Difference equations

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1 Difference calculus

We will study sequences of real numbers. Recall that the sequence of real numbers is every function $a : \mathbb{N}_0 \to \mathbb{R}$. The arguments of this function we write usually as indexes and we arrange the values in the "sequence" $(a_n) = (a_0, a_1, a_2, a_3, \dots, a_i, \dots)$.

Remark 1 (important sequences).

- Each term in the *arithmetic sequence* can be obtained from the preceding one by adding some fixed value d, called difference. For example, the sequence $(-2, 1, 4, 7, \dots)$ is an arithmetic sequence with difference $d = 3$. The formula for a general term a_n of the arithmetic sequence with the first term a_0 and the difference d is

$$a_n = a_0 + nd.$$

- Each term in the *geometric sequence* can be obtained from the preceding one by multiplication by the fixed nonzero value q called quotient. The sequence $\left(84, 21, \frac{21}{4}, \frac{21}{16}, \dots\right)$ is a geometric sequence with quotient $\frac{1}{4}$. The formula for a general term a_n of the geometric sequence with quotient q and the first term a_0 is

$$a_n = a_0 q^n.$$

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- Each term in the *geometric sequence* can be obtained from the preceding one by multiplication by the fixed nonzero value q called quotient. The sequence $\left(84, 21, \frac{21}{4}, \frac{21}{16}, \dots\right)$ is a geometric sequence with quotient $\frac{1}{4}$. The formula for a general term a_n of the geometric sequence with quotient q and the first term a_0 is
$$a_n = a_0 q^n.$$

The fact whether a sequence is increasing or decreasing can be recognized from the difference of two consecutive terms. This is the motivation for the following definition.

Definition (difference of a sequence). Let a_n, a_{n+1} be two consecutive terms of the sequence (a_n). The difference $a_{n+1} - a_n$ is called a *(forward) difference of the sequence* (a_n) *at the point* n and denoted by Δa_n. Hence

$$\Delta a_n = a_{n+1} - a_n. \tag{1}$$

Remark 2. The letter Δ is a capital Greek letter “delta”.

Example 1. Consider the sequence $(a_n) = (n^2)$. The difference Δa_n is given by the relation

$$\Delta a_n = (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$$

for all $n \in \mathbb{N}_0$.

2 General difference equations

A differential equation is a relationship between an unknown function y *and the derivative of this function* y'*.* Recall that the derivative of a function is a quantity which describes, how fast the function y changes. An information about changes of terms in a sequence are included into its differences. Hence it is natural to expect that *we will be interested a relationship between a given sequence and its difference – difference equations*.

Definition (difference equation of the first kind). Let f be a real function. The equation

$$\Delta y_n = f(n, y_n) \tag{2}$$

is called a *first order difference equation of the first kind*.

Let us rewrite the difference Δy_n into $y_{n+1} - y_n$

$$y_{n+1} - y_n = f(n, y_n)$$

and let us solve the resulting equation for y_{n+1}, i.e.

$$y_{n+1} = f(n, y_n) + y_n.$$

This is a recurrence equation for the terms of the sequence (y_n). Given y_0, it is possible after a finite number of steps to find y_n for arbitrary n. This relation is equivalent to (2) and it is called a difference equation of the second kind.

Definition (difference equation of the second kind). Let φ be a real function of two variables. The equation

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Definition. Under a *solution* of equation (3) we understand every sequence defined on $\mathbb{N}_0$ which satisfies (3) for all $n \in \mathbb{N}_0$.

The problem to find a solution of (3) has infinitely many solutions (since there are infinitely many of possibilities for the starting value y_0). If there is a formula containing a real variable C, say

$$y_n = y(n, C),$$

such that for every C this formula defines a solution of (3), then this formula is called a *general solution* of equation (3).

Definition (particular solution). The problem to find the solution of (3) which for given numbers $\alpha \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$ satisfies an *initial condition*

$$y_\alpha = \beta \tag{4}$$

is called an *initial value problem* for equation (3), shortly IVP. The solution of this IVP is called a *particular solution*.

The particular solution can be usually obtained from the general solution for a convenient choice of the constant C (like for differential equations).

3 Linear difference equation

Definition (first order linear ΔR with constant coefficients)**.** Let $q \in \mathbb{R} \setminus \{0\}$. The difference equation

$$L_\Delta[y] = y_{n+1} - qy_n = f(n) \tag{5}$$

is called a *first order linear difference equation with constant coefficient*, shortly LΔR.

This equation is said to be *homogeneous* if $f(n) \equiv 0$ on $\mathbb{N}_0$ and *nonhomogeneous* otherwise.

Definition. The first order homogeneous difference equation

$$L_\Delta[y] = y_{n+1} - qy_n = 0 \tag{6}$$

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The operator L_Δ is a linear operator: for real numbers $C_1, C_2 \in \mathbb{R}$ and sequences a_n, b_n we have

$$L_\Delta[C_1 a_n + C_2 b_n] = C_1 L_\Delta[a_n] + C_2 L_\Delta[b_n]$$

Theorem 1 (general solution of nonhomogeneous equation). Let $\overline{y}_n$ be a particular solution of the nonhomogeneous linear equation

$$L_\Delta[\overline{y}_n] = f(n) \tag{5}$$

and Y_n be a general solution of the associated homogeneous equation

$$L_\Delta[Y_n] = 0$$

Then a general solution y_n of nonhomogeneous equation (5) is the function

$$y_n = \overline{y}_n + Y_n. \tag{7}$$

Proof.

$$L_\Delta[y_n] = L_\Delta[\overline{y}_n + Y_n] = \underbrace{L_\Delta[\overline{y}_n]}_{f(n)} + \underbrace{L_\Delta[Y_n]}_{0} = f(n) + 0 = f(n)$$

(Cobweb cycles in economics)

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4 Homogeneous linear difference equations

We start by investigating the homogeneous difference equations

$$y_{n+1} - qy_n = 0. \tag{6}$$

Remark 3 (trivial solution)**.** The zero sequence $y_n = 0$ for all $n \in \mathbb{N}_0$ is a solution of the homogeneous equation, as can be verified by direct substitution. This solution is called a *trivial solution*.

Remark 4 (solution of homogeneous equation)**.** Homogeneous linear difference equation is a recurrence equation $y_{n+1} = qy_n$ for the coefficients of geometric sequence and the general solution of this equation is $y_n = Cq^n$, where C is an arbitrary real constant.

Solve $\Delta y_n = y_n$.

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We simplify the “delta operator”

Solve $\Delta y_n = y_n$.

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Solve $\Delta y_n = y_n$.

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$$y_n = C(2)^n,\ C \in \mathbb{R}.$$

The equation $y_{n+1} - qy_n = 0$ has a solution $y_n = Cq^n$, where C is arbitrary real number.

Solve $\Delta y_n = y_n$.

$$y_{n+1} - y_n = y_n$$

$$y_{n+1} - 2y_n = 0$$

$$y_n = C(2)^n,\ C \in \mathbb{R}.$$

The problem is solved.

5 Nonomogeneous difference equations.

According to the linearity of the equation it is sufficient to find the particular solution of nonhomogeneous equation – hence the situation is about the same as for linear differential equations.
To find this particular solution in some special cases we introduce a method of "educated guessing" of the particular solution based on the following theorem.

Theorem 2 (particular solution)**.** Let ρ be a real number and $P_s(n)$ an s-degree polynomial of the variable n. Consider the first order linear difference equation with constant coefficient

$$y_{n+1} - qy_n = \rho^n P_s(n). \tag{8}$$

A particular solution of this equation can be found in the form

$$\overline{y}_n = \rho^n Q_s(n) n^r, \tag{9}$$

where $Q_s(n)$ is an s-degree polynomial of variable n. In this notation $r = 1$ if $\rho = q$ and $r = 0$ otherwise. The coefficients of the polynomial can be found by substitution (9) with undetermined coefficients into (8).

Write the form of the particular solution.

$$y_{n+1} - qy_n = P_s(n)\rho^n$$

$$y_n = \begin{cases} \boxed{Q_s(n)n\rho^n} & \rho = q \\ \boxed{Q_s(n)\rho^n} & \rho \neq q \end{cases}$$

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- $y_{n+1} + y_n = -2$

Write the form of the particular solution.

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- $y_{n+1} + y_n = 3(-2)^n$

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- $y_{n+1} - 2y_n = -2^n$

Write the form of the particular solution.

$$y_{n+1} - qy_n = P_s(n)\rho^n \qquad y_n = \begin{cases} Q_s(n)n\rho^n & \rho = q \\ Q_s(n)\rho^n & \rho \neq q \end{cases}$$

- $y_{n+1} + y_n = -2$ $\qquad y_n = a$
- $y_{n+1} + y_n = 3(-2)^n$ $\qquad y_n = a(-2)^n$
- $y_{n+1} + y_n = n - 1$ $\qquad y_n = an + b$
- $y_{n+1} - y_n = 4n$ $\qquad y_n = n(an + b)$
- $y_{n+1} - y_n = n^2 2^n$ $\qquad y_n = (an^2 + bn + c)2^n$
- $y_{n+1} - y_n = n^2(-2)^n$ $\qquad y_n = (an^2 + bn + c)(-2)^n$
- $y_{n+1} - 2y_n = n^2 2^n$ $\qquad y_n = n(an^2 + bn + c)2^n$
- $y_{n+1} - 2y_n = n^2$ $\qquad y_n = an^2 + bn + c$
- $y_{n+1} - 2y_n = -2^n$ $\qquad y_n = an2^n$

Write the form of the particular solution.

$$y_{n+1} - qy_n = P_s(n)\rho^n \qquad y_n = \begin{cases} Q_s(n)n\rho^n & \rho = q \\ Q_s(n)\rho^n & \rho \neq q \end{cases}$$

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- $y_{n+1} - y_n = 4n$ $\qquad y_n = n(an + b)$
- $y_{n+1} - y_n = n^2 2^n$ $\qquad y_n = (an^2 + bn + c)2^n$
- $y_{n+1} - y_n = n^2(-2)^n$ $\qquad y_n = (an^2 + bn + c)(-2)^n$
- $y_{n+1} - 2y_n = n^2 2^n$ $\qquad y_n = n(an^2 + bn + c)2^n$
- $y_{n+1} - 2y_n = n^2$ $\qquad y_n = an^2 + bn + c$
- $y_{n+1} - 2y_n = -2^n$ $\qquad y_n = an2^n$
- $y_{n+1} - 2y_n = (-2)^n$

Write the form of the particular solution.

$$y_{n+1} - qy_n = P_s(n)\rho^n \qquad y_n = \begin{cases} \boxed{Q_s(n)n\rho^n} & \rho = q \\ \boxed{Q_s(n)\rho^n} & \rho \neq q \end{cases}$$

- $y_{n+1} + y_n = -2$ — $y_n = a$
- $y_{n+1} + y_n = 3(-2)^n$ — $y_n = a(-2)^n$
- $y_{n+1} + y_n = n - 1$ — $y_n = an + b$
- $y_{n+1} - y_n = 4n$ — $y_n = n(an + b)$
- $y_{n+1} - y_n = n^2 2^n$ — $y_n = (an^2 + bn + c)2^n$
- $y_{n+1} - y_n = n^2(-2)^n$ — $y_n = (an^2 + bn + c)(-2)^n$
- $y_{n+1} - 2y_n = n^2 2^n$ — $y_n = n(an^2 + bn + c)2^n$
- $y_{n+1} - 2y_n = n^2$ — $y_n = an^2 + bn + c$
- $y_{n+1} - 2y_n = -2^n$ — $y_n = an2^n$
- $y_{n+1} - 2y_n = (-2)^n$ — $y_n = a(-2)^n$

Write the form of the particular solution.

$$y_{n+1} - qy_n = P_s(n)\rho^n \qquad y_n = \begin{cases} Q_s(n)n\rho^n & \rho = q \\ Q_s(n)\rho^n & \rho \neq q \end{cases}$$

- $y_{n+1} + y_n = -2$ $\qquad y_n = a$
- $y_{n+1} + y_n = 3(-2)^n$ $\qquad y_n = a(-2)^n$
- $y_{n+1} + y_n = n - 1$ $\qquad y_n = an + b$
- $y_{n+1} - y_n = 4n$ $\qquad y_n = n(an + b)$
- $y_{n+1} - y_n = n^2 2^n$ $\qquad y_n = (an^2 + bn + c)2^n$
- $y_{n+1} - y_n = n^2(-2)^n$ $\qquad y_n = (an^2 + bn + c)(-2)^n$
- $y_{n+1} - 2y_n = n^2 2^n$ $\qquad y_n = n(an^2 + bn + c)2^n$
- $y_{n+1} - 2y_n = n^2$ $\qquad y_n = an^2 + bn + c$
- $y_{n+1} - 2y_n = -2^n$ $\qquad y_n = an2^n$
- $y_{n+1} - 2y_n = (-2)^n$ $\qquad y_n = a(-2)^n$
- $2y_{n+1} - 4y_n = n^2 3^n$

Write the form of the particular solution.

$$y_{n+1} - qy_n = P_s(n)\rho^n \qquad y_n = \begin{cases} Q_s(n)n\rho^n & \rho = q \\ Q_s(n)\rho^n & \rho \neq q \end{cases}$$

- $y_{n+1} + y_n = -2$ $\qquad y_n = a$
- $y_{n+1} + y_n = 3(-2)^n$ $\qquad y_n = a(-2)^n$
- $y_{n+1} + y_n = n - 1$ $\qquad y_n = an + b$
- $y_{n+1} - y_n = 4n$ $\qquad y_n = n(an + b)$
- $y_{n+1} - y_n = n^2 2^n$ $\qquad y_n = (an^2 + bn + c)2^n$
- $y_{n+1} - y_n = n^2(-2)^n$ $\qquad y_n = (an^2 + bn + c)(-2)^n$
- $y_{n+1} - 2y_n = n^2 2^n$ $\qquad y_n = n(an^2 + bn + c)2^n$
- $y_{n+1} - 2y_n = n^2$ $\qquad y_n = an^2 + bn + c$
- $y_{n+1} - 2y_n = -2^n$ $\qquad y_n = an2^n$
- $y_{n+1} - 2y_n = (-2)^n$ $\qquad y_n = a(-2)^n$
- $2y_{n+1} - 4y_n = n^2 3^n$ $\qquad y_n = (an^2 + bn + c)3^n$

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$y_{n+1} - 2y_n = 0 \quad \Rightarrow$

We start with the correspnding homogeneous equation.

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$y_{n+1} - 2y_n = 0$ $\Rightarrow$ $Y_n = C2^n$

The general solution of $y_{n+1} - ay_n = 0$ is $y_n = Ca^n$, $C \in \mathbb{R}$. We have $a = 2$

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$y_{n+1} - 2y_n = 0 \quad \Rightarrow \quad Y_n = C2^n$

$$y_{n+1} - 2y_n = 2n2^n$$

We look for the particular solution of the nonhomogeneous equation.

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$y_{n+1} - 2y_n = 0 \quad \Rightarrow \quad Y_n = C2^n$

$$y_{n+1} - 2y_n = 2n2^n$$

$\overline{y}_n = n(an + b)2^n$

The right-hand side is $2n2^n$. It is a product of the linear polynomial and the power function 2^n. In the notation

$$y_{n+1} - qy_n = P(n)\rho^n$$

we have $\rho = q$ and the particular solution is in the form

$$\overline{y}_n = n \cdot (\text{linear polynomial}) \cdot 2^n.$$

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$y_{n+1} - 2y_n = 0 \quad \Rightarrow \quad Y_n = C2^n$

$$y_{n+1} - 2y_n = 2n2^n$$

$\overline{y}_n = n(an + b)2^n$

$\overline{y}_{n+1} = (n + 1)(a(n + 1) + b)2^{n+1}$

We have to substitute the particular solution in to the nonhomogeneous equation. To do this it is necessary to find y_{n+1}.

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$$y_{n+1} - 2y_n = 0 \qquad \Rightarrow \qquad Y_n = C2^n$$

$$y_{n+1} - 2y_n = 2n2^n$$

$$\overline{y}_n = n(an + b)2^n$$

$$\overline{y}_{n+1} = (n+1)(a(n+1) + b)2^{n+1} = (n+1)(an + a + b)2^n 2$$
$$= 2(an^2 + (a+b)n + an + a + b)2^n$$

We simplify. . .

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$$y_{n+1} - 2y_n = 0 \qquad \Rightarrow \qquad Y_n = C2^n$$

$$y_{n+1} - 2y_n = 2n2^n$$

$$\overline{y}_n = n(an+b)2^n$$

$$\overline{y}_{n+1} = (n+1)(a(n+1)+b)2^{n+1} = (n+1)(an+a+b)2^n 2$$
$$= 2(an^2 + (a+b)n + an + a + b)2^n$$

$$2(an^2 + (a+b)n + an + a + b)2^n - 2n(an+b)2^n = 2n2^n$$

…and substitute for y_n and y_{n+1}.

Solve ΔE $\quad y_{n+1} - 2y_n = 2n2^n$

$$y_{n+1} - 2y_n = 0 \quad \Rightarrow \quad Y_n = C2^n$$

$$y_{n+1} - 2y_n = 2n2^n$$

$$\overline{y}_n = n(an+b)2^n$$

$$\overline{y}_{n+1} = (n+1)(a(n+1)+b)2^{n+1} = (n+1)(an+a+b)2^n2$$
$$= 2(an^2+(a+b)n+an+a+b)2^n$$

$$2(an^2+(a+b)n+an+a+b)2^n - 2n(an+b)2^n = 2n2^n$$

$$(an^2+(a+b)n+an+a+b) - n(an+b) = n$$

We divide the equation by the factor 2^n...

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$$y_{n+1} - 2y_n = 0 \qquad \Rightarrow \qquad Y_n = C2^n$$

$$y_{n+1} - 2y_n = 2n2^n$$

$$\overline{y}_n = n(an+b)2^n$$

$$\begin{aligned}\overline{y}_{n+1} &= (n+1)(a(n+1)+b)2^{n+1} = (n+1)(an+a+b)2^n2\\ &= 2(an^2+(a+b)n+an+a+b)2^n\end{aligned}$$

$$2(an^2+(a+b)n+an+a+b)2^n - 2n(an+b)2^n = 2n2^n$$

$$(an^2+(a+b)n+an+a+b) - n(an+b) = n$$

$$2an + a + b = n$$

. . . and simpify.

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$$y_{n+1} - 2y_n = 0 \quad \Rightarrow \quad Y_n = C2^n$$

$$y_{n+1} - 2y_n = 2n2^n$$

$$\overline{y}_n = n(an+b)2^n$$

$$\begin{aligned}\overline{y}_{n+1} &= (n+1)(a(n+1)+b)2^{n+1} = (n+1)(an+a+b)2^n 2\\ &= 2(an^2+(a+b)n+an+a+b)2^n\end{aligned}$$

$$2(an^2+(a+b)n+an+a+b)2^n - 2n(an+b)2^n = 2n2^n$$

$$(an^2+(a+b)n+an+a+b) - n(an+b) = n$$

$$2an+a+b=n \quad \Rightarrow \quad \left[2a=1, \quad a+b=0\right]$$

The coefficient at the same powers of n have to be identical.

$$2an + a + b = 1 \cdot n + 0$$

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$$y_{n+1} - 2y_n = 0 \quad \Rightarrow \quad Y_n = C2^n$$

$$y_{n+1} - 2y_n = 2n2^n$$

$$\overline{y}_n = n(an+b)2^n$$

$$\begin{aligned}\overline{y}_{n+1} &= (n+1)(a(n+1)+b)2^{n+1} = (n+1)(an+a+b)2^n 2\\ &= 2(an^2 + (a+b)n + an + a + b)2^n\end{aligned}$$

$$2(an^2 + (a+b)n + an + a + b)2^n - 2n(an+b)2^n = 2n2^n$$

$$(an^2 + (a+b)n + an + a + b) - n(an+b) = n$$

$$2an + a + b = n \quad \Rightarrow \quad \left[2a = 1, \quad a + b = 0\right] \quad \Rightarrow \quad \left[a = \frac{1}{2}, \quad b = -\frac{1}{2}\right]$$

We solve these equations for a and b.

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$$y_{n+1} - 2y_n = 0 \quad \Rightarrow \quad Y_n = C2^n$$

$$y_{n+1} - 2y_n = 2n2^n$$

$$\overline{y}_n = n(an+b)2^n$$

$$\begin{aligned}\overline{y}_{n+1} &= (n+1)(a(n+1)+b)2^{n+1} = (n+1)(an+a+b)2^n 2\\ &= 2(an^2 + (a+b)n + an + a + b)2^n\end{aligned}$$

$$2(an^2 + (a+b)n + an + a + b)2^n - 2n(an+b)2^n = 2n2^n$$

$$(an^2 + (a+b)n + an + a + b) - n(an+b) = n$$

$$2an + a + b = n \quad \Rightarrow \quad \left[2a = 1, \quad a + b = 0\right] \quad \Rightarrow \quad \left[a = \frac{1}{2}, \quad b = -\frac{1}{2}\right]$$

$$y_n = n\left(\frac{1}{2}n - \frac{1}{2}\right) + C2^n$$

The particular solution is $\overline{y}_n = n\left(\frac{1}{2}n - \frac{1}{2}\right)2^n$. The general solution is the sum of this particular equations and the general solution of the corresponding homogeneous equation.

Solve ΔE $y_{n+1} - 2y_n = 2n2^n$

$$y_{n+1} - 2y_n = 0 \quad \Rightarrow \quad Y_n = C2^n$$

$$y_{n+1} - 2y_n = 2n2^n$$

$$\overline{y}_n = n(an+b)2^n$$

$$\overline{y}_{n+1} = (n+1)(a(n+1)+b)2^{n+1} = (n+1)(an+a+b)2^n 2$$
$$= 2(an^2 + (a+b)n + an + a + b)2^n$$

$$2(an^2 + (a+b)n + an + a + b)2^n - 2n(an+b)2^n = 2n2^n$$

$$(an^2 + (a+b)n + an + a + b) - n(an+b) = n$$

$$2an + a + b = n \quad \Rightarrow \quad \left[2a = 1, \quad a + b = 0\right] \quad \Rightarrow \quad \left[a = \frac{1}{2}, \quad b = -\frac{1}{2}\right]$$

$$y_n = n\left(\frac{1}{2}n - \frac{1}{2}\right) + C2^n = \frac{1}{2}n(n-1) + C2^n$$

We simpify the general solution.

Solve ΔE $\quad y_{n+1} - 2y_n = 2n2^n$

$$y_{n+1} - 2y_n = 0 \qquad \Rightarrow \qquad Y_n = C2^n$$

$$y_{n+1} - 2y_n = 2n2^n$$

$$\overline{y}_n = n(an+b)2^n$$

$$\begin{aligned}\overline{y}_{n+1} &= (n+1)(a(n+1)+b)2^{n+1} = (n+1)(an+a+b)2^n 2\\ &= 2(an^2+(a+b)n+an+a+b)2^n\end{aligned}$$

$$2(an^2+(a+b)n+an+a+b)2^n - 2n(an+b)2^n = 2n2^n$$

$$(an^2+(a+b)n+an+a+b) - n(an+b) = n$$

$$2an+a+b = n \quad \Rightarrow \quad \left[2a=1, \quad a+b=0\right] \quad \Rightarrow \quad \left[a=\frac{1}{2}, \quad b=-\frac{1}{2}\right]$$

$$y_n = n\left(\frac{1}{2}n - \frac{1}{2}\right) + C2^n = \frac{1}{2}n(n-1) + C2^n$$

The problem is solved.

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$$y_{n+1} + y_n = 0$$

We start with the corresponding homogeneous equation.

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$y_{n+1} + y_n = 0 \quad \Rightarrow \quad Y_n = C(-1)^n$

The general solution of $y_{n+1} - qy_n = 0$ is $y_n = Cq^n$, $C \in \mathbb{R}$. We have $q = -1$.

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$$y_{n+1} + y_n = 0 \qquad \Rightarrow \qquad Y_n = C(-1)^n$$

$$y_{n+1} + y_n = 2(-3)^n$$

Let us continue with the nonhomogeneous equation.

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$$y_{n+1} + y_n = 0 \qquad \Rightarrow \qquad Y_n = C(-1)^n$$

$$y_{n+1} + y_n = 2(-3)^n$$

$\overline{y}_n = a(-3)^n$

The right-hand side is $2(-3)^n$. Is is a product of the constant polynomial and the power function $(-3)^n$. In the notation

$$y_{n+1} - q y_n = P(n)\rho^n$$

we have $\rho \neq q$ and the particular solution is in the form

$$\overline{y}_n = (\text{constant polynomial}) \cdot (-3)^n.$$

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$$y_{n+1} + y_n = 0 \qquad \Rightarrow \qquad Y_n = C(-1)^n$$

$$y_{n+1} + y_n = 2(-3)^n$$

$$\overline{y}_n = a(-3)^n \qquad \Rightarrow \qquad \overline{y}_{n+1} = a(-3)^{n+1}$$

We evaluate $\overline{y}_{n+1}$...

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$$y_{n+1} + y_n = 0 \quad \Rightarrow \quad Y_n = C(-1)^n$$

$$y_{n+1} + y_n = 2(-3)^n$$

$$\overline{y}_n = a(-3)^n \quad \Rightarrow \quad \overline{y}_{n+1} = a(-3)^{n+1} = -3a(-3)^n$$

. . . simplify, . . .

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$$y_{n+1} + y_n = 0 \quad \Rightarrow \quad Y_n = C(-1)^n$$

$$y_{n+1} + y_n = 2(-3)^n$$

$$\overline{y}_n = a(-3)^n \quad \Rightarrow \quad \overline{y}_{n+1} = a(-3)^{n+1} = -3a(-3)^n$$

$$-3a(-3)^n + a(-3)^n = 2(-3)^n$$

. . . and substitute into the nonhomogeneous equation.

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$$y_{n+1} + y_n = 0 \quad \Rightarrow \quad Y_n = C(-1)^n$$

$$y_{n+1} + y_n = 2(-3)^n$$

$$\overline{y}_n = a(-3)^n \quad \Rightarrow \quad \overline{y}_{n+1} = a(-3)^{n+1} = -3a(-3)^n$$

$$-3a(-3)^n + a(-3)^n = 2(-3)^n$$

$$-3a + a = 2$$

We divide by the factor $(-3)^n$.

Solve ΔE $y_{n+1}+y_n=2(-3)^n$

$$y_{n+1}+y_n=0 \quad \Rightarrow \quad Y_n=C(-1)^n$$

$$y_{n+1}+y_n=2(-3)^n$$

$$\overline{y}_n=a(-3)^n \quad \Rightarrow \quad \overline{y}_{n+1}=a(-3)^{n+1}=-3a(-3)^n$$

$$-3a(-3)^n+a(-3)^n=2(-3)^n$$

$$-3a+a=2$$

$$a=-1$$

The solution is $a=-1$.

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$$y_{n+1} + y_n = 0 \quad \Rightarrow \quad Y_n = C(-1)^n$$

$$y_{n+1} + y_n = 2(-3)^n$$

$$\overline{y}_n = a(-3)^n \quad \Rightarrow \quad \overline{y}_{n+1} = a(-3)^{n+1} = -3a(-3)^n$$

$$-3a(-3)^n + a(-3)^n = 2(-3)^n$$

$$-3a + a = 2$$

$$a = -1$$

$$y_n = -(-3)^n + C(-1)^n$$

We use the value of a in the suggested form of particular solution. The particular solution is $\overline{y}_n = -(-3)^n$. The general solution is the sum of this particular solution and the general solution of the corresponding homogeneous equation.

Solve ΔE $y_{n+1} + y_n = 2(-3)^n$

$$y_{n+1} + y_n = 0 \quad \Rightarrow \quad Y_n = C(-1)^n$$

$$y_{n+1} + y_n = 2(-3)^n$$

$$\overline{y}_n = a(-3)^n \quad \Rightarrow \quad \overline{y}_{n+1} = a(-3)^{n+1} = -3a(-3)^n$$

$$-3a(-3)^n + a(-3)^n = 2(-3)^n$$

$$-3a + a = 2$$

$$a = -1$$

$$y_n = -(-3)^n + C(-1)^n$$

The problem is solved.

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$y_{n+1} + 4y_n = 0$

We start with the corresponding homogeneous equation.

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$y_{n+1} + 4y_n = 0 \quad \Rightarrow \quad Y_n = C(-4)^n$

The general solution of $y_{n+1} - qy_n = 0$ is $y_n = Cq^n$, $C \in \mathbb{R}$. We have $q = -4$.

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$y_{n+1} + 4y_n = 0 \quad \Rightarrow \quad Y_n = C(-4)^n$

$$y_{n+1} + 4y_n = 5n^2 - 1$$

We continue with the nonhomogeneous equation.

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$y_{n+1} + 4y_n = 0$ $\Rightarrow$ $Y_n = C(-4)^n$

$y_{n+1} + 4y_n = 5n^2 - 1$

$\overline{y}_n = an^2 + bn + c$

The right hand side is a quadratic polynomial. We can consider the power function on the right hand side as $(1)^n$. In the notation

$$y_{n+1} - qy_n = P(n)\rho^n$$

we have $\rho \neq q$ and the particular solution is in the form

$$\overline{y}_n = (\text{quadratic polynomial}) \cdot (1)^n.$$

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$y_{n+1} + 4y_n = 0 \quad \Rightarrow \quad Y_n = C(-4)^n$

$$y_{n+1} + 4y_n = 5n^2 - 1$$

$\overline{y}_n = an^2 + bn + c$

$\overline{y}_{n+1} = a(n+1)^2 + b(n+1) + c$

We evaluate $\overline{y}_{n+1}$...

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$y_{n+1} + 4y_n = 0 \quad \Rightarrow \quad Y_n = C(-4)^n$

$$y_{n+1} + 4y_n = 5n^2 - 1$$

$\overline{y}_n = an^2 + bn + c$

$\overline{y}_{n+1} = a(n+1)^2 + b(n+1) + c = an^2 + 2an + bn + a + b + c$

...simplify,...

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$y_{n+1} + 4y_n = 0 \quad \Rightarrow \quad Y_n = C(-4)^n$

$y_{n+1} + 4y_n = 5n^2 - 1$

$\overline{y}_n = an^2 + bn + c$

$\overline{y}_{n+1} = a(n+1)^2 + b(n+1) + c = an^2 + 2an + bn + a + b + c$

$an^2 + 2an + bn + a + b + c + 4(an^2 + bn + c) = 5n^2 - 1$

. . . and substitute into the nonhomogeneous equation.

Solve ΔE $y_{n+1}+4y_n=5n^2-1$

$y_{n+1}+4y_n=0 \quad \Rightarrow \quad Y_n=C(-4)^n$

$$y_{n+1}+4y_n=5n^2-1$$

$\overline{y}_n=an^2+bn+c$

$\overline{y}_{n+1}=a(n+1)^2+b(n+1)+c=an^2+2an+bn+a+b+c$

$$an^2+2an+bn+a+b+c+4(an^2+bn+c)=5n^2-1$$

$$5an^2+2an+5bn+a+b+5c=5n^2-1$$

We simplify the obtained equation. . .

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$$y_{n+1} + 4y_n = 0 \quad \Rightarrow \quad Y_n = C(-4)^n$$

$$y_{n+1} + 4y_n = 5n^2 - 1$$

$$\overline{y}_n = an^2 + bn + c$$

$$\overline{y}_{n+1} = a(n+1)^2 + b(n+1) + c = an^2 + 2an + bn + a + b + c$$

$$an^2 + 2an + bn + a + b + c + 4(an^2 + bn + c) = 5n^2 - 1$$

$$5an^2 + 2an + 5bn + a + b + 5c = 5n^2 - 1$$

$$5a = 5$$

$$2a + 5b = 0$$

$$a + b + 5c = -1$$

...and write the system for a, b and c.

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$$y_{n+1} + 4y_n = 0 \quad \Rightarrow \quad Y_n = C(-4)^n$$

$$y_{n+1} + 4y_n = 5n^2 - 1$$

$$\overline{y}_n = an^2 + bn + c$$

$$\overline{y}_{n+1} = a(n+1)^2 + b(n+1) + c = an^2 + 2an + bn + a + b + c$$

$$an^2 + 2an + bn + a + b + c + 4(an^2 + bn + c) = 5n^2 - 1$$

$$5an^2 + 2an + 5bn + a + b + 5c = 5n^2 - 1$$

$$\begin{aligned} 5a &= 5 \\ 2a + 5b &= 0 \\ a + b + 5c &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= 1 \\ b &= -\frac{2}{5} \\ c &= -\frac{8}{25} \end{aligned}$$

We solve the linear system for a, b and c.

Solve ΔE $y_{n+1} + 4y_n = 5n^2 - 1$

$$y_{n+1} + 4y_n = 0 \quad \Rightarrow \quad Y_n = C(-4)^n$$

$$y_{n+1} + 4y_n = 5n^2 - 1$$

$$\overline{y}_n = an^2 + bn + c$$

$$\overline{y}_{n+1} = a(n+1)^2 + b(n+1) + c = an^2 + 2an + bn + a + b + c$$

$$an^2 + 2an + bn + a + b + c + 4(an^2 + bn + c) = 5n^2 - 1$$

$$5an^2 + 2an + 5bn + a + b + 5c = 5n^2 - 1$$

$$\begin{aligned} 5a &= 5 \\ 2a + 5b &= 0 \\ a + b + 5c &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= 1 \\ b &= -\frac{2}{5} \\ c &= -\frac{8}{25} \end{aligned}$$

$$y_n = n^2 - \frac{2}{5}n - \frac{8}{25} + C(-4)^n$$

We use that constants in the suggested form of the particular solution and add the general solution of the corresponding homogeneous equation.

Solve ΔE $y_{n+1}+4y_n=5n^2-1$

$y_{n+1}+4y_n=0 \quad\Rightarrow\quad Y_n=C(-4)^n$

$$y_{n+1}+4y_n=5n^2-1$$

$\overline{y}_n=an^2+bn+c$

$\overline{y}_{n+1}=a(n+1)^2+b(n+1)+c=an^2+2an+bn+a+b+c$

$$an^2+2an+bn+a+b+c+4(an^2+bn+c)=5n^2-1$$

$$5an^2+2an+5bn+a+b+5c=5n^2-1$$

$$\begin{aligned}5a&=5\\2a+5b&=0\\a+b+5c&=-1\end{aligned} \quad\Rightarrow\quad \begin{aligned}a&=1\\b&=-\frac{2}{5}\\c&=-\frac{8}{25}\end{aligned}$$

$$y_n=n^2-\frac{2}{5}n-\frac{8}{25}+C(-4)^n$$

The problem is solved.

Solve

$$\Delta y_n = y_n - 2n - 3.$$

Solution: We convert equation into the form without differences

$$y_{n+1} - y_n = y_n - 2n - 3,$$
$$y_{n+1} - 2y_n = -2n - 3. \quad (10)$$

The associated homogeneous equation is equation

$$y_{n+1} - 2y_n = 0 \quad (11)$$

with general solution $Y_n = K \cdot 2^n$. The particular solution from Theorem 2 is

$$\overline{y}_n = 1^n(an + b) = an + b.$$

In this notation $\overline{y}_{n+1} = a(n+1) + b = an + a + b$ holds and hence after substitution into (10) we have

$$an + a + b - 2(an + b) = -2n - 3.$$

From here we obtain the system

$$-a = -2,$$
$$a - b = -3,$$

with solution $a = 2$, $b = 5$. General solution of the equation is the sequence $y_n = Y_n + \overline{y}_n = K \cdot 2^n + 2n + 5.$

Solve IVP

$$y_{n+1} + 2y_n = (-2)^n(n-1) \qquad y_1 = -2. \tag{12}$$

Solution: The associated homogeneous equation is

$$y_{n+1} + 2y_n = 0$$

with general solution

$$Y_n = C(-2)^n, \qquad C \in \mathbb{R}.$$

Particular solution from Theorem 2 is

$$\overline{y}_n = (-2)^n n(an+b) = (-2)^n(an^2+bn)$$

and for $\overline{y}_{n+1}$ we have (after simplification)

$$\overline{y}_{n+1} = (-2)^{n+1}(n+1)[a(n+1)+b] = (-2)^{n+1}(an^2+2an+bn+a+b).$$

Substitution into (12) yields:

$$(-2)^{n+1}(an^2+2an+bn+a+b) + 2(-2)^n(an^2+bn) = (-2)^n(n-1).$$

We divide equation by the factor $(-2)^n$

$$(-2)(an^2 + 2an + bn + a + b) + 2(an^2 + bn) = n - 1$$

and simplify. Remark that the terms containing the power n^2 cancel. We obtain equation

$$-4an - 2a - 2b = n - 1.$$

From the coefficients at the powers n^1 and n^0 we write the system

$$\begin{aligned} -4a &= 1, \\ -2a - 2b &= -1, \end{aligned}$$

with solution $a = -\frac{1}{4}$, $b = \frac{3}{4}$. The particular solution is a sequence $\overline{y}_n = (-2)^n n\Big(-\frac{1}{4}n + \frac{3}{4}\Big)$. The general solution is

$$\begin{aligned} y_n = Y_n + \overline{y}_n &= C(-2)^n + (-2)^n n\Big(-\frac{1}{4}n + \frac{3}{4}\Big) \\ &= \frac{1}{4}(-2)^n(4C - n^2 + 3n). \end{aligned}$$

We have to find the value of the constant C which makes the initial condition hold true. From the initial condition we substitute $n = 1$ and $y_n = -2$ and obtain the equation

$$-2 = \frac{1}{4}(-2)(4C - 1 + 3)$$

with solution $C = \dfrac{1}{2}$. Solution of the initial value problem is the sequence

$$y_n = \frac{1}{4}(-2)^n(2 - n^2 + 3n).$$

FINISHED.