The primitive function to e^{-x^2}.

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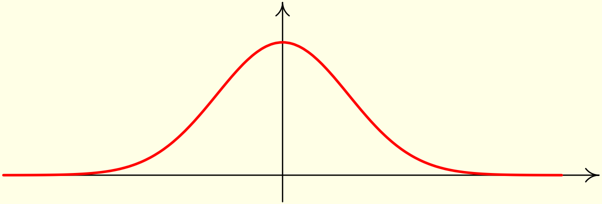
The problem to find $\int e^{-x^2}\,\mathrm{d}x$ seems to be easy, but, surprisingly, the converse is true.

Function $y = e^{-x^2}$:



The primitive function to e^{-x^2} exists, but it is not an elementary function. We find at least Taylor expansion to this primitive function about $x = 0$. Since all derivatives of $y(x) = e^x$ are e^x, we have

$$y^{(n)}(0) = e^x \Big|_{x=0} = e^0 = 1,$$

and the n-degree Taylor polynomial is

$$T_n(x) = e^0 + \sum_{i=1}^{n} \frac{y^{(n)}(0)}{n!} x^n = 1 + \sum_{i=1}^{n} \frac{1}{n!} x^n.$$

$$e^x \approx 1 + \sum_{i=1}^{n} \frac{1}{n!} x^n$$

(Taylor expansion)

$$e^{-x^2} \approx 1 + \sum_{i=1}^{n} (-1)^n \frac{1}{n!} x^{2n}$$

(replace x by $-x^2$)

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(integrate)

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$$\int e^{-x^2}\,\mathrm{d}x \approx x + \sum_{i=1}^{n} (-1)^n \frac{1}{n!(2n+1)} x^{2n+1} \qquad \text{(integrate)}$$

It can be shown that the infinite series

$$F(x) = x + \sum_{i=1}^{\infty} (-1)^n \frac{1}{n!(2n+1)} x^{2n+1}$$

is for every $x \in \mathbb{R}$ a well-defined differentiable function and $F'(x) = e^{-x^2}$.

$$e^x \approx 1 + \sum_{i=1}^{n} \frac{1}{n!} x^n$$ (Taylor expansion)

$$e^{-x^2} \approx 1 + \sum_{i=1}^{n} (-1)^n \frac{1}{n!} x^{2n}$$ (replace x by $-x^2$)

$$\int e^{-x^2} \, \mathrm{d}x \approx x + \sum_{i=1}^{n} (-1)^n \frac{1}{n!(2n+1)} x^{2n+1}$$ (integrate)

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is for every $x \in \mathbb{R}$ a well-defined differentiable function and $F'(x) = e^{-x^2}$. Hence $F(x)$ is an antiderivative to e^{-x^2}. This antiderivative cannot be written in closed finite form using basic elemetary function. The function

$$\frac{2}{\sqrt{\pi}} \int e^{-x^2} \, \mathrm{d}x = \frac{2}{\sqrt{\pi}} \left[x + \sum_{i=1}^{\infty} (-1)^n \frac{1}{n!(2n+1)} x^{2n+1} \right]$$

is one of the most famous nonelementary functions – the *error function*.