aligned}$$

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+\ x_4&=1\\ x_1+2x_2+\ x_3-2x_4&=1\\ 3x_1+4x_2-\ x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right)$$

We write the augmented matrix.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ & & & & \\ & & & & \\ & & & & \end{array}\right)$$

We choose the second row as a pivot row, since $a_{21} = 1$.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) \overset{(-2)}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & -2 & -4 & 5 & -1\\ & & & & \\ & & & & \end{array}\right)$$

$(-2)R_2 + R_1$

Solve the system $\begin{aligned} 2x_1+2x_2-2x_3+\ x_4 &= 1\\ x_1+2x_2+\ x_3-2x_4 &= 1\\ 3x_1+4x_2-\ x_3+2x_4 &= 5\\ x_1+3x_2+3x_3-2x_4 &= 4 \end{aligned}$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) (-3) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & -2 & -4 & 5 & -1\\ 0 & -2 & -4 & 8 & 2\\ & & & & \end{array}\right)$$

$(-3)R_2 + R_3$

Solve the system

$$\begin{aligned} 2x_1+2x_2-2x_3+x_4 &= 1\\ x_1+2x_2+x_3-2x_4 &= 1\\ 3x_1+4x_2-x_3+2x_4 &= 5\\ x_1+3x_2+3x_3-2x_4 &= 4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{rrrr|r} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) \overset{(-1)}{\sim} \left(\begin{array}{rrrr|r} 1 & 2 & 1 & -2 & 1\\ 0 & -2 & -4 & 5 & -1\\ 0 & -2 & -4 & 8 & 2\\ 0 & 1 & 2 & 0 & 3 \end{array}\right)$$

$(-1)R_2 + R_4$

Solve the system

$$\begin{aligned} 2x_1+2x_2-2x_3+\ x_4 &= 1\\ x_1+2x_2+\ x_3-2x_4 &= 1\\ 3x_1+4x_2-\ x_3+2x_4 &= 5\\ x_1+3x_2+3x_3-2x_4 &= 4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & -2 & -4 & 5 & -1\\ 0 & -2 & -4 & 8 & 2\\ 0 & 1 & 2 & 0 & 3 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ & & & & \\ & & & & \end{array}\right)$$

The first row remains and the last row will be the next pivot row, since $a_{42} = 1$, which is better than $a_{22} = a_{23} = -2$.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & -2 & -4 & 5 & -1\\ 0 & -2 & -4 & 8 & 2\\ 0 & 1 & 2 & 0 & 3 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 5 & 5\\ & & & & \end{array}\right)$$

$2R_4 + R_2$

Solve the system

$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & -2 & -4 & 5 & -1\\ 0 & -2 & -4 & 8 & 2\\ 0 & 1 & 2 & 0 & 3 \end{array}\right)_{2} \sim$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 5 & 5\\ 0 & 0 & 0 & 8 & 8 \end{array}\right)$$

$2R_4 + R_3$

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & -2 & -4 & 5 & -1\\ 0 & -2 & -4 & 8 & 2\\ 0 & 1 & 2 & 0 & 3 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 5 & 5\\ 0 & 0 & 0 & 8 & 8 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ & & & & \\ & & & & \end{array}\right)$$

The first two rows remain.

Solve the system
$$\begin{aligned}2x_1+2x_2-2x_3+x_4&=1\\x_1+2x_2+x_3-2x_4&=1\\3x_1+4x_2-x_3+2x_4&=5\\x_1+3x_2+3x_3-2x_4&=4\end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c}2&2&-2&1&1\\1&2&1&-2&1\\3&4&-1&2&5\\1&3&3&-2&4\end{array}\right) \sim \left(\begin{array}{cccc|c}1&2&1&-2&1\\0&-2&-4&5&-1\\0&-2&-4&8&2\\0&1&2&0&3\end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c}1&2&1&-2&1\\0&1&2&0&3\\0&0&0&5&5\\0&0&0&8&8\end{array}\right) \sim \left(\begin{array}{cccc|c}1&2&1&-2&1\\0&1&2&0&3\\0&0&0&1&1\\0&0&0&1&1\end{array}\right)$$

The last two rows can be divided. We divide the third row by the number 5 and the last row by 8.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & -2 & -4 & 5 & -1\\ 0 & -2 & -4 & 8 & 2\\ 0 & 1 & 2 & 0 & 3 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 5 & 5\\ 0 & 0 & 0 & 8 & 8 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

The last two rows are the same. We keep only one of them.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 2 & 2 & -2 & 1 & 1\\ 1 & 2 & 1 & -2 & 1\\ 3 & 4 & -1 & 2 & 5\\ 1 & 3 & 3 & -2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & -2 & -4 & 5 & -1\\ 0 & -2 & -4 & 8 & 2\\ 0 & 1 & 2 & 0 & 3 \end{array}\right) \sim$$
$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 5 & 5\\ 0 & 0 & 0 & 8 & 8 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1\\ & & & & \end{array}\right)$$

Hence we omit the last row.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

x_1 x_2 x_3 x_4

- We row-reduced the augmented matrix of the system.
- rank $(A) = 3$, rank $(A^*) = 3$, $n = 4$
- The system possesses infinitely many solutions with one parameter.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

x_1 x_2 x_3 $x_4 = 1$

$x_4 = 1$

We write the equation corresponding to the last row.

Solve the system

$$\begin{aligned} 2x_1+2x_2-2x_3+x_4 &= 1\\ x_1+2x_2+x_3-2x_4 &= 1\\ 3x_1+4x_2-x_3+2x_4 &= 5\\ x_1+3x_2+3x_3-2x_4 &= 4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

x_1 x_2 x_3 $x_4 = 1$

$x_4 = 1$

$x_2 + 2x_3 = 3,$

We write the equation corresponding to the middle row.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+\ x_4 &= 1\\ x_1+2x_2+\ x_3-2x_4 &= 1\\ 3x_1+4x_2-\ x_3+2x_4 &= 5\\ x_1+3x_2+3x_3-2x_4 &= 4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

x_1 x_2 x_3 is free; $x_4 = 1$

$x_4 = 1$

$x_2 + 2x_3 = 3$, x_3 is free $\Rightarrow$

- We have two unknowns in this equation. We choose one of them to be free.
- We choose x_3 to be free and solve the equation for x_2, since this is easier than solving for x_3.

Solve the system
$$\begin{aligned}2x_1+2x_2-2x_3+\ x_4&=1\\ x_1+2x_2+\ x_3-2x_4&=1\\ 3x_1+4x_2-\ x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4\end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1\end{array}\right)$$

x_1 $\quad$ $x_2 = 3 - 2x_3;$ $\quad$ x_3 is free; $\quad$ $x_4 = 1$

$x_4 = 1$

$x_2 + 2x_3 = 3$, x_3 is free $\Rightarrow$ $x_2 = 3 - 2x_3$

We have x_2.

Solve the system

$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

x_1 $\quad$ $x_2 = 3 - 2x_3;$ $\quad$ x_3 is free; $\quad$ $x_4 = 1$

$x_4 = 1$

$x_2 + 2x_3 = 3$, x_3 is free $\Rightarrow x_2 = 3 - 2x_3$

$x_1 + 2x_2 + x_3 - 2x_4 = 1$

We write the equation corresponding to the first row.

Solve the system

$$\begin{aligned} 2x_1+2x_2-2x_3+x_4 &= 1\\ x_1+2x_2+x_3-2x_4 &= 1\\ 3x_1+4x_2-x_3+2x_4 &= 5\\ x_1+3x_2+3x_3-2x_4 &= 4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

x_1 $\qquad$ $x_2 = 3 - 2x_3$; $\qquad$ x_3 is free; $\qquad$ $x_4 = 1$

$x_4 = 1$

$x_2 + 2x_3 = 3$, x_3 is free $\Rightarrow$ $x_2 = 3 - 2x_3$

$x_1 + 2x_2 + x_3 - 2x_4 = 1$

$x_1 + 2(3 - 2x_3) + x_3 - 2 \cdot 1 = 1$

We substitute for x_2 and x_4. The variable x_3 is free.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

x_1 $\qquad$ $x_2 = 3 - 2x_3$; $\qquad$ x_3 is free; $\qquad$ $x_4 = 1$

$x_4 = 1$

$x_2 + 2x_3 = 3$, x_3 is free $\Rightarrow$ $x_2 = 3 - 2x_3$

$x_1 + 2x_2 + x_3 - 2x_4 = 1$

$x_1 + 2(3 - 2x_3) + x_3 - 2 \cdot 1 = 1$

$x_1 - 4x_3 + x_3 + 4 = 1$

We simplify.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

x_1 $\quad$ $x_2 = 3 - 2x_3$; $\quad$ x_3 is free; $\quad$ $x_4 = 1$

$x_4 = 1$

$x_2 + 2x_3 = 3$, x_3 is free $\Rightarrow$ $x_2 = 3 - 2x_3$

$$x_1 + 2x_2 + x_3 - 2x_4 = 1$$
$$x_1 + 2(3 - 2x_3) + x_3 - 2 \cdot 1 = 1$$
$$x_1 - 4x_3 + x_3 + 4 = 1$$
$$x_1 - 3x_3 = -3$$

We simplify.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+\ x_4 &= 1\\ x_1+2x_2+\ x_3-2x_4 &= 1\\ 3x_1+4x_2-\ x_3+2x_4 &= 5\\ x_1+3x_2+3x_3-2x_4 &= 4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$x_1 = -3 + 3x_3;$ $x_2 = 3 - 2x_3;$ x_3 is free; $x_4 = 1$

$x_4 = 1$

$x_2 + 2x_3 = 3$, x_3 is free $\Rightarrow x_2 = 3 - 2x_3$

$x_1 + 2x_2 + x_3 - 2x_4 = 1$

$x_1 + 2(3 - 2x_3) + x_3 - 2 \cdot 1 = 1$

$x_1 - 4x_3 + x_3 + 4 = 1$

$x_1 - 3x_3 = -3$

$x_1 = 3x_3 - 3$

x_1 is known.

Solve the system
$$\begin{aligned} 2x_1+2x_2-2x_3+x_4&=1\\ x_1+2x_2+x_3-2x_4&=1\\ 3x_1+4x_2-x_3+2x_4&=5\\ x_1+3x_2+3x_3-2x_4&=4 \end{aligned}$$

$$A^* \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 1\\ 0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$x_1 = -3 + 3x_3$; $x_2 = 3 - 2x_3$; x_3 is free; $x_4 = 1$

$x_4 = 1$

$x_2 + 2x_3 = 3$, x_3 is free $\Rightarrow x_2 = 3 - 2x_3$

The solution is

$$\begin{aligned} x_1 &= -3 + 3x_3\\ x_2 &= 3 - 2x_3\\ x_4 &= 1 \end{aligned}$$

where x_3 is a free variable.

6 Determinants

Definition (determinant). Let A be an $n \times n$ square matrix. Under a *determinant of the matrix* A we understand the real number $\mathbf{det}\, \boldsymbol{A}$ which is assigned to the matrix by the following three–step recursive algorithm.

1. If the matrix A is 1×1 matrix, i.e. if $A = (a_{11})$, then $\det A = a_{11}$.

2. Suppose that the determinant of $(n-1) \times (n-1)$ matrix is defined. Denote by M_{ij} the determinant of the $(n-1) \times (n-1)$ matrix which has arisen from the matrix A by omitting the i-th row and the j-th column. We define *cofactor* A_{ij} of the element a_{ij} in the matrix A as the product $A_{ij} = (-1)^{i+j} M_{ij}$.

3. Finally, we define the determinant of the matrix A by the relation

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \tag{13}$$

where $i \in \{1, 2, \ldots n\}$ is the index of arbitrary row.

Remark 10 (notation). Determinant of the matrix A is denoted also by $|A|$. If $A = (a_{ij})$, we write also $|a_{ij}|$ instead of $|(a_{ij})|$.

Remark 11 (Is the definition correct?)**.** The formula (13) is called an *expansion of the determinant along the i-th row*. This formula allows to write the determinant of the $n \times n$ matrix in terms of n determinants of $(n-1) \times (n-1)$ matrices. Each of these determinants can be written in terms of determinants of the $(n-2) \times (n-2)$ matrices and so on. We end after a finite number of steps when we obtain determinants of 1×1 matrices. It should be noted that the index i in the expansion can be chosen arbitrary. The proof of this fact can be found in the literature under the name *Laplace theorem*. In this sense the expansion (13) is called the *Laplace expansion of the determinant along the i-th row*.

Many of the most important properties of matrices depend on the fact whether the determinant of the matrix equals zero or not. It is fruitful to distinguish these cases by the following definition.

Definition (regular and singular matrix)**.** Let A be a square matrix. The matrix A is said to be *singular* if $\det A = 0$ and it is said to be *regular* in the opposite case.

Evaluate the determinant

$$\begin{vmatrix} a & b \\ i & j \end{vmatrix}$$

Evaluate the determinant

$$\begin{vmatrix} a & b \\ i & j \end{vmatrix} = a(-1)^{1+1}|j|$$

Laplace expansion along the first row.

Evaluate the determinant

$$\begin{vmatrix} a & b \\ i & j \end{vmatrix} = a(-1)^{1+1}|j| + b(-1)^{1+2}|i|$$

Laplace expansion along the first row.

Evaluate the determinant

$$\begin{vmatrix} a & b \\ i & j \end{vmatrix} = a(-1)^{1+1}|j| + b(-1)^{1+2}|i| = aj - bi$$

We simplify.

Evaluate the determinant

$$\begin{vmatrix} a & b \\ i & j \end{vmatrix} = a(-1)^{1+1}|j| + b(-1)^{1+2}|i| = aj - bi$$

The rule: multiply the main diagonal and subtract the product in the auxiliary diagonal.

Evaluate the following determinant

$$\begin{vmatrix} a & b & c \\ i & j & k \\ x & y & z \end{vmatrix} = a(-1)^{1+1}\begin{vmatrix} j & k \\ y & z \end{vmatrix} + b(-1)^{1+2}\begin{vmatrix} i & k \\ x & z \end{vmatrix} + c(-1)^{1+3}\begin{vmatrix} i & j \\ x & y \end{vmatrix}$$

$$= a(jz - ky) - b(iz - kx) + c(iy - jx)$$

$$= ajz - aky - biz + bkx + ciy - cjx$$

The rule: $\begin{vmatrix} a & b & c \\ i & j & k \\ x & y & z \end{vmatrix} = ajz + iyc + xbk - (cjx + kya + zbi)$

Evaluate the following determinant

$$\begin{vmatrix} a & b & c \\ i & j & k \\ x & y & z \end{vmatrix} = a(-1)^{1+1}\begin{vmatrix} j & k \\ y & z \end{vmatrix} + b(-1)^{1+2}\begin{vmatrix} i & k \\ x & z \end{vmatrix} + c(-1)^{1+3}\begin{vmatrix} i & j \\ x & y \end{vmatrix}$$

$$= a(jz - ky) - b(iz - kx) + c(iy - jx)$$

$$= ajz - aky - biz + bkx + ciy - cjx$$

The rule:
$$\begin{vmatrix} a & b & c \\ i & j & k \\ x & y & z \end{vmatrix} = ajz + iyc + xbk - (cjx + kya + zbi)$$
$$\begin{matrix} a & b & c \\ i & j & k \end{matrix}$$

Theorem 7 (operations preserving the value of the determinant)**.** The following operations preserve the value of the determinant:

1. Leaving one row (column) without any change and adding arbitrary multiples of this row (column) to the remaining rows (columns).

2. Transposition of the matrix.

Remark 12 (Laplace expansion for columns)**.** Theorem 7 shows that changing rows of the determinant into columns (and vice versa) does not change the value of the determinant. Thus all statements concerning the determinant and rows can be reformulated also for columns. Among others, the Laplace expansion along a column reads as follows: for an arbitrary column index $j \in \{1, 2, \ldots, n\}$ we have

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj},$$

where A_{ij} is the ij-th cofactor of the matrix A. This formula is called the *Laplace expansion along the j-th column.*

Theorem 8 (another operations with determinant). The following operations change the value of the determinant in a known way:

1. Interchange two rows (or two columns) of the determinant changes the sign of the determinant.
2. Dividing an arbitrary row (an arbitrary column) by a nonzero number a decreases the value of the determinant a-times.

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix}$$

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} & & & \\ 1 & 4 & 3 & -1 \\ & & & \\ & & & \end{vmatrix}$$

The second row is the pivot row.

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix} \overset{(-2)}{=} \begin{vmatrix} 0 & -8 & -9 & 5 \\ & 4 & 3 & -1 \\ & & & \\ & & & \end{vmatrix}$$

We adjust the first row. We don't interchange rows!

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix} \overset{(-1)}{=} \begin{vmatrix} 0 & -8 & -9 & 5 \\ & 4 & 3 & -1 \\ 0 & -8 & 5 & 1 \\ & & & \end{vmatrix}$$

We adjust the third row.

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -8 & -9 & 5 \\ 1 & 4 & 3 & -1 \\ 0 & -8 & 5 & 1 \\ 0 & 3 & -1 & 2 \end{vmatrix}$$

The last row remains.

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -8 & -9 & 5 \\ 1 & 4 & 3 & -1 \\ 0 & -8 & 5 & 1 \\ 0 & 3 & -1 & 2 \end{vmatrix} = 1.(-1)^{2+1}. \begin{vmatrix} -8 & -9 & 5 \\ -8 & 5 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

- We expand the determinant along the first column.
- The red element remains and it will be multiplied by $(-1)^{\text{row + column}}$.
- We omit the first column and the second row.

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -8 & -9 & 5 \\ 1 & 4 & 3 & -1 \\ 0 & -8 & 5 & 1 \\ 0 & 3 & -1 & 2 \end{vmatrix} = 1.(-1)^{2+1}. \begin{vmatrix} -8 & -9 & 5 \\ -8 & 5 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= -1\Big[-8.5.2 + (-8)(-1).5 + 3.(-9).1 - (5.5.3 + 1.(-1).(-8) + 2.(-9).(-8))\Big]$$

We evaluate the 3×3 determinant by the rule:

$$\begin{vmatrix} -8 & -9 & 5 \\ -8 & 5 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$
$$\begin{matrix} -8 & -9 & 5 \\ -8 & 5 & 1 \end{matrix}$$

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -8 & -9 & 5 \\ 1 & 4 & 3 & -1 \\ 0 & -8 & 5 & 1 \\ 0 & 3 & -1 & 2 \end{vmatrix} = 1.(-1)^{2+1}. \begin{vmatrix} -8 & -9 & 5 \\ -8 & 5 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= -1\Big[-8.5.2 + (-8)(-1).5 + 3.(-9).1$$

$$- (5.5.3 + 1.(-1).(-8) + 2.(-9).(-8))\Big]$$

$$= -1\Big[-80 + 40 - 27 - (75 + 8 + 144)\Big]$$

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -8 & -9 & 5 \\ 1 & 4 & 3 & -1 \\ 0 & -8 & 5 & 1 \\ 0 & 3 & -1 & 2 \end{vmatrix} = 1.(-1)^{2+1}. \begin{vmatrix} -8 & -9 & 5 \\ -8 & 5 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= -1\Big[-8.5.2 + (-8)(-1).5 + 3.(-9).1$$

$$- (5.5.3 + 1.(-1).(-8) + 2.(-9).(-8))\Big]$$

$$= -1\Big[-80 + 40 - 27 - (75 + 8 + 144)\Big]$$

$$= -\Big[-67 - 227\Big]$$

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & -3 & 3 \\ 1 & 4 & 3 & -1 \\ 1 & -4 & 8 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -8 & -9 & 5 \\ 1 & 4 & 3 & -1 \\ 0 & -8 & 5 & 1 \\ 0 & 3 & -1 & 2 \end{vmatrix} = 1.(-1)^{2+1}. \begin{vmatrix} -8 & -9 & 5 \\ -8 & 5 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= -1\Big[-8.5.2 + (-8)(-1).5 + 3.(-9).1 - (5.5.3 + 1.(-1).(-8) + 2.(-9).(-8))\Big]$$

$$= -1\Big[-80 + 40 - 27 - (75 + 8 + 144)\Big]$$

$$= -\Big[-67 - 227\Big] = 294$$

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} =$$

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} =$$

- The last row contains only one nonzero element.
- We use the Laplace expansion along the last row.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

- The only nonzero element is $a_{55} = 1$.
- The Laplace expansion starts with this element.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5}$$

We continue with the factor $(-1)^{\text{row+column}}$.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} \quad \end{vmatrix}$$

And the determinant 4×4 ...

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

which arises from the preceding determinant by omitting the row 5 and column 5.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

- The product on the begin of the determinant equals one.
- We take a common factor 2 from the second row and from the third row.
- From both rows there arise number 2 in the front of determinant.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} \\ \\ \\ \\ \end{vmatrix}$$

We use the third row as a pivot row.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{-1} \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} & & & \\ & & & \\ \color{blue}{1} & \color{blue}{1} & \color{blue}{1} & \color{blue}{-1} \\ & & & \end{vmatrix}$$

Pivot row remains. We keep the order of rows!

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ & & & \\ 1 & 1 & 1 & -1 \\ & & & \end{vmatrix}$$

$R_1 + R_3$

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}-1} \\ -5 & -4 & -7 & 5 \end{vmatrix} \overset{(-1)}{=} \begin{vmatrix} -2 & -4 & 4 & 0 \\ {\color{blue}3} & {\color{blue}0} & {\color{blue}0} & {\color{blue}0} \\ 1 & 1 & 1 & -1 \\ & & & \end{vmatrix}$$

$R_2 - R_3$

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} \overset{5}{=} 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$R_4 + 5R_3$

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot$$

We repeat the starting number . . .

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{3+4}$$

and continue with the Laplace expansion along the last column: (the only nonzero element) $\cdot (-1)^{\text{row+column}}$

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{3+4} \Bigg| \qquad\qquad \Bigg|$$

and the determinant of 3×3 matrix ...

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{3+4} \begin{vmatrix} -2 & -4 & 4 \\ 3 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

which arises by omitting the row number three and column number four.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{3+4} \begin{vmatrix} -2 & -4 & 4 \\ 3 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 4$$

The product before the determinant equals 4.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{3+4} \begin{vmatrix} -2 & -4 & 4 \\ 3 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 4 \cdot 3 \cdot (-1)^{2+1}$$

We use the expansion along the third row:
(the only nonzero element) $\cdot\, (-1)^{\text{row+column}}$ ...

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{3+4} \begin{vmatrix} -2 & -4 & 4 \\ 3 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 4 \cdot 3 \cdot (-1)^{2+1} \begin{vmatrix} -4 & 4 \\ 1 & -2 \end{vmatrix}$$

times the corresponding 2×2 determinant.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{3+4} \begin{vmatrix} -2 & -4 & 4 \\ 3 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 4 \cdot 3 \cdot (-1)^{2+1} \begin{vmatrix} -4 & 4 \\ 1 & -2 \end{vmatrix} = -12$$

The product on the front is -12 ...

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{3+4} \begin{vmatrix} -2 & -4 & 4 \\ 3 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 4 \cdot 3 \cdot (-1)^{2+1} \begin{vmatrix} -4 & 4 \\ 1 & -2 \end{vmatrix} = -12 \cdot (8 - 4)$$

and the 2×2 determinant can be evaluated by the rule $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1\cdot(-1)^{5+5}\cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1\cdot 2\cdot 2\cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4\cdot(-1)\cdot(-1)^{3+4} \begin{vmatrix} -2 & -4 & 4 \\ 3 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 4\cdot 3\cdot(-1)^{2+1} \begin{vmatrix} -4 & 4 \\ 1 & -2 \end{vmatrix} = -12\cdot(8-4) = -48$$

We finished.

Evaluate the determinant.

$$\begin{vmatrix} -3 & -5 & 3 & 1 & 2 \\ 8 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & -2 & 4 \\ -5 & -4 & -7 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{5+5} \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 8 & 2 & 2 & -2 \\ 2 & 2 & 2 & -2 \\ -5 & -4 & -7 & 5 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 2 \cdot \begin{vmatrix} -3 & -5 & 3 & 1 \\ 4 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -5 & -4 & -7 & 5 \end{vmatrix} = 4 \begin{vmatrix} -2 & -4 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{3+4} \begin{vmatrix} -2 & -4 & 4 \\ 3 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 4 \cdot 3 \cdot (-1)^{2+1} \begin{vmatrix} -4 & 4 \\ 1 & -2 \end{vmatrix} = -12 \cdot (8-4) = -48$$