# Second order linear diferential equations

Robert Mařík

May 9, 2006

# Contents

# 1 Second order linear differential equation

**Definition (second order linear differential equation).** Let $p$, $q$ and $f$ be functions continuous on the interval $I$. The equation

$$L[y] := y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

is said to be a *second order linear differential equation*. Under a *solution* of this equation we understand every function which has the second derivative on the interval $I$ and satisfies (1) for every $x \in I$.

Linearity of $L[\cdot]$. For every pair of two times differentiable functions $y_1$ and $y_2$ and every pair of real numbers $C_1$ and $C_2$ the following relation holds

$$L[C_1 y_1 + C_2 y_2] = C_1 L[y_1] + C_2 L[y_2].$$

Consider the 2-nd order linear differential equation $y'' = 0$. It is easy to see that every linear function $y = C_1 x + C_2$ satisfies this equation. This reveals the fact that one constant in the general solution and one initial condition is no more sufficient for higher order differential equations.

**Definition (initial value problem).** Let $\alpha \in I$ be a number from the interval $I$ and $\beta$, $\gamma$ be arbitrary real numbers. The problem to find the solution of equation (1) which satisfies *initial conditions*

$$\begin{cases} y(\alpha) = \beta \\ y'(\alpha) = \gamma \end{cases} \tag{2}$$

is called an *initial value problem* for equation (1), shortly IVP.

The solution of the IVP is called a *particular solution*.

**Definition (general solution).** There exists a single formula $y = y(x, C_1, C_2)$ such that every solution of equation (1) can be obtained from this formula for a convenient choice of the real numbers $C_1$, $C_2$. This formula is called a *general solution* of equation (1)

Consider the 2-nd order linear differential equation $y'' = 0$. It is easy to see that every linear function $y = C_1 x + C_2$ satisfies this equation. This reveals the fact that one constant in the general solution and one initial condition is no more sufficient for higher order differential equations.

**Definition (initial value problem).** Let $\alpha \in I$ be a number from the interval $I$ and $\beta$, $\gamma$ be arbitrary real numbers. The problem to find the solution of equation (1) which satisfies *initial conditions*

$$\begin{cases} y(\alpha) = \beta \\ y'(\alpha) = \gamma \end{cases} \tag{2}$$

is called an *initial value problem* for equation (1), shortly IVP.

The solution of the IVP is called a *particular solution*.

**Definition (general solution).** There exists a single formula $y = y(x, C_1, C_2)$ such that every solution of equation (1) can be obtained from this formula for a convenient choice of the real numbers $C_1$, $C_2$. This formula is called a *general solution* of equation (1)

Consider the 2-nd order linear differential equation $y'' = 0$. It is easy to see that every linear function $y = C_1x + C_2$ satisfies this equation. This reveals the fact that one constant in the general solution and one initial condition is no more sufficient for higher order differential equations.

**Definition (initial value problem).** Let $\alpha \in I$ be a number from the interval $I$ and $\beta$, $\gamma$ be arbitrary real numbers. The problem to find the solution of equation (1) which satisfies *initial conditions*

$$\begin{cases} y(\alpha) = \beta \\ y'(\alpha) = \gamma \end{cases} \tag{2}$$

is called an *initial value problem* for equation (1), shortly IVP.

The solution of the IVP is called a *particular solution*.

**Definition (general solution).** There exists a single formula $y = y(x, C_1, C_2)$ such that every solution of equation (1) can be obtained from this formula for a convenient choice of the real numbers $C_1$, $C_2$. This formula is called a *general solution* of equation (1)

**Definition (special types of 2nd order LDE).** Equation (1) is said to be *homogeneous* if $f(x) = 0$ for all $x \in I$ and *nonhomogeneous* otherwise.

Equation (1) is said to be an *equation with constant coefficients* if the functions $p(x)$ and $q(x)$ are constant functions.

**Remark 1 (existence and uniqueness of the solution).** Every IVP for equation (1) has a unique solution defined on the interval $I$.

**Remark 2 (trivial solution).** The constant function $y(x) \equiv 0$ is solution of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

as can be verified by direct substitution. This solution is called a *trivial solution* of the homogeneous equation.

**Definition (associated homogeneous equation).** Consider nonhomogeneous equation (1). Homogeneous equation

$$y'' + p(x)y' + q(x)y = 0. \tag{3}$$

with the left-hand side identical with equation (1) is called a *homogeneous equation associated to the nonhomogeneous equation* (1).

Like in the case of the first order linear differential equation, there exists a relationship between the solution of nonhomogeneous equation and the solution of the associated homogeneous equations, as follows from linearity of the operator $L[\cdot]$.

Let us study the homogeneous equation (3) first.

# 2 Homogeneous equation

Concerning the second order linear differential equation, we will show that the set of all solutions forms a kind of vector space. Let us introduce the concept of linear combinations and linear independence of functions first.

$$\boxed{y'' + p(x)y' + q(x)y = 0} \tag{H}$$

$$L[y](x) = y''(x) + p(x)y'(x) + q(x)y(x)$$

**Theorem 1.** If $y_1(x)$ and $y_2(x)$ are solutions of (H), then the function

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is a solution of the same equation for all real constants $C_1$, $C_2$.

*Proof:*

$$L[C_1 y_1 + C_2 y_2] = C_1 \underbrace{L[y_1]}_{0} + C_2 \underbrace{L[y_2]}_{0} = 0$$

Motivation. Let $y_1(x)$ and $y_2(x)$ be solutions of the homogeneous LDE. The function

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \tag{4}$$

is a solution of the same equation and we find conditions which ensure, that (4) is a general solution, i.e. any partiucular solution can be obtained from (4) by convenient choice of constants $C_1$, $C_2$. Differentiating (4) we get

$$y'(x) = C_1 y_1'(x) + C_2 y_2'(x)$$

and initial conditions $y(\alpha) = \beta$, $y'(\alpha) = \gamma$ yield the following linear system for $C_1$, $C_2$

$$\begin{aligned} \beta &= C_1 y_1(\alpha) + C_2 y_2(\alpha), \\ \gamma &= C_1 y_1'(\alpha) + C_2 y_2'(\alpha). \end{aligned} \tag{5}$$

From the linear algebra it is known that this system possesses a unique solution iff the matrix $\begin{pmatrix} y_1(\alpha) & y_2(\alpha) \\ y_1'(\alpha) & y_2'(\alpha) \end{pmatrix}$, is regular ( $\iff$ it has a full rank $\iff$ the determinant of this matrix is a nonzero number $\iff$ one of the rows is not a constant multiple of the other row). This is a motivation for the following following definition.

**Definition (linear (in-)dependence of functions).** Let $y_1$ and $y_2$ be functions defined on $I$. Suppose that neither $y_1$ nor $y_2$ equal zero everywhere on $I$. The functions $y_1$ and $y_2$ are said to be *linearly dependent* on the interval $I$ if there exists a number $k \in \mathbb{R}$ such that either $y_1(x) = ky_2(x)$ or $y_2(x) = ky_1(x)$ holds for all $x \in I$. In the opposite case the functions are said to be *linearly independent*.

**Theorem 2 (general solution of homogeneous equation).** Let $y_1$ and $y_2$ be linearly independent nontrivial solutions of the homogeneous equation (3). The function $y$ defined by the relation

$$y(x, C_1, C_2) = C_1 y_1(x) + C_2 y_2(x), \qquad C_1 \in \mathbb{R},\ C_2 \in \mathbb{R} \tag{6}$$

is general solution of (3) on the interval $I$.

**Definition (fundamental system of solutions).** The pair $y_1$ and $y_2$ of the functions from the preceding theorem is called a *fundamental system of the solutions* of equation (3).

In general, we can find the fundamental system of solutions only in the case of second order LDE with constant coefficients.

Consider the 2-nd order homogeneous LDE with *constant coefficients*

$$y'' + py' + qy = 0, \tag{7}$$

where $p, q \in \mathbb{R}$. Let us mention the following simple fact: Substituting $y = e^{zx}$, where $z$ is a real number, the the derivatives are $y' = ze^{zx}$, $y'' = z^2 e^{zx}$ and subsstituting into (7) we get

$$y'' + py' + q = z^2 e^{zx} + pze^{zx} + qe^{zx} = e^{zx}(z^2 + pz + q).$$

The exponential function on the right is positive and the function $y = e^{zx}$ is a solution of (7) iff

$$z^2 + pz + q = 0. \tag{8}$$

Equation (8) allows to solve (7) without any integration or differentiation.

**Definition (characteristic equation).** The quadratic equation (8) with unknown $z$ is called a *characteristic equation for equation* (7).

Consider the 2-nd order homogeneous LDE with *constant coefficients*

$$y'' + py' + qy = 0, \tag{7}$$

where $p, q \in \mathbb{R}$. Let us mention the following simple fact: Substituting $y = e^{zx}$, where $z$ is a real number, the the derivatives are $y' = ze^{zx}$, $y'' = z^2 e^{zx}$ and subsstituting into (7) we get

$$y'' + py' + q = z^2 e^{zx} + pze^{zx} + qe^{zx} = e^{zx}(z^2 + pz + q).$$

The exponential function on the right is positive and the function $y = e^{zx}$ is a solution of (7) iff

$$z^2 + pz + q = 0. \tag{8}$$

Equation (8) allows to solve (7) without any integration or differentiation.

**Definition (characteristic equation).** The quadratic equation (8) with unknown $z$ is called a *characteristic equation for equation (7)*.

**Theorem 3 (general solution of the equation with constant coefficients).** Consider the homogeneous second order ODE with constant coefficients

$$y'' + py' + qy = 0, \qquad p, q \in \mathbb{R} \quad (7)$$

and the quadratic auxiliary equation

$$z^2 + pz + q = 0 \quad (8)$$

in unknown $z$.

- If $z_1, z_2 \in \mathbb{R}$ are mutually different real zeros of the auxiliary equation (8), we put $y_1(x) = e^{z_1 x}$ and $y_2(x) = e^{z_2 x}$.
- If $z_1 \in \mathbb{R}$ is a double real zero of the auxiliary equation (8), we put $y_1(x) = e^{z_1 x}$ and $y_2(x) = x e^{z_1 x}$.
- If $z_{1,2} = \alpha \pm i\beta \notin \mathbb{R}$ are complex zeros of the auxiliary equation (8), we put $y_1(x) = e^{\alpha x} \cos(\beta x)$ and $y_2(x) = e^{\alpha x} \sin(\beta x)$.

The functions $y_1$ and $y_2$ form a fundamental system of solutions of equation (7). A general solution of equation (7) is

$$y(x, C_1, C_2) = C_1 y_1(x) + C_2 y_2(x),$$

where $C_{1,2}$ are arbitrary real numbers.

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$z^2 + 1 = 0 \qquad \Rightarrow$

We write the characteristic equation...

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow$

...and solve.

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$

The solution is in the set of complex numbers.

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

$$y_1(x) = \sin x$$

$$y_2(x) = \cos x$$

The real part of the roots of characteristic equation is $\alpha = 0$, the imaginary part is $\beta = 1$. The fundamental system is

$$y_1(x) = e^{\alpha x}\cos(\beta x)$$

and

$$y_2(x) = e^{\alpha x}\sin(\beta x).$$

Solve IVP $y'' + y = 0 \qquad y(0) = 1,\ y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

Fundamental system: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

Now we have the fundamental system...

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

Fundamental system: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

General solution: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

…and we can write the general solution. It is a linear combination of the functions from the fundamental system.

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

Fundamental system: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

General solution: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

$y'(x) = C_1 \cos x - C_2 \sin x$

We continue with initial conditions. We find $y'$...

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

Fundamental system: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

General solution: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

$y'(x) = C_1 \cos x - C_2 \sin x$

$$1 = C_1 \sin 0 + C_2 \cos 0$$

…and substitute for $y$…

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

Fundamental system: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

General solution: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

$y'(x) = C_1 \cos x - C_2 \sin x$

$$1 = C_1 \sin 0 + C_2 \cos 0$$
$$-1 = C_1 \cos 0 - C_2 \sin 0$$

...and for $y'$.

Solve IVP $\quad y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$

Fundamental system: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

General solution: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

$y'(x) = C_1 \cos x - C_2 \sin x$

$$\left.\begin{aligned} 1 &= C_1 \sin 0 + C_2 \cos 0 \\ -1 &= C_1 \cos 0 - C_2 \sin 0 \end{aligned}\right\} \Rightarrow C_1 = -1, \quad C_2 = 1$$

We solve the linear system.

Solve IVP $y'' + y = 0 \quad y(0) = 1,\ y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

Fundamental system: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

General solution: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

$y'(x) = C_1 \cos x - C_2 \sin x$

$$\left.\begin{aligned} 1 &= C_1 \sin 0 + C_2 \cos 0 \\ -1 &= C_1 \cos 0 - C_2 \sin 0 \end{aligned}\right\} \Rightarrow C_1 = -1, \quad C_2 = 1$$

Solution of IVP: $y(x) = -\sin x + \cos x$

Finaly we use the values of $C_1$ and $C_2$ in the general solution. This gives the particular solution of the initial value problem.

Solve DE $4y'' + 4y' + y = 0.$

Solve DE $4y'' + 4y' + y = 0.$

$4z^2 + 4z + 1 = 0$

We write the characteristic equation...

Solve DE $4y'' + 4y' + y = 0.$

$4z^2 + 4z + 1 = 0$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4}$$

... and solve it. The solution of the quadratic equation

$$az^2 + bz + c = 0$$

is given by the formula

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solve DE $4y'' + 4y' + y = 0.$

$$4z^2 + 4z + 1 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8}$$

Solve DE $4y'' + 4y' + y = 0.$

$$4z^2 + 4z + 1 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8} = -\frac{1}{2} \text{ \dots root of multiplicity 2}$$

The characteristic equation has a double root $z_{1,2} = -\frac{1}{2}$.

Solve DE $4y'' + 4y' + y = 0.$

$4z^2 + 4z + 1 = 0$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8} = -\frac{1}{2}$$ ... root of multiplicity 2

Fundamental system: $\begin{cases} y_1 = e^{-\frac{x}{2}} \\ y_2 = xe^{-\frac{x}{2}} \end{cases}$

In the case of double root $z$ of characteristic equation the fundamental system is

$$y_1(x) = e^{zx}, \qquad y_2(x) = xe^{zx}.$$

Solve DE $4y'' + 4y' + y = 0.$

$4z^2 + 4z + 1 = 0$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8} = -\frac{1}{2} \text{ . . . root of multiplicity 2}$$

Fundamental system: $\begin{cases} y_1 = e^{-\frac{x}{2}} \\ y_2 = xe^{-\frac{x}{2}} \end{cases}$

General solution: $y(x) = C_1 e^{-\frac{x}{2}} + C_2 x e^{-\frac{x}{2}}$ , $C_1, C_2 \in \mathbb{R}$

The general solution is a linear combination of the functions in the fundamental system.

Solve DE $4y'' + 4y' + y = 0.$

$4z^2 + 4z + 1 = 0$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8} = -\frac{1}{2}$$ …root of multiplicity 2

Fundamental system: $\begin{cases} y_1 = e^{-\frac{x}{2}} \\ y_2 = xe^{-\frac{x}{2}} \end{cases}$

General solution: $y(x) = C_1 e^{-\frac{x}{2}} + C_2 x e^{-\frac{x}{2}} = e^{-\frac{x}{2}}(C_1 + C_2 x)$, $C_1, C_2 \in \mathbb{R}$

We simplify the general solution and finish.

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$z^2 + 4z + 29 = 0$

The equation is linear second order homogeneous equation. We start with the characteristic equation.

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1}$$

The solution of the equation

$$az^2 + bz + c = 0$$

is given by the formula

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2}$$

We simplify ...

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

...and find the roots. We use the fact that

$$\sqrt{-100} = \sqrt{100}\sqrt{-1} = 10\sqrt{-1} = 10i.$$

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$y_1(x) = e^{-2x} \cos(5x)$ $\qquad$ $y_2(x) = e^{-2x} \sin(5x)$

From the roots of characteristic equation we establish the fundamental system. The real part of the roots of characteristic equation is $\alpha = -2$, the imaginary part is $\beta = 5$. The fundamental system is

$$y_1(x) = e^{\alpha x} \cos(\beta x) \quad \text{and} \quad y_2(x) = e^{\alpha x} \sin(\beta x).$$

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$y_1(x) = e^{-2x}\cos(5x) \qquad y_2(x) = e^{-2x}\sin(5x)$$

$$y(x) = C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x)$$

The general solution is a linear combination of the functions from the fundamental system.

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$y_1(x) = e^{-2x}\cos(5x) \qquad y_2(x) = e^{-2x}\sin(5x)$$

$$y(x) = C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x)$$

$$y'(x) = C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] + C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right]$$

We find the derivative $y'$. We have to use the product rule

$$(uv)' = u'v + uv'.$$

When differentiating $e^{-2x}$ and $\sin(5x)$ we use the chain rule.

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$y_1(x) = e^{-2x}\cos(5x)$ $\qquad$ $y_2(x) = e^{-2x}\sin(5x)$

$$y(x) = C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x)$$

$$y'(x) = C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] + C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right]$$

$$0 = C_1 + 0C_2$$

We substitute for $y$...

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$y_1(x) = e^{-2x}\cos(5x) \qquad y_2(x) = e^{-2x}\sin(5x)$$

$$y(x) = C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x)$$

$$y'(x) = C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] + C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right]$$

$$0 = C_1 + 0C_2$$
$$10 = -2C_1 + 5C_2$$

…and for $y'$.

Solve IVP $\quad y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$y_1(x) = e^{-2x}\cos(5x) \qquad y_2(x) = e^{-2x}\sin(5x)$$

$$\begin{aligned} y(x) =& C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x) \\ y'(x) =& C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] \\ &+ C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right] \end{aligned}$$

$$\begin{aligned} 0 &= C_1 + 0C_2 \\ 10 &= -2C_1 + 5C_2 \end{aligned} \Rightarrow C_1 = 0,\ C_2 = 2$$

We solve the linear system for $C_1$ and $C_2$.

Solve IVP $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$y_1(x) = e^{-2x}\cos(5x) \qquad y_2(x) = e^{-2x}\sin(5x)$$

$$\begin{aligned} y(x) =& C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x) \\ y'(x) =& C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] \\ &+ C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right] \end{aligned}$$

$$\begin{aligned} 0 &= C_1 + 0C_2 \\ 10 &= -2C_1 + 5C_2 \end{aligned} \Rightarrow C_1 = 0,\ C_2 = 2$$

$$y_p(x) = 2e^{-2x}\sin(5x)$$

We use the coefficients $C_1$ and $C_2$. The problem is resolved.

# 3 Nonhomogeneous equation

The following theorem suggests a method how to find a general solution of the 2nd order LDE. This theorem shows that this general solution has a very special form — we can write this solution from three special functions, as formula (9) shows.

**Theorem 4 (general solution of nonhomogeneous second order LDE).** Let $y_1(x)$ and $y_2(x)$ be fundamental system of solutions of the homogeneous LDE (3) and $y_p(x)$ be an arbitrary particular solution of the nonhomogeneous LDE (1). Then the function

$$y(x, C_1, C_2) = C_1 y_1(x) + C_2 y_2(x) + y_p(x), \qquad C_1 \in \mathbb{R},\ C_2 \in \mathbb{R} \tag{9}$$

is a general solution of the nonhomogeneous LDE (1).

## 3.1 Ingenious guessing

In this chapter we use the method of inspired guessing of particular solution. The main idea is to look for a particular solution of the nonhomogeneous equation is a special form involving parameters and adjust the parameters to make the equation hold true for all $x$. Following this method, it is necessary to look for the particular solution in correct form and this method works for some special right hand sides only. Details can be found in textbooks on differential equations.

Solve $y'' + 4y = xe^x$.

Hint: One of the particular solutions can be found in the form

$$y(x) = \text{linear polynomial} \cdot e^x$$

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

We find the associated homogeneous equation, its characteristic equation and its two independent solutions.

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$y = (ax + b)e^x$

We follow the hint and look for the particular solution in the form

$$y(x) = (\text{linear function}) \cdot e^x.$$

We write the most general fucntion of this type and adjust the parameters $a$ and $b$ to make the equation true for all real $x$

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$$y = (ax + b)e^x$$
$$y' = ae^x + (ax + b)e^x = (ax + a + b)e^x$$

We have to find the second derivative. We find the first derivative by the product rule

$$(uv)' = u'v + uv'$$

and simplify.

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$$y = (ax + b)e^x$$
$$y' = ae^x + (ax + b)e^x = (ax + a + b)e^x$$
$$y'' = ae^x + (ax + a + b)e^x = (ax + 2a + b)e^x$$

We find the second derivative and simplify. We use the product rule again.

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$y = (ax + b)e^x$ $\quad y' = (ax + a + b)e^x$ $\quad y'' = (ax + 2a + b)e^x$

$$\overbrace{(ax + 2a + b)e^x}^{y''} + 4\overbrace{(ax + b)e^x}^{y} = xe^x$$

We plug the function and its second derivative into the equation.

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$$y = (ax+b)e^x \qquad y' = (ax+a+b)e^x \qquad y'' = (ax+2a+b)e^x$$

$$\overbrace{(ax+2a+b)e^x}^{y''} + 4\overbrace{(ax+b)e^x}^{y} = xe^x$$

$$(ax+2a+b) + 4(ax+b) = x$$

We divide the equation through by the factor $e^x$. This converts the equation into equation between two polynomials.

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$$y = (ax+b)e^x \qquad y' = (ax+a+b)e^x \qquad y'' = (ax+2a+b)e^x$$

$$\overbrace{(ax+2a+b)e^x}^{y''} + 4\overbrace{(ax+b)e^x}^{y} = xe^x$$

$$(ax + 2a + b) + 4(ax + b) = x$$

$$x(a + 4a) + (2a + b + 4b) = x + 0$$

We collect like powers of $x$.

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$$y = (ax + b)e^x \qquad y' = (ax + a + b)e^x \qquad y'' = (ax + 2a + b)e^x$$

$$\overbrace{(ax + 2a + b)e^x}^{y''} + 4\overbrace{(ax + b)e^x}^{y} = xe^x$$

$$(ax + 2a + b) + 4(ax + b) = x$$

$$x(a + 4a) + (2a + b + 4b) = x + 0$$

$$5a = 1$$

$$2a + 5b = 0$$

Two polynomials are equal iff the corresponding coefficients are equal. This yields a system of two equations.

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$$y = (ax+b)e^x \qquad y' = (ax+a+b)e^x \qquad y'' = (ax+2a+b)e^x$$

$$\overbrace{(ax+2a+b)e^x}^{y''} + 4\overbrace{(ax+b)e^x}^{y} = xe^x$$

$$(ax+2a+b) + 4(ax+b) = x$$

$$x(a+4a) + (2a+b+4b) = x + 0$$

$$\begin{aligned} 5a &= 1 \\ 2a + 5b &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} a &= \frac{1}{5} \\ b &= -\frac{2}{5}a = -\frac{2}{25} \end{aligned}$$

We solve the system and find $a$ and $b$.

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$$y = (ax + b)e^x \qquad y' = (ax + a + b)e^x \qquad y'' = (ax + 2a + b)e^x$$

$$\overbrace{(ax + 2a + b)e^x}^{y''} + 4\overbrace{(ax + b)e^x}^{y} = xe^x$$

$$(ax + 2a + b) + 4(ax + b) = x$$

$$x(a + 4a) + (2a + b + 4b) = x + 0$$

$$\begin{aligned} 5a &= 1 \\ 2a + 5b &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= \frac{1}{5} \\ b &= -\frac{2}{5}a = -\frac{2}{25} \end{aligned}$$

$$y(x) = \left(\frac{1}{5}x - \frac{2}{25}\right)e^x + C_1 \cos 2x + C_2 \sin 2x$$

We find the particular and the general solution.

Solve $y'' + 4y = xe^x$.

$$y'' + 4y = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} y_1(x) = \cos 2x \\ y_2(x) = \sin 2x \end{cases}$$

$$y = (ax + b)e^x \qquad y' = (ax + a + b)e^x \qquad y'' = (ax + 2a + b)e^x$$

$$\overbrace{(ax + 2a + b)e^x}^{y''} + 4\overbrace{(ax + b)e^x}^{y} = xe^x$$

$$(ax + 2a + b) + 4(ax + b) = x$$

$$x(a + 4a) + (2a + b + 4b) = x + 0$$

$$\begin{aligned} 5a &= 1 \\ 2a + 5b &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} a &= \frac{1}{5} \\ b &= -\frac{2}{5}a = -\frac{2}{25} \end{aligned}$$

$$y(x) = \left(\frac{1}{5}x - \frac{2}{25}\right)e^x + C_1 \cos 2x + C_2 \sin 2x$$

Finished!

Solve $y'' - 3y' + 2y = x^2 - 4$.

Hint: One of the particular solutios is quadratic function.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

We find the associated homogeneous equation, the characteristic equation and the fundamental system of the associated homogeneous equation.

Solve $y'' - 3y' + 2y = x^2 - 4.$

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c$$

Following the hint, we look for a particular solution. We write the most general quadratic function and adjust its parameters $a$, $b$ and $c$ to make the equation hold true for every $x$.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c \qquad y' = 2ax + b$$

We find the derivatives of the function $y$.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c \qquad y' = 2ax + b \qquad y'' = 2a$$

We find the derivatives of the function $y$.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c \qquad y' = 2ax + b \qquad y'' = 2a$$

$$\overbrace{2a}^{y''} - 3\,\overbrace{(2ax + b)}^{y'} + 2\,\overbrace{(ax^2 + bx + c)}^{y} = x^2 - 4$$

We substitute the functino and its derivatives into the equation.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c \qquad y' = 2ax + b \qquad y'' = 2a$$

$$\overbrace{2a}^{y''} - 3\overbrace{(2ax + b)}^{y'} + 2\overbrace{(ax^2 + bx + c)}^{y} = x^2 - 4$$

$$x^2(2a) + x(2b - 6a) + (2a - 3b + 2c) = 1 \cdot x^2 + 0 \cdot x - 4$$

We collect like powers of $x$ and get the equation between two polynomials.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c \qquad y' = 2ax + b \qquad y'' = 2a$$

$$\overbrace{2a}^{y''} - 3\overbrace{(2ax + b)}^{y'} + 2\overbrace{(ax^2 + bx + c)}^{y} = x^2 - 4$$

$$x^2(2a) + x(2b - 6a) + (2a - 3b + 2c) = 1 \cdot x^2 + 0 \cdot x - 4$$

$$2a = 1$$

$$2b - 6a = 0$$

$$2a - 3b + 2c = -4$$

Two polynomials are equal for every $x$ iff the coefficients at like powers of $x$ are equal.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c \qquad y' = 2ax + b \qquad y'' = 2a$$

$$\overbrace{2a}^{y''} - 3\overbrace{(2ax+b)}^{y'} + 2\overbrace{(ax^2+bx+c)}^{y} = x^2 - 4$$

$$x^2(2a) + x(2b - 6a) + (2a - 3b + 2c) = 1 \cdot x^2 + 0 \cdot x - 4$$

$$\begin{aligned} 2a &= 1 \\ 2b - 6a &= 0 \\ 2a - 3b + 2c &= -4 \end{aligned} \quad \Rightarrow \quad a = \frac{1}{2}$$

We solve the system of linear equations. We find $a$ from the first eqaution.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c \qquad y' = 2ax + b \qquad y'' = 2a$$

$$\overbrace{2a}^{y''} - 3\,\overbrace{(2ax + b)}^{y'} + 2\,\overbrace{(ax^2 + bx + c)}^{y} = x^2 - 4$$

$$x^2(2a) + x(2b - 6a) + (2a - 3b + 2c) = 1 \cdot x^2 + 0 \cdot x - 4$$

$$\begin{aligned} 2a &= 1 \\ 2b - 6a &= 0 \\ 2a - 3b + 2c &= -4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= \frac{1}{2} \\ b &= 3a = \frac{3}{2} \end{aligned}$$

We find the value of $b$ from the second equation.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c \qquad y' = 2ax + b \qquad y'' = 2a$$

$$\overbrace{2a}^{y''} - 3\overbrace{(2ax + b)}^{y'} + 2\overbrace{(ax^2 + bx + c)}^{y} = x^2 - 4$$

$$x^2(2a) + x(2b - 6a) + (2a - 3b + 2c) = 1 \cdot x^2 + 0 \cdot x - 4$$

$$\begin{aligned} 2a &= 1 \\ 2b - 6a &= 0 \\ 2a - 3b + 2c &= -4 \end{aligned} \Rightarrow \quad a = \frac{1}{2} \quad b = 3a = \frac{3}{2}$$

$$c = (-4 - 2a + 3b)\frac{1}{2} = \left(-4 - 1 + \frac{9}{2}\right)\frac{1}{2} = -\frac{1}{4}$$

We find the value of $c$ from the last.

Solve $y'' - 3y' + 2y = x^2 - 4$.

$$y'' - 3y' + 2y = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} 1 \\ 2 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}$$

$$y = ax^2 + bx + c \qquad y' = 2ax + b \qquad y'' = 2a$$

$$\overbrace{2a}^{y''} - 3\overbrace{(2ax+b)}^{y'} + 2\overbrace{(ax^2+bx+c)}^{y} = x^2 - 4$$

$$x^2(2a) + x(2b - 6a) + (2a - 3b + 2c) = 1 \cdot x^2 + 0 \cdot x - 4$$

$$\begin{aligned} 2a &= 1 \\ 2b - 6a &= 0 \\ 2a - 3b + 2c &= -4 \end{aligned} \Rightarrow \begin{aligned} a &= \frac{1}{2} \\ b &= 3a = \frac{3}{2} \\ c &= (-4 - 2a + 3b)\frac{1}{2} = \left(-4 - 1 + \frac{9}{2}\right)\frac{1}{2} = -\frac{1}{4} \end{aligned}$$

$$y = C_1 e^x + C_2 e^{2x} + \frac{1}{2}x^2 + \frac{3}{2}x - \frac{1}{4}$$

Finished!

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

Hint: Find the particular solution in the form

$$y(x) = (\text{linear}) \cdot e^{-x}.$$

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

We find independent solutoins of the associated homogeneous equation.

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

$$y = e^{-x}(ax + b)$$

We write the particular solution with parameters which have to be adjusted to make the equation hold true for every $x$.

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

$$y = e^{-x}(ax + b)$$

$$y' = -e^{-x}(ax + b) + e^{-x}a = e^{-x}(a - b - ax)$$

We differentiate by the product rule and simplify.

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

$$y = e^{-x}(ax+b)$$

$$y' = -e^{-x}(ax+b) + e^{-x}a = e^{-x}(a - b - ax)$$

$$y'' = -e^{-x}(a - b - ax) + e^{-x}(-a) = e^{-x}(ax + b - 2a)$$

We differentiate by the product rule and simplify.

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

$$y = e^{-x}(ax+b) \qquad y' = e^{-x}(a-b-ax) \qquad y'' = e^{-x}(ax+b-2a)$$

$$\overbrace{e^{-x}(ax+b-2a)}^{y''} + \overbrace{e^{-x}(a-b-ax)}^{y'} - 6\,\overbrace{e^{-x}(ax+b)}^{y} = e^{-x}(x+1)$$

We plug the function and its derivatives into the equation.

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

$$y = e^{-x}(ax+b) \qquad y' = e^{-x}(a-b-ax) \qquad y'' = e^{-x}(ax+b-2a)$$

$$\overbrace{e^{-x}(ax+b-2a)}^{y''} + \overbrace{e^{-x}(a-b-ax)}^{y'} - 6\overbrace{e^{-x}(ax+b)}^{y} = e^{-x}(x+1)$$

$$ax + b - 2a + a - b - ax - 6ax - 6b = x + 1$$

$$x(-6a) - 6b - a = x + 1$$

We divide the equation through the common factor $e^{-x}$.

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

$$y = e^{-x}(ax+b) \qquad y' = e^{-x}(a-b-ax) \qquad y'' = e^{-x}(ax+b-2a)$$

$$\overbrace{e^{-x}(ax+b-2a)}^{y''} + \overbrace{e^{-x}(a-b-ax)}^{y'} - 6\overbrace{e^{-x}(ax+b)}^{y} = e^{-x}(x+1)$$

$$ax+b-2a+a-b-ax-6ax-6b = x+1$$

$$x(-6a) - 6b - a = x+1$$

$$-6a = 1$$

$$-6b - a = 1$$

We write the equations for coefficients of the polynomial.

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

$$y = e^{-x}(ax+b) \qquad y' = e^{-x}(a-b-ax) \qquad y'' = e^{-x}(ax+b-2a)$$

$$\overbrace{e^{-x}(ax+b-2a)}^{y''} + \overbrace{e^{-x}(a-b-ax)}^{y'} - 6\overbrace{e^{-x}(ax+b)}^{y} = e^{-x}(x+1)$$

$$ax+b-2a+a-b-ax-6ax-6b = x+1$$

$$x(-6a)-6b-a = x+1$$

$$\begin{aligned} -6a &= 1 \\ -6b-a &= 1 \end{aligned} \Rightarrow \begin{aligned} a &= -\frac{1}{6} \\ b &= \frac{-a-1}{6} = -\frac{5}{36} \end{aligned}$$

We find $a$ and $b$.

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

$$y = e^{-x}(ax+b) \qquad y' = e^{-x}(a-b-ax) \qquad y'' = e^{-x}(ax+b-2a)$$

$$\overbrace{e^{-x}(ax+b-2a)}^{y''} + \overbrace{e^{-x}(a-b-ax)}^{y'} - 6\overbrace{e^{-x}(ax+b)}^{y} = e^{-x}(x+1)$$

$$ax+b-2a+a-b-ax-6ax-6b = x+1$$

$$x(-6a) - 6b - a = x+1$$

$$\begin{aligned} -6a &= 1 \\ -6b - a &= 1 \end{aligned} \Rightarrow \begin{aligned} a &= -\frac{1}{6} \\ b &= \frac{-a-1}{6} = -\frac{5}{36} \end{aligned}$$

$$y(x) = -\left(\frac{1}{6}x + \frac{5}{36}\right)e^{-x} + C_1 e^{2x} + C_2 e^{-3x}$$

We find the particular and general solution.

Solve $y'' + y' - 6y = e^{-x}(x+1)$.

$$y'' + y' - 6y = 0 \Rightarrow z^2 + z - 6 \Rightarrow z_{1,2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow \begin{cases} y_1(x) = e^{2x} \\ y_2(x) = e^{-3x} \end{cases}$$

$$y = e^{-x}(ax+b) \qquad y' = e^{-x}(a-b-ax) \qquad y'' = e^{-x}(ax+b-2a)$$

$$\overbrace{e^{-x}(ax+b-2a)}^{y''} + \overbrace{e^{-x}(a-b-ax)}^{y'} - 6\,\overbrace{e^{-x}(ax+b)}^{y} = e^{-x}(x+1)$$

$$ax+b-2a+a-b-ax-6ax-6b = x+1$$

$$x(-6a)-6b-a = x+1$$

$$\begin{aligned} -6a &= 1 \\ -6b-a &= 1 \end{aligned} \Rightarrow \begin{aligned} a &= -\frac{1}{6} \\ b &= \frac{-a-1}{6} = -\frac{5}{36} \end{aligned}$$

$$y(x) = -\left(\frac{1}{6}x + \frac{5}{36}\right)e^{-x} + C_1\,e^{2x} + C_2\,e^{-3x}$$

## 3.2 Variation of constants

If the method of ingenious guessing fails (i.e. if we are not able to guess the correct form of the particular solution) then we can use the following method of variation of constants. We start with the solution of the associated homogeneous equation, replace the constants by functions and find functions which make the equation true.

**Theorem 5 (solution of the nonhomogeneous LDE with constant coefficients).** Consider the second order LDE with constant coefficients

$$y'' + py' + qy = f(x). \tag{10}$$

Let $y_1(x)$ and $y_2(x)$ be a fundamental system of solutions of the associated homogeneous equation (7) from Theorem 3. Let $A(x)$ and $B(x)$ be differentiable functions with derivatives $A'(x)$ and $B'(x)$ which satisfy the system of equations

$$\boxed{\begin{aligned} A'(x)y_1(x) + B'(x)y_2(x) &= 0,\\ A'(x)y_1'(x) + B'(x)y_2'(x) &= f(x).\end{aligned}} \tag{11}$$

The function $y_p(x)$ defined by the relation

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) \tag{12}$$

is a particular solution of (10). The function

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x), \quad C_1 \in \mathbb{R}, C_2 \in \mathbb{R},$$

is a general solution of (10).

Remark 3 (Cramer's rule, wronskian). The linear system (11) possesses always a unique solution $A(x)$, $B(x)$. Among others, the system can be solved by the Cramer's rule. The determinant of the coefficient matrix is

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

This determinant is called a *wronskian of the functions $y_1(x)$ and $y_2(x)$* and it is always nonzero. With the auxiliary determinants

$$W_1 = \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix}, \qquad W_2 = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix},$$

we obtain the formula for unknowns $A'(x)$ and $B'(x)$ in the form

$$A'(x) = \frac{W_1}{W}, \qquad B'(x) = \frac{W_2}{W}.$$

The Cramer's rule is very effective especially when the fundamental system of the solutions is formed by the functions $\sin \beta x$ and $\cos \beta x$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y'' - 4y' + 4y = 0$$

The equation is not homogeneous. We start with the corresponding homogeneous equation.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad z^2 - 4z + 4 = 0$$

We write the characteristic equation for the homogeneous equation...

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad z^2 - 4z + 4 = 0 \quad \Rightarrow \quad z_{1,2} = \frac{4 - \sqrt{16 - 4.1.4}}{2.1}$$

. . . and solve it.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad z^2 - 4z + 4 = 0 \quad \Rightarrow \quad z_{1,2} = \frac{4 - \sqrt{16 - 4.1.4}}{2.1} = 2$$

The characteristic equation has a double root $z_{1,2} = 2$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad z^2 - 4z + 4 = 0 \quad \Rightarrow \quad z_{1,2} = \frac{4 - \sqrt{16 - 4.1.4}}{2.1} = 2$$

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

The fundamental system is in the case of double root $z$ given by the functions

$$y_1 = e^{zx}, \quad y_2 = xe^{zx}.$$

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$$

We look for the particular solution in this form.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1 + 2x)$

We find the derivatives $y_1'(x)$ and $y_2'(x)$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$$\begin{aligned} A'e^{2x} + B'xe^{2x} &= 0 \\ 2A'e^{2x} + B'(1+2x)e^{2x} &= e^{-x} \end{aligned}$$

We write the system for the coefficients $A'(x)$ and $B'(x)$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1 + 2x)$

$$\begin{aligned} A'e^{2x} + B'xe^{2x} &= 0 \\ 2A'e^{2x} + B'(1 + 2x)e^{2x} &= e^{-x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'x &= 0 \\ 2A' + B'(1 + 2x) &= e^{-3x} \end{aligned}$$

We divide both equations by the factor $e^{2x}$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$$\begin{aligned} A'e^{2x} + B'xe^{2x} &= 0 \\ 2A'e^{2x} + B'(1+2x)e^{2x} &= e^{-x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'x &= 0 \\ 2A' + B'(1+2x) &= e^{-3x} \end{aligned}$$

$$B' = e^{-3x}$$

We multiply the first equation by $(-2)$ and add to the second equation. We obtain

$$B' = e^{-3x}.$$

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$$\begin{aligned} A'e^{2x} + B'xe^{2x} &= 0 \\ 2A'e^{2x} + B'(1+2x)e^{2x} &= e^{-x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'x &= 0 \\ 2A' + B'(1+2x) &= e^{-3x} \end{aligned}$$

$B' = e^{-3x}$

$A' = -xe^{-3x}$

We put $B' = e^{-3x}$ to the first equation and obtain

$$A' + xe^{-3x} = 0.$$

We solve this equation for $A'$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$ $\qquad$ $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$$B' = e^{-3x} \qquad A' = -xe^{-3x}$$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

We integrate by parts:

$$\int xe^{-3x}\,\mathrm{d}x \quad \boxed{\begin{array}{ll} u = x & u' = 1 \\ v' = e^{-3x} & v = -\frac{1}{3}e^{-3x} \end{array}} = -\frac{1}{3}e^{-3x}x - \int -\frac{1}{3}e^{-3x}\,\mathrm{d}x$$

$$= -\frac{1}{3}e^{-3x}x - \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)e^{-3x}$$

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$B' = e^{-3x}$

$A' = -xe^{-3x}$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

The integral for $B$ is easy.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$$B' = e^{-3x} \qquad A' = -xe^{-3x}$$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$y_p = Ay_1 + By_2$

We return to the formula for the particular solution.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$ $\qquad$ $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$$B' = e^{-3x} \qquad A' = -xe^{-3x}$$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$$y_p = Ay_1 + By_2 = \underbrace{\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)}_{A} \cdot \underbrace{e^{2x}}_{y_1} \underbrace{-\frac{1}{3}e^{-3x}}_{B} \cdot \underbrace{xe^{2x}}_{y_2}$$

We know all the function here: $A, B, y_1, y_2$. We substitute...

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$ $\qquad$ $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$$B' = e^{-3x} \qquad A' = -xe^{-3x}$$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$$y_p = Ay_1 + By_2 = \underbrace{\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)}_{A} \cdot \underbrace{e^{2x}}_{y_1} \underbrace{-\frac{1}{3}e^{-3x}}_{B} \cdot \underbrace{xe^{2x}}_{y_2} = \frac{1}{9}e^{-x}$$

…and simplify.

$$\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)e^{2x} - \frac{1}{3}e^{-3x}xe^{2x} = \frac{1}{3}xe^{-x} + \frac{1}{9}e^{-x} - \frac{1}{3}xe^{-x}.$$

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$ $\qquad y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$$B' = e^{-3x} \qquad A' = -xe^{-3x}$$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$$y_p = Ay_1 + By_2 = \underbrace{\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)}_{A} \cdot \underbrace{e^{2x}}_{y_1} \underbrace{-\frac{1}{3}e^{-3x}}_{B} \cdot \underbrace{xe^{2x}}_{y_2} = \frac{1}{9}e^{-x}$$

$$y = C_1e^{2x} + C_2xe^{2x} + \frac{1}{9}e^{-x}, \qquad C_1, C_2 \in \mathbb{R}$$

We write the solution as a sum of particular solution and linear combination of functions from fundamental system.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$ $\qquad$ $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}$, $y_2'(x) = e^{2x}(1+2x)$

$$B' = e^{-3x} \qquad A' = -xe^{-3x}$$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$$y_p = Ay_1 + By_2 = \underbrace{\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)}_{A} \cdot \underbrace{e^{2x}}_{y_1} \underbrace{-\frac{1}{3}e^{-3x}}_{B} \cdot \underbrace{xe^{2x}}_{y_2} = \frac{1}{9}e^{-x}$$

$$y = C_1e^{2x} + C_2xe^{2x} + \frac{1}{9}e^{-x}, \qquad C_1, C_2 \in \mathbb{R}$$

The problem has been solved.

Solve DE $y'' - 5y' + 6y = xe^x$.

Solve DE $y'' - 5y' + 6y = xe^x$.

$$y'' - 5y' + 6y = 0$$

We start with the homogeneous equation...

Solve DE $y'' - 5y' + 6y = xe^x$.

$$y'' - 5y' + 6y = 0 \qquad \Rightarrow \qquad z^2 - 5z + 6 = 0$$

...and its characteristic equation.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

The roots of the equation are real and distinct.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

We write the fundamental system...

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

...and the derivatives of the functions from that fundamental system.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

We will use the variation of parameters.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A'e^{2x} + B'e^{3x} = 0$$
$$2A'e^{2x} + 3B'e^{3x} = xe^x$$

The functions $A$ and $B$ satisfy the following relations...

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

. . . which are equivalent to this linear system.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$$A' = -B'e^x$$

We solve the first equation with respect to $A'$...

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$$A' = -B'e^x$$

$$B'e^x = xe^{-x}$$

...and substitute into the second equation. We obtain

$$2(-B'e^{-x}) + 3B'e^x = xe^{-x}$$

which is equivalent to the blue expression.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$$A' = -B'e^x$$

$$B'e^x = xe^{-x}$$

$$B' = xe^{-2x}$$

We can find $B'$ ...

Solve DE $y'' - 5y' + 6y = xe^x$.

$$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$$

$$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$$

$$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$$

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$$A' = -B'e^x$$

$$B'e^x = xe^{-x}$$

$$B' = xe^{-2x}$$

$$A' = -B'e^x = -xe^{-x}$$

…and $A'$.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^{x} &= 0 \\ 2A' + 3B'e^{x} &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}$, $B' = xe^{-2x}$

We will look for $A(x)$ and $B(x)$ from $A'$ and $B'$.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}$, $B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}$,

We integrate by parts

$$A = -\int xe^{-x}\,\mathrm{d}x \quad \boxed{\begin{matrix} u = x & u' = 1 \\ v' = e^{-x} & v = -e^{-x} \end{matrix}} \quad = -\left(-xe^{-x} + \int e^{-x}\,\mathrm{d}x\right)$$

$$= -(-xe^{-x} - e^{-x}) = (x+1)e^{-x}$$

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}$, $B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}$, $\qquad B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$$B(x) = \int xe^{-2x}\,\mathrm{d}x \quad \begin{vmatrix} u = x & u' = 1 \\ v' = e^{-2x} & v = -\frac{1}{2}e^{-2x} \end{vmatrix}$$

$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,\mathrm{d}x = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^{x} &= 0 \\ 2A' + 3B'e^{x} &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}$, $B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}$, $\qquad$ $B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

We return to the formula for the particular equation, ...

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^{x} &= 0 \\ 2A' + 3B'e^{x} &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x},\ B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}, \qquad B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) = \underbrace{(x+1)e^{-x}}_{A}\underbrace{e^{2x}}_{y_1}\underbrace{-\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}}_{B}\underbrace{e^{3x}}_{y_2}$$

...substitute...

Solve DE $y'' - 5y' + 6y = xe^x$.

$$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}$, $B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}$, $\qquad$ $B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) = \underbrace{(x+1)e^{-x}}_{A}\underbrace{e^{2x}}_{y_1}\underbrace{-\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}}_{B}\underbrace{e^{3x}}_{y_2}$$

$$= \frac{1}{4}e^x(2x+3)$$

...and simplify.

**Solve DE** $y'' - 5y' + 6y = xe^x.$

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

The general solution is the sum of the particular solution and the general solution of the corresponding homogeneous equation. The general solution of the corresponding homogeneous equation is a linear combination of the functions from the fundamental system.

$A(x) = (x+1)e^{-x}, \qquad B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) = \underbrace{(x+1)e^{-x}}_{A}\underbrace{e^{2x}}_{y_1}\underbrace{-\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}}_{B}\underbrace{e^{3x}}_{y_2}$$
$$= \frac{1}{4}e^x(2x+3)$$

$$y = C_1e^{2x} + C_2e^{3x} + \frac{1}{4}e^x(2x+3), \qquad C_1, C_2 \in \mathbb{R}$$

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2,\ z_2 = 3$

$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^{x} &= 0 \\ 2A' + 3B'e^{x} &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x},\ B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}, \qquad B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) = \underbrace{(x+1)e^{-x}}_{A}\underbrace{e^{2x}}_{y_1} \underbrace{-\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}}_{B}\underbrace{e^{3x}}_{y_2}$$
$$= \frac{1}{4}e^x(2x+3)$$

$$y = C_1e^{2x} + C_2e^{3x} + \frac{1}{4}e^x(2x+3), \qquad C_1, C_2 \in \mathbb{R}$$

The problem is resolved.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y'' + y = 0$

We start with the corresponding homogeneous equation...

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$$y'' + y = 0 \qquad \Rightarrow \qquad z^2 + 1 = 0$$

...and its characteristic equation.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y'' + y = 0 \quad \Rightarrow \quad z^2 + 1 = 0 \quad \Rightarrow \quad z_1 = i,\ z_2 = -i$

The characteristic equation has complex roots.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y'' + y = 0 \quad \Rightarrow \quad z^2 + 1 = 0 \quad \Rightarrow \quad z_1 = i,\ z_2 = -i$

$y_1(x) = \cos x,$ $\quad y_2(x) = \sin x$

We write the fundamental system.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x$, $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

We look for the particular solution in this form.

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x$, $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A' \cos x + B' \sin x = 0$$
$$-A' \sin x + B' \cos x = \frac{\cos x}{\sin x}$$

The functions $A$ and $B$ have to satisfy this linear system.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $\quad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A' \cos x + B' \sin x = 0$$
$$-A' \sin x + B' \cos x = \frac{\cos x}{\sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

We will solve this system by Cramer's rule. We find the determinant of the coefficients matrix. This determinant is called *wronskian*.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A' \cos x + B' \sin x = 0$$
$$-A' \sin x + B' \cos x = \frac{\cos x}{\sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \dfrac{\cos x}{\sin x} & \cos x \end{vmatrix} = -\cos x;$$

We evaluate the auxiliary determinants. . .

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'\cos x + B'\sin x &= 0 \\ -A'\sin x + B'\cos x &= \frac{\cos x}{\sin x} \end{aligned}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \dfrac{\cos x}{\sin x} & \cos x \end{vmatrix} = -\cos x; \qquad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \dfrac{\cos x}{\sin x} \end{vmatrix} = \frac{\cos^2 x}{\sin x}$$

We evaluate the auxiliary determinants...

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A'\cos x + B'\sin x = 0$$
$$-A'\sin x + B'\cos x = \frac{\cos x}{\sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \dfrac{\cos x}{\sin x} & \cos x \end{vmatrix} = -\cos x; \qquad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \dfrac{\cos x}{\sin x} \end{vmatrix} = \frac{\cos^2 x}{\sin x}$$

$$A' = \frac{W_1}{W} = -\cos x$$

...and use the formula of Cramer for $A'$...

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A' \cos x + B' \sin x = 0$$
$$-A' \sin x + B' \cos x = \frac{\cos x}{\sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \dfrac{\cos x}{\sin x} & \cos x \end{vmatrix} = -\cos x; \qquad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \dfrac{\cos x}{\sin x} \end{vmatrix} = \frac{\cos^2 x}{\sin x}$$

$$A' = \frac{W_1}{W} = -\cos x \qquad B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$$

... and for $B'$.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

We integrate. The integral for $A$ is easy.

Solve DE $\quad y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$$B = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x \sin x}{1-\cos^2 x}\,\mathrm{d}x$$

The integral for $B$ is more complicated. The odd power of the goniometric function is in the denominator. We have to multiply and divide by $\sin x$ and use the formula $\cos^2 + \sin^2 x = 1$. This gives

$$B(x) = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x}{\sin^2 x}\sin x\,\mathrm{d}x = \int \frac{\cos^2 x}{1-\cos^2 x}\sin x\,\mathrm{d}x$$

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \frac{W_1}{W} = -\cos x$ $B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$

$A = -\sin x$

$$B = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x \sin x}{1-\cos^2 x}\,\mathrm{d}x = \int \frac{t^2}{t^2-1}\,\mathrm{d}t = \int 1 + \frac{1}{t^2-1}\,\mathrm{d}t$$

Now we use the substitution $\begin{array}{c} \cos x = t \\ \sin x\,\mathrm{d}x = -\,\mathrm{d}t \end{array}$. This gives

$$B(x) = \int \frac{t^2}{1-t^2}(-1)\,\mathrm{d}t = \int \frac{t^2}{t^2-1}\,\mathrm{d}t.$$

Further we divide the numerator $t^2$ by the denominator $(t^2-1)$.

Solve DE $\quad y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$$B = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x \sin x}{1-\cos^2 x}\,\mathrm{d}x = \int \frac{t^2}{t^2-1}\,\mathrm{d}t = \int 1 + \frac{1}{t^2-1}\,\mathrm{d}t$$

$$= t + \frac{1}{2}\ln\frac{1-t}{1+t}$$

We expand the fraction $\dfrac{1}{t^2-1}$ into partial fractions, integrate and add logarithms. This gives

$$B(x) = t + \int \frac{1}{2}\frac{1}{t-1} - \frac{1}{2}\frac{1}{t+1}\,\mathrm{d}t = t + \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1|$$

$$= t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| = t + \frac{1}{2}\ln\frac{1-t}{1+t}$$

Solve DE $\quad y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A' = \frac{W_1}{W} = -\cos x \qquad B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$$

$A = -\sin x$

$$B = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x \sin x}{1-\cos^2 x}\,\mathrm{d}x = \int \frac{t^2}{t^2-1}\,\mathrm{d}t = \int 1 + \frac{1}{t^2-1}\,\mathrm{d}t$$

$$= t + \frac{1}{2}\ln\frac{1-t}{1+t} = \cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}$$

We use the back substitution $\cos x = t$.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$$B = \cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}$$

Now both $A$ and $B$ are known.

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \frac{W_1}{W} = -\cos x$ $B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$

$A = -\sin x$

$B = \cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}$

$y_p(x) = Ay_1 + By_2$

The particular solution we looked in this form.

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \frac{W_1}{W} = -\cos x$ $B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$

$A = -\sin x$

$B = \cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}$

$$y_p(x) = Ay_1 + By_2 = \underbrace{-\sin x}_{A}\underbrace{\cos x}_{y_1} + \underbrace{\left[\cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}\right]}_{B}\underbrace{\sin x}_{y_2}$$

We can substitute...

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$B = \cos x + \dfrac{1}{2}\ln\dfrac{1-\cos x}{1+\cos x}$

$$y_p(x) = Ay_1 + By_2 = \underbrace{-\sin x}_{A}\underbrace{\cos x}_{y_1} + \underbrace{\left[\cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}\right]}_{B}\underbrace{\sin x}_{y_2}$$

$$= \frac{1}{2}\sin x \ln\frac{1-\cos x}{1+\cos x}$$

...and simplify

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x$, $\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$$B = \cos x + \frac{1}{2} \ln \frac{1-\cos x}{1+\cos x}$$

$$y_p(x) = Ay_1 + By_2 = \underbrace{-\sin x}_{A} \underbrace{\cos x}_{y_1} + \underbrace{\left[\cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}\right]}_{B} \underbrace{\sin x}_{y_2}$$

$$= \frac{1}{2} \sin x \ln \frac{1-\cos x}{1+\cos x}$$

$$y(x) = C_1 \cos x + C_2 \sin x + \frac{\sin x}{2} \ln \frac{1-\cos x}{1+\cos x}$$

The general solution is a sum of the general solution of the homogeneous system and the particular solution of nonhomogeneous system.

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \frac{W_1}{W} = -\cos x \qquad B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$

$A = -\sin x$

$B = \cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}$

$$y_p(x) = Ay_1 + By_2 = \underbrace{-\sin x}_{A}\underbrace{\cos x}_{y_1} + \underbrace{\left[\cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}\right]}_{B}\underbrace{\sin x}_{y_2}$$

$$= \frac{1}{2}\sin x \ln\frac{1-\cos x}{1+\cos x}$$

$$y(x) = C_1\cos x + C_2\sin x + \frac{\sin x}{2}\ln\frac{1-\cos x}{1+\cos x}$$

The problem is resolved.

Solve $y'' + 2y' + y = e^{-x} \ln x$.

Solve $y'' + 2y' + y = e^{-x} \ln x$.

$$y'' + 2y' + y = 0 \Rightarrow z^2 + 2z + 1 = 0 \Rightarrow z_{1,2} = -1 \Rightarrow \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

We find two independent solutions of the associated homogeneous equation.

Solve $y'' + 2y' + y = e^{-x}\ln x$.

$$y'' + 2y' + y = 0 \;\Rightarrow\; z^2 + 2z + 1 = 0 \;\Rightarrow\; z_{1,2} = -1 \;\Rightarrow\; \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

$$y = A(x)e^{-x} + B(x)xe^{-x}$$

We look for the particular solution in the following form.

Solve $y'' + 2y' + y = e^{-x}\ln x$.

$$y'' + 2y' + y = 0 \;\Rightarrow\; z^2 + 2z + 1 = 0 \;\Rightarrow\; z_{1,2} = -1 \;\Rightarrow\; \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

$$y = A(x)e^{-x} + B(x)xe^{-x}$$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-2x}$$

We find wronskian.

Solve $y'' + 2y' + y = e^{-x} \ln x$.

$$y'' + 2y' + y = 0 \Rightarrow z^2 + 2z + 1 = 0 \Rightarrow z_{1,2} = -1 \Rightarrow \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

$$y = A(x)e^{-x} + B(x)xe^{-x}$$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\ln x & (1-x)e^{-x} \end{vmatrix} = -xe^{-2x}\ln x$$

We find determinant $W_1$.

Solve $y'' + 2y' + y = e^{-x} \ln x$.

$$y'' + 2y' + y = 0 \Rightarrow z^2 + 2z + 1 = 0 \Rightarrow z_{1,2} = -1 \Rightarrow \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

$$y = A(x)e^{-x} + B(x)xe^{-x}$$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{-x} \\ e^{-x} \ln x & (1-x)e^{-x} \end{vmatrix} = -xe^{-2x} \ln x$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x} \ln x \end{vmatrix} = e^{-2x} \ln x$$

We find determinant $W_2$.

Solve $y'' + 2y' + y = e^{-x} \ln x$.

$$y'' + 2y' + y = 0 \ \Rightarrow\ z^2 + 2z + 1 = 0 \ \Rightarrow\ z_{1,2} = -1 \ \Rightarrow\ \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

$$y = A(x)e^{-x} + B(x)xe^{-x}$$

$W = e^{-2x}$ $\qquad W_1 = -xe^{-2x} \ln x$ $\qquad W_2 = e^{-2x} \ln x$

$$A = \int \frac{W_1}{W}\,\mathrm{d}x = -\int x \ln x \,\mathrm{d}x = -\frac{x^2}{2} \ln x + \int \frac{x}{2}\,\mathrm{d}x$$

$$= -\frac{x^2}{2} \ln x + \frac{x^2}{4} = \frac{x^2}{4}(1 - 2\ln x)$$

We find $A'$ and integrate to get $A$. We integarte by parts with

| | |
|---|---|
| $u = \ln x$ | $u' = \frac{1}{x}$ |
| $v' = x$ | $v = \frac{x^2}{2}$ |

Solve $y'' + 2y' + y = e^{-x} \ln x$.

$$y'' + 2y' + y = 0 \Rightarrow z^2 + 2z + 1 = 0 \Rightarrow z_{1,2} = -1 \Rightarrow \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

$$y = A(x)e^{-x} + B(x)xe^{-x}$$

$$W = e^{-2x} \qquad W_1 = -xe^{-2x} \ln x \qquad W_2 = e^{-2x} \ln x$$

$$A = \int \frac{W_1}{W} \,\mathrm{d}x = -\int x \ln x \,\mathrm{d}x = -\frac{x^2}{2} \ln x + \int \frac{x}{2} \,\mathrm{d}x$$

$$= -\frac{x^2}{2} \ln x + \frac{x^2}{4} = \frac{x^2}{4}(1 - 2\ln x)$$

$$B = \int \ln x \,\mathrm{d}x = x \ln x - \int 1 \,\mathrm{d}x = x \ln x - x = x(\ln x - 1)$$

We find $B'$ and integrate to get $B$. We integarte by parts with

$$\begin{array}{|ll|} u = \ln x & u' = \frac{1}{x} \\ v' = 1 & v = x \end{array}$$

Solve $y'' + 2y' + y = e^{-x} \ln x$.

$$y'' + 2y' + y = 0 \;\Rightarrow\; z^2 + 2z + 1 = 0 \;\Rightarrow\; z_{1,2} = -1 \;\Rightarrow\; \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

$$y = A(x)e^{-x} + B(x)xe^{-x}$$

$$W = e^{-2x} \qquad W_1 = -xe^{-2x} \ln x \qquad W_2 = e^{-2x} \ln x$$

$$A = \int \frac{W_1}{W}\,\mathrm{d}x = -\int x \ln x\,\mathrm{d}x = -\frac{x^2}{2}\ln x + \int \frac{x}{2}\,\mathrm{d}x$$
$$= -\frac{x^2}{2}\ln x + \frac{x^2}{4} = \frac{x^2}{4}(1 - 2\ln x)$$
$$B = \int \ln x\,\mathrm{d}x = x\ln x - \int 1\,\mathrm{d}x = x\ln x - x = x(\ln x - 1)$$

$$y_p(x) = \frac{x^2}{4}(1 - 2\ln x)e^{-x} + x(\ln x - 1)xe^{-x}$$

We find the particular solution.

Solve $y'' + 2y' + y = e^{-x} \ln x$.

$$y'' + 2y' + y = 0 \;\Rightarrow\; z^2 + 2z + 1 = 0 \;\Rightarrow\; z_{1,2} = -1 \;\Rightarrow\; \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

$$y = A(x)e^{-x} + B(x)xe^{-x}$$

$$W = e^{-2x} \qquad W_1 = -xe^{-2x} \ln x \qquad W_2 = e^{-2x} \ln x$$

$$\begin{aligned} A &= \int \frac{W_1}{W}\,\mathrm{d}x = -\int x \ln x\,\mathrm{d}x = -\frac{x^2}{2}\ln x + \int \frac{x}{2}\,\mathrm{d}x \\ &= -\frac{x^2}{2}\ln x + \frac{x^2}{4} = \frac{x^2}{4}(1 - 2\ln x) \\ B &= \int \ln x\,\mathrm{d}x = x\ln x - \int 1\,\mathrm{d}x = x\ln x - x = x(\ln x - 1) \end{aligned}$$

$$y_p(x) = \frac{x^2}{4}(1 - 2\ln x)e^{-x} + x(\ln x - 1)xe^{-x} = \frac{1}{4}x^2 e^{-x}(2\ln x - 3)$$

We simplify.

Solve $y'' + 2y' + y = e^{-x} \ln x$.

$$y'' + 2y' + y = 0 \;\Rightarrow\; z^2 + 2z + 1 = 0 \;\Rightarrow\; z_{1,2} = -1 \;\Rightarrow\; \begin{cases} y_1 = e^{-x} \\ y_2 = xe^{-x} \end{cases}$$

$$y = A(x)e^{-x} + B(x)xe^{-x}$$

$$W = e^{-2x} \qquad W_1 = -xe^{-2x} \ln x \qquad W_2 = e^{-2x} \ln x$$

$$A = \int \frac{W_1}{W}\,dx = -\int x \ln x\,dx = -\frac{x^2}{2} \ln x + \int \frac{x}{2}\,dx$$
$$= -\frac{x^2}{2} \ln x + \frac{x^2}{4} = \frac{x^2}{4}(1 - 2\ln x)$$
$$B = \int \ln x\,dx = x \ln x - \int 1\,dx = x \ln x - x = x(\ln x - 1)$$
$$y_p(x) = \frac{x^2}{4}(1 - 2\ln x)e^{-x} + x(\ln x - 1)xe^{-x} = \frac{1}{4}x^2e^{-x}(2\ln x - 3)$$
$$y(x) = \frac{1}{4}x^2e^{-x}(2\ln x - 3) + C_1e^{-x} + C_2xe^{-x}$$

Finished!

# Further reading

- http://www.sosmath.com/diffeq/second/linear/secondlinear.html
- http://www.sosmath.com/diffeq/second/homolinear/homolinear.html
- http://www.sosmath.com/diffeq/second/nonhomo/nonhomo.html
- http://www.chass.utoronto.ca/~osborne/MathTutorial/