Limits and L'Hospital rule

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Find $\lim_{x\to 0} \frac{\arcsin x}{1-e^x}$

$$\lim_{x\to 0} \frac{\arcsin x}{1-e^x} = \boxed{\frac{0}{0}}$$

We substitute. Since $\arcsin 0 = 0$ and $e^0 = 1$, we have an indeterminate form.

Find $\lim_{x\to 0} \frac{\arcsin x}{1-e^x}$

$$\lim_{x\to 0} \frac{\arcsin x}{1-e^x} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=}$$

The l'Hospital rule can be used for this limit.

Find $\lim_{x\to 0} \frac{\arcsin x}{1-e^x}$

$$\lim_{x\to 0} \frac{\arcsin x}{1-e^x} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{-e^x}$$

According ot the l'Hospital rule, we have

$$\lim_{x\to 0} \frac{\arcsin x}{1-e^x} = \lim_{x\to 0} \frac{(\arcsin x)'}{(1-e^x)'},$$

provided the second limit exists (finite or infinite).

Find $\lim_{x\to 0} \frac{\arcsin x}{1-e^x}$

$$\lim_{x\to 0} \frac{\arcsin x}{1-e^x} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{-e^x} = -1$$

We substitute. We get

$$\lim_{x\to 0} \frac{\frac{1}{\sqrt{1-0}}}{-1} = \frac{1}{-1} = -1$$

Find $\lim_{x\to 0} \frac{\arcsin x}{1-e^x}$

$$\lim_{x\to 0} \frac{\arcsin x}{1-e^x} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{-e^x} = -1$$

Finished!

Find $\lim_{x\to\infty} \frac{x \ln x}{x^2 + x + 1}$

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$$\lim_{x\to\infty} \frac{x \ln x}{x^2 + x + 1}$$

We start with the limit.

Find $\lim_{x\to\infty} \frac{x\ln x}{x^2+x+1}$

$$\lim_{x\to\infty} \frac{x\ln x}{x^2+x+1} = \boxed{\frac{\infty}{\infty}}$$

We substitute. We have

$$\frac{\infty \ln \infty}{\infty} = \frac{\infty}{\infty}.$$

This is an indeterminate form.

Find $\lim_{x\to\infty} \frac{x \ln x}{x^2 + x + 1}$

$$\lim_{x\to\infty} \frac{x \ln x}{x^2 + x + 1} = \boxed{\frac{\infty}{\infty}} \stackrel{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\ln x + 1}{2x + 1}$$

We use the l'Hospital rule. When differentiating we obtain

$$\lim_{x\to\infty} \frac{(x \ln x)'}{(x^2 + x + 1)'} = \lim_{x\to\infty} \frac{\ln x + x\frac{1}{x}}{2x + 1}.$$

Find $\lim_{x\to\infty} \frac{x \ln x}{x^2 + x + 1}$

$$\lim_{x\to\infty} \frac{x \ln x}{x^2 + x + 1} = \boxed{\frac{\infty}{\infty}} \overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\ln x + 1}{2x + 1} = \boxed{\frac{\infty}{\infty}}$$

We substitute. We have

$$\frac{\ln \infty + 1}{\infty} = \frac{\infty}{\infty}.$$

Find $\lim\limits_{x\to\infty} \dfrac{x\ln x}{x^2+x+1}$

$$\lim_{x\to\infty} \frac{x\ln x}{x^2+x+1} = \boxed{\frac{\infty}{\infty}} \overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\ln x + 1}{2x+1} = \boxed{\frac{\infty}{\infty}} \overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2}$$

We continue with the l'Hospital rule.

Find $\lim_{x\to\infty} \frac{x \ln x}{x^2+x+1}$

$$\lim_{x\to\infty} \frac{x \ln x}{x^2+x+1} = \boxed{\frac{\infty}{\infty}} \overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\ln x + 1}{2x+1} = \boxed{\frac{\infty}{\infty}} \overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2} = 0$$

We substitute. We obtain

$$\frac{\frac{1}{\infty}}{2} = \frac{0}{2} = 0$$

Find $\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x}$

Find $\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x}$

$$\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x}$$

We start with the limit.

Find $\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x}$

$$\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x} = \boxed{\frac{0}{0}}$$

We substitute. We have

$$\frac{0 - \sin 0}{\sin^3 0} = \frac{0}{0}.$$

Find $\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x}$

$$\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1 - \cos x}{3\sin^2 x \cos x}$$

We use l'Hospital rule. The chain rule gives

$$(\sin^3(x))' = 3\sin^2 x(\sin x)' = 3\sin^2 x \cos x.$$

Find $\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x}$

$$\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1 - \cos x}{3 \sin^2 x \cos x} = \boxed{\frac{0}{0}}$$

We substitute. Since $\boxed{\cos 0 = 1}$ and $\boxed{\sin 0 = 0}$, we have still $\boxed{\frac{0}{0}}$.

Find $\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x}$

$$\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x} = \boxed{\frac{0}{0}} \stackrel{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1 - \cos x}{3\sin^2 x \cos x} = \boxed{\frac{0}{0}}$$

$$\stackrel{\text{l'H.r}}{=} \lim_{x\to 0} \frac{\sin x}{6\sin x \cos^2 x - 3\sin^3 x}$$

We use the l'Hospital rule once more. In the denominator we have

$$(3\sin^2 x \times \cos x)' = 3.2\sin x \cos x \cos x + 3\sin^2 x(-\sin x)$$

(product rule and chain rule).

Find $\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x}$

$$\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1 - \cos x}{3 \sin^2 x \cos x} = \boxed{\frac{0}{0}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{\sin x}{6 \sin x \cos^2 x - 3 \sin^3 x} = \boxed{\frac{0}{0}}$$

We substitute. Still $\boxed{\frac{0}{0}}$.

Find $\lim_{x\to 0} \frac{x-\sin x}{\sin^3 x}$

$$\lim_{x\to 0} \frac{x-\sin x}{\sin^3 x} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1-\cos x}{3\sin^2 x \cos x} = \boxed{\frac{0}{0}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{\sin x}{6\sin x \cos^2 x - 3\sin^3 x} = \boxed{\frac{0}{0}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{\cos x}{6\cos^3 x - 6.2.\sin^2 x \cos x - 9\sin^2 x \cos x}$$

We use the l'Hospital rule once more.

Find $\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x}$

$$\lim_{x\to 0} \frac{x - \sin x}{\sin^3 x} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1 - \cos x}{3\sin^2 x \cos x} = \boxed{\frac{0}{0}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{\sin x}{6\sin x \cos^2 x - 3\sin^3 x} = \boxed{\frac{0}{0}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{\cos x}{6\cos^3 x - 6.2.\sin^2 x \cos x - 9\sin^2 x \cos x} = \frac{1}{6}$$

The last function is continuous at $x = 0$. Really, the substitution into the limit gives the well-defined result. The problem is solved.

Find $\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3}$

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$$\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3}$$

We start with the limit ...

Find $\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3}$

$$\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2}$$

...and substitute. Remember that $\boxed{\operatorname{arctg} 0 = 0}$. We have an indeterminate form and we can use the l'Hospital rule.

Find $\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3}$

$$\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2} = \boxed{\frac{0}{0}}$$

We substitute and obtain again the indeterminate form.

Find $\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3}$

$$\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2} = \boxed{\frac{0}{0}}$$

$$\overset{\text{simplify}}{=} \lim_{x\to 0} \frac{1}{3(1+x^2)}$$

The next application of l'Hospital rule is possible, but it would yield complicated calculations. We prefer simplification of the fraction

$$\frac{1 - \frac{1}{1+x^2}}{3x^2} = \frac{\frac{x^2}{1+x^2}}{3x^2} = \frac{x^2}{(1+x^2)3x^2} = \frac{1}{3(1+x^2)}$$

Find $\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3}$

$$\lim_{x\to 0} \frac{x - \operatorname{arctg} x}{x^3} = \boxed{\frac{0}{0}} \overset{\text{l'H.r}}{=} \lim_{x\to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2} = \boxed{\frac{0}{0}}$$

$$\overset{\text{simplify}}{=} \lim_{x\to 0} \frac{1}{3(1+x^2)} = \frac{1}{3}$$

We substitute. The function is continuous at $x = 0$ and the limit can be evaluated by the substitution.

Find $\lim_{x\to 0^+} x \ln x$.

Find $\lim_{x\to 0^+} x\ln x$.

$$\lim_{x\to 0^+} x\ln x$$

We start with the limit and substitute.

Find $\lim_{x\to 0^+} x\ln x$.

$$\lim_{x\to 0^+} x\ln x = \boxed{0\times(-\infty)} \stackrel{\text{simplify}}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

The substitution gives and indeterminate form. We have to write the function in the limit as a fraction. To do this we write $x = \frac{1}{\frac{1}{x}}$ and multiply with logarithm.

Find $\lim_{x\to 0^+} x \ln x$.

$$\lim_{x\to 0^+} x \ln x = \boxed{0 \times (-\infty)} \stackrel{\text{simplify}}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \boxed{\frac{-\infty}{\infty}}$$

Now we have indeterminate form for which l' Hospital rule can be used.

Find $\lim_{x\to 0^+} x\ln x.$

$$\lim_{x\to 0^+} x\ln x = \boxed{0\times(-\infty)} \overset{\text{simplify}}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \boxed{\frac{-\infty}{\infty}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

We use l' Hospital rule.

Find $\lim_{x\to 0^+} x\ln x$.

$$\lim_{x\to 0^+} x\ln x = \boxed{0\times(-\infty)} \overset{\text{simplify}}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \boxed{\frac{-\infty}{\infty}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \overset{\text{simplify}}{=} \lim_{x\to 0^+} -x$$

We simplify. The function in the limit is continuous at $x = 0$.

Find $\lim_{x\to 0^+} x \ln x.$

$$\lim_{x\to 0^+} x \ln x = \boxed{0 \times (-\infty)} \overset{\text{simplify}}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \boxed{\frac{-\infty}{\infty}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \overset{\text{simplify}}{=} \lim_{x\to 0^+} -x = 0$$

The limit of continuous function can be evaluated by direct substitution.

Find $\lim\limits_{x\to 0^+} x\ln x$.

$$\lim_{x\to 0^+} x\ln x = \boxed{0\times(-\infty)} \overset{\text{simplify}}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \boxed{\frac{-\infty}{\infty}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \overset{\text{simplify}}{=} \lim_{x\to 0^+} -x = 0$$

The problem is solved.

Find $\lim\limits_{x\to\infty} x^2 e^{-x^2}$

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$$\lim_{x\to\infty} x^2 e^{-x^2}$$

We start with the limit.

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Find $\lim_{x\to\infty} x^2 e^{-x^2}$

$$\lim_{x\to\infty} x^2 e^{-x^2} = \boxed{\infty \times 0}$$

We substitute. More precisely, we evaluate separately the following limits

$$\lim_{x\to\infty} x^2 \qquad \text{and} \qquad \lim_{x\to\infty} e^{-x^2}.$$

We obtain an indeterminate form.

Find $\lim_{x\to\infty} x^2 e^{-x^2}$

$$\lim_{x\to\infty} x^2 e^{-x^2} = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{x^2}{e^{x^2}}$$

We have to convert the function inside the limit into fraction. We use the identity

$$e^{-x^2} = \frac{1}{e^{x^2}}$$

Find $\lim_{x\to\infty} x^2 e^{-x^2}$

$$\lim_{x\to\infty} x^2 e^{-x^2} = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{x^2}{e^{x^2}} = \boxed{\frac{\infty}{\infty}}$$

The limit has the form required by the l' Hospital rule.

Find $\lim_{x\to\infty} x^2 e^{-x^2}$

$$\lim_{x\to\infty} x^2 e^{-x^2} = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{x^2}{e^{x^2}} = \boxed{\frac{\infty}{\infty}} \overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{2x}{2xe^{x^2}}$$

We use l' Hospital rule.

Find $\lim_{x\to\infty} x^2 e^{-x^2}$

$$\lim_{x\to\infty} x^2 e^{-x^2} = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{x^2}{e^{x^2}} = \boxed{\frac{\infty}{\infty}} \overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{2x}{2xe^{x^2}}$$

$$\overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{1}{e^{x^2}}$$

We simplify.

Find $\lim_{x\to\infty} x^2 e^{-x^2}$

$$\lim_{x\to\infty} x^2 e^{-x^2} = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{x^2}{e^{x^2}} = \boxed{\frac{\infty}{\infty}} \overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{2x}{2xe^{x^2}}$$

$$\overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{1}{e^{x^2}} = \frac{1}{\infty}$$

We substitute (in the sense of limits). Hence we evaluate separately the limit of the numerator and the denominator.

Find $\lim_{x\to\infty} x^2 e^{-x^2}$

$$\lim_{x\to\infty} x^2 e^{-x^2} = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{x^2}{e^{x^2}} = \boxed{\frac{\infty}{\infty}} \overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{2x}{2xe^{x^2}}$$

$$\overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{1}{e^{x^2}} = \frac{1}{\infty} = 0$$

We obtain well–defined expression which equals zero. The problem is solved.

Find $\lim_{x\to\infty} x(\operatorname{arctg} x - \frac{\pi}{2})$

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Find $\lim_{x\to\infty} x(\operatorname{arctg} x - \frac{\pi}{2})$

$$\lim_{x\to\infty} x\left(\operatorname{arctg} x - \frac{\pi}{2}\right) = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{\operatorname{arctg} x - \frac{\pi}{2}}{\frac{1}{x}}$$

Find $\lim_{x\to\infty} x(\operatorname{arctg} x - \frac{\pi}{2})$

$$\lim_{x\to\infty} x\left(\operatorname{arctg} x - \frac{\pi}{2}\right) = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{\operatorname{arctg} x - \frac{\pi}{2}}{\frac{1}{x}} = \boxed{\frac{0}{0}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}}$$

Find $\lim_{x\to\infty} x(\operatorname{arctg} x - \frac{\pi}{2})$

$$\lim_{x\to\infty} x\left(\operatorname{arctg} x - \frac{\pi}{2}\right) = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{\operatorname{arctg} x - \frac{\pi}{2}}{\frac{1}{x}} = \boxed{\frac{0}{0}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \boxed{\frac{0}{0}} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{-x^2}{1+x^2}$$

Find $\lim_{x\to\infty} x(\operatorname{arctg} x - \frac{\pi}{2})$

$$\lim_{x\to\infty} x\left(\operatorname{arctg} x - \frac{\pi}{2}\right) = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{\operatorname{arctg} x - \frac{\pi}{2}}{\frac{1}{x}} = \boxed{\frac{0}{0}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \boxed{\frac{0}{0}} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{-x^2}{1+x^2} = \boxed{\frac{-\infty}{\infty}}$$

$$= \lim_{x\to\infty} \frac{-x^2}{x^2}$$

Find $\lim_{x\to\infty} x(\operatorname{arctg} x - \frac{\pi}{2})$

$$\lim_{x\to\infty} x\left(\operatorname{arctg} x - \frac{\pi}{2}\right) = \boxed{\infty \times 0} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{\operatorname{arctg} x - \frac{\pi}{2}}{\frac{1}{x}} = \boxed{\frac{0}{0}}$$

$$\overset{\text{l'H.r}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \boxed{\frac{0}{0}} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{-x^2}{1+x^2} = \boxed{\frac{-\infty}{\infty}}$$

$$= \lim_{x\to\infty} \frac{-x^2}{x^2} = -1$$