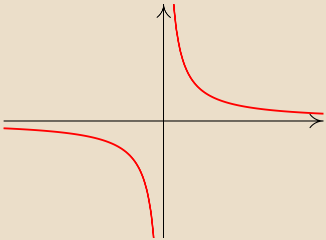
Basic elementary functions

July 20, 2006

1. Reciprocal value $y = \dfrac{1}{x}$

$\mathrm{Dom}(\frac{1}{x}) = \mathbb{R} \setminus \{0\}$,
$\mathrm{Im}(\frac{1}{x}) = \mathbb{R} \setminus \{0\}$,
there is neither x- nor y- intercept



The function $\frac{1}{x}$ is one-to-one, unbounded. The lines $x = 0$ and $y = 0$ are the vertical and horizontal asymptote, respectively.

$$\left(\frac{1}{x}\right)' = -x^{-2} = -\frac{1}{x^2}$$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to \infty} \frac{1}{x} = 0$$

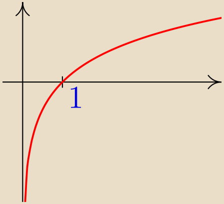
$$\lim_{x\to -\infty} \frac{1}{x} = 0$$

The function is its own inverse, i.e.

$$y = \frac{1}{x} \iff x = \frac{1}{y}.$$

2. Natural logarithm $y = \ln x$

$\mathrm{Dom}(\ln) = (0, \infty)$,
$\mathrm{Im}(\ln) = \mathbb{R}$,
x-intercept: $x = 1$



$\ln$ is an increasing function, concave down, one-to-one, with the vertical asymptote $x = 0$

$$(\ln x)' = \frac{1}{x}$$

$$\int \ln x \, \mathrm{d}x = x \ln x - x + C$$

$\ln 1 = 0$,
$\ln e = 1$,
$\ln e^x = x$ for all $x \in \mathbb{R}$

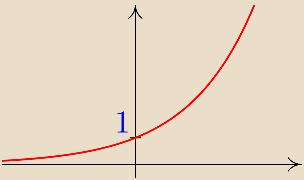
$$\ln(0+) = \lim_{x \to 0^+} \ln(x) = -\infty$$
$$\ln \infty = \lim_{x \to \infty} \ln(x) = \infty$$

The inverse function is $y = e^x$. Thus

$$y = \ln x \iff e^y = x.$$

3. Natural exponential function $y = e^x$

$\mathrm{Dom}(\exp) = \mathbb{R}$,
$\mathrm{Im}(\exp) = (0, \infty)$,
no x-intercept,
y-intercept is $y = 1$



$\exp$ is an increasing function, bounded below, concave up, one-to-one, with the horizontal asymptote $y = 0$ at $-\infty$

$(e^x)' = e^x$

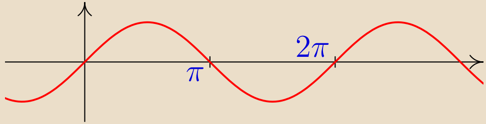
$$\int e^x \,\mathrm{d}x = e^x + C$$

$e^0 = 1$,
$e^{\ln x} = x$ for all $x > 0$
$e^{-\infty} = \lim\limits_{x\to-\infty} e^x = 0$
$e^{\infty} = \lim\limits_{x\to\infty} e^x = \infty$

The inverse function is $y = \ln x$. Thus

$$y = e^x \iff \ln(y) = x.$$

4. Sine function $y = \sin x$



$\operatorname{Dom}(\sin) = \mathbb{R}$,
$\operatorname{Im}(\sin) = [-1, 1]$,
$\sin$ is an odd 2π-periodic function
x-intercepts are $x = k\pi$, where k is an arbitrary integer.

$$(\sin x)' = \cos x$$
$$\int \sin x \, \mathrm{d}x = -\cos x + C$$

$\lim\limits_{x\to\pm\infty} \sin x$ does not exist
The $\sin$ function is one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and possesses on this interval the inverse function $y = \arcsin x$. Thus

$$y = \sin x \iff \arcsin y = x.$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

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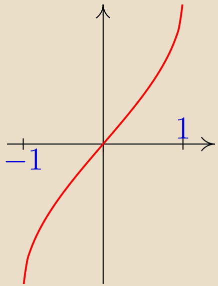
5. Inverse sine function $y = \arcsin x$

$\mathrm{Dom}(\arcsin) = [-1, 1]$,
$\mathrm{Im}(\arcsin) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



x-intercept is $x = 0$, since $\arcsin 0 = 0$

$\arcsin$ is an odd, increasing and bounded function

$$\arcsin x = \frac{\pi}{2} - \arccos x$$

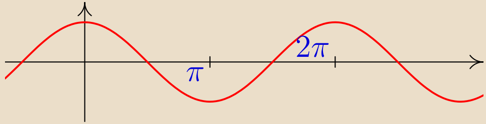
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

The $\arcsin$ function is one-to-one and the inverse function is the $\sin$ function. Thus

$$y = \arcsin x \iff \sin y = x.$$

x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arcsin x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

6. Cosine function $y = \cos x$



$\mathrm{Dom}(\cos) = \mathbb{R}$,
$\mathrm{Im}(\cos) = [-1, 1]$
x-intercepts are $x = \dfrac{\pi}{2} + k\pi$, where k is an arbitrary integer.
$\cos$ is a bounded, 2π-periodic, even function

$$(\cos x)' = -\sin x$$
$$\int \cos x \, \mathrm{d}x = \sin x + C$$

$\lim\limits_{x\to\pm\infty} \cos x$ does not exist
The $\cos$ function is one-to-one on $[0, \pi]$ and possesses on this interval the inverse function $y = \arccos x$. Thus

$$y = \cos x \iff \arccos y = x.$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

$y = \frac{1}{x}$
$y = \ln x$
$y = e^x$
$y = \sin x$
$y = \arcsin x$
$y = \cos x$
$y = \arccos x$
$y = \mathrm{tg}\, x$
$y = \mathrm{arctg}\, x$
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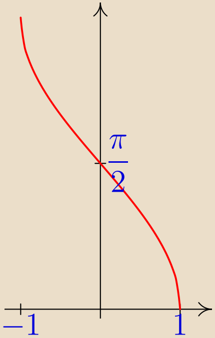
7. Inverse cosine function $y = \arccos x$

$\mathrm{Dom}(\arccos) = [-1, 1]$,
$\mathrm{Im}(\arccos) = [0, \pi]$



x-intercept is $x = 1$, since $\arccos 1 = 0$

y-intercept is $y = \frac{\pi}{2}$, since $\arccos = \frac{\pi}{2}$

$\arccos$ is a decreasing and bounded function

$$\arccos x = \frac{\pi}{2} - \arcsin x$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

The $\arccos$ function is one-to-one and the inverse function is $y = \cos x$. Thus

$$y = \arccos x \iff \cos y = x.$$

x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arccos x$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{6}$	0

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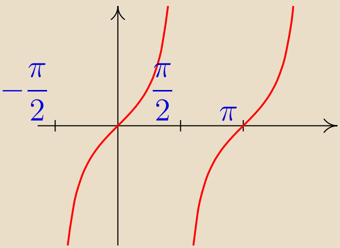
8. Tangent function $y = \operatorname{tg} x$

$\operatorname{Dom}(\operatorname{tg}) = \mathbb{R} \setminus \{\frac{\pi}{2} + k\frac{\pi}{2}; k \in \mathbb{Z}\}$,
$\operatorname{Im}(\operatorname{tg}) = \mathbb{R}$



$\operatorname{tg} x = \dfrac{\sin x}{\cos x}$
x-intercepts are the same as for the sine function.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\operatorname{tg} x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

tg is an odd, π-periodic function

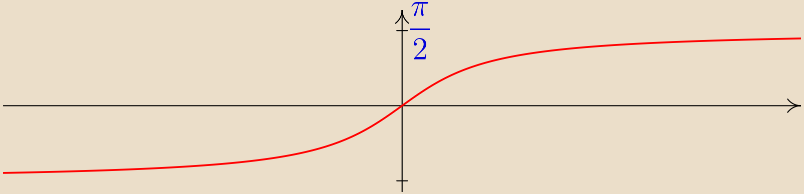
$$(\operatorname{tg} x)' = -\frac{1}{\cos^2 x}$$

$$\int \operatorname{tg} x \, dx = -\ln|\cos x| + C$$

The tg function is one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and possesses on this interval the inverse function $y = \operatorname{arctg} x$. Thus

$$y = \operatorname{tg} x \iff \operatorname{arctg} y = x.$$

9. Inverse tagent function $y = \operatorname{arctg} x$



$\operatorname{Dom}(\operatorname{arctg}) = \mathbb{R}$,

$\operatorname{Im}(\operatorname{arctg}) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

x-intercept is $x = 0$, since $\operatorname{arctg} 0 = 0$

arctg is an increasing, one-to-one, odd and bounded function

$\operatorname{arctg}\infty = \lim\limits_{x\to\infty} \operatorname{arctg} x = \frac{\pi}{2}$

$\operatorname{arctg}(-\infty) = -\frac{\pi}{2}$

The function possesses horizontal asymptotes $y = \frac{\pi}{2}$ at $+\infty$ and $y = -\frac{\pi}{2}$ at $-\infty$.

$$(\operatorname{arctg} x)' = \frac{1}{1+x^2}$$

The arctg function is one-to-one and the inverse function is $y = \operatorname{tg} x$. Thus

$$y = \operatorname{arctg} x \iff \operatorname{tg} y = x.$$

x	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
$\operatorname{arctg} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$

Important formulas:

$$\ln x + \ln y = \ln(xy)$$
$$\ln x - \ln y = \ln \frac{x}{y}$$
$$r \ln x = \ln x^r$$

$$e^x e^y = e^{x+y}$$
$$\frac{e^x}{e^y} = e^{x-y}$$
$$(e^x)^r = e^{rx}$$

$$\sin(x + 2\pi) = \sin x$$
$$\sin(-x) = -\sin(x)$$
$$\sin(2x) = 2 \sin x \cos x$$
$$\sin^2 x = 1 - \cos^2 x$$
$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$
$$\sin(x) = \sin(\pi - x)$$

$$\cos(x + 2\pi) = \cos x$$
$$\cos(-x) = \cos(x)$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$\cos^2 x = 1 - \sin^2 x$$
$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$