First order diferential equations

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Contents

1 First order differential equations

The basic problem of the integral calculus:

Given a function $f(x)$ defined on the interval I, find the function $y(x)$ defined on I which satisfies

$$y'(x) = f(x) \tag{1}$$

for every $x \in I$.

The solution to this problem is known:

$$y(x) = \int f(x)\,\mathrm{d}x + C, \tag{2}$$

where $\int f(x)\,\mathrm{d}x$ is an arbitrary antiderivative of the function f and C is an arbitrary constant.

(Bridge) (Dog curve)

A slight modification of the basic problem of the integral calculus:

> *Given a function $f(x)$ defined on the interval I and given numbers $\alpha \in I$ and $\beta \in \mathbb{R}$, find the function $y(x)$ defined on I which satisfies (1) for every $x \in I$ and $y(\alpha) = \beta$.*

The solution is also known: We solve the problem (1) and obtain (2). In the relation (2) we substitute (after evaluation of the integral, of course) $x = \alpha$ and $y = \beta$. Then we solve the obtained equation for C and substitute the value of C into (2).

Example 1. Find the function y which satisfies

$$y' = 2x, \quad y(1) = 2.$$

Solution: Integration of the equation gives $y(x) = \int 2x \,\mathrm{d}x = x^2 + C$.
From the condition $y(1) = 2$ we substitute $x = 1$ and $y = 2$ in the expression $y = x^2 + C$. We obtain

$$2 = 1^2 + C$$

and hence $C = 1$. The solution of the problem is the function $y(x) = x^2 + 1$.

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From now consider that the uknown function y is also present on the right hand side of equation (1). (social diffusion)

Definition (ordinary differential equation). Let $\varphi(x, y)$ be a function of two variables. The equation

$$y' = \varphi(x, y), \tag{3}$$

is called the *first order ordinary differential equation, explicitly solved for y'*, shortly *ordinary differential equation* or *ODE*.

Definition (solution of ODE). Under a *solution* of the ODE on the interval I we understand every function $y = y(x)$ which has the following properties

- $y(x)$ is differentiable on I,
- the function $\varphi(x, y(x))$ is defined for all $x \in I$,
- the relation $y' = \varphi(x, y(x))$ holds for all $x \in I$.

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- the relation $y' = \varphi(x, y(x))$ holds for all $x \in I$.

Definition (initial value problem). Let α, β be real numbers, $\alpha \in I$. The problem to find the function $y = y(x)$ which satisfies equation (3) and an *initial condition*

$$y(\alpha) = \beta \tag{4}$$

is called an *initial value problem* (shortly *IVP*).

The solution of the IVP (3), (4) is called a *particular solution*.

The graph of the particular solution is called an *integral curve*.

Example 2. The solution of equation $y' = \frac{1}{2}(x^2 + y^2)$ passing through the point $[2, 1]$ has the slope $\frac{1}{2}(2^2 + 1^2) = \frac{5}{2}$ in this point. Hence the line

$$y = \frac{5}{2}(x - 2) + 1$$

is a tangent to this integral curve in $[2, 1]$.

Remark 1 (geometric interpretation, slope field). Roughly speaking, the differential equation states:

> *If an integral curve goes through the point* $[x_0, y_0]$*, it follows necessarily the direction* $\varphi(x_0, y_0)$ *at this point.*

The simplest way how to visualize this idea is to draw a short mark at several points to indicate the direction associated with that point. This can be done in a systematic way on the computer. As a result, we obtain a *slope field* of the equation. This slope field has the property that the linear elements in this field are tangent to the integral curve which passes through the same point.
The initial condition (4) requires that the graph of the particular solution goes through the point $[\alpha, \beta]$.

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2 DE with separated variables

Definition (ODE with separated variables). ODE in the form

$$y' = f(x)g(y), \tag{5}$$

where f and g are continuous functions on some opena intervals is said to be *ordinary differential equation with separated variables.*

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

We write the derivavative y' as a quotient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\frac{1}{y}\,\mathrm{d}y = \cos x\ \mathrm{d}x$$

- Using cross multiplication we convert expressions with variable x to one of the sides and expressions with y to the other side of the equation.
- From the conditition $y(0) = 0.1$ we know that the solution is a nonzero fucntion (at least in some neighborhood of the point $x = 0$).

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \ \mathrm{d}x$$

We write integrals to both sides of the equation. The variable of integration on the left is y and on the right x.

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \ \ \mathrm{d}x$$

$$\ln y = \sin x + C$$

- We evaluate the integrals. The function y is positive (at least in some neighborhood of the point $x = 0$). We consider only one of the constants of integration.
- We obtain an equation which describes all solutions of the equation $y' = y \cdot \cos x$.

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\cdot\cos x$$

$$\int\frac{1}{y}\,\mathrm{d}y = \int\cos x\ \mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

We use the initial condition $y(0) = 0.1$ to find the value of the constant of integration.

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= y \cdot \cos x \\
\int \frac{1}{y}\,\mathrm{d}y &= \int \cos x \ \ \mathrm{d}x \\
\ln y &= \sin x + C \\
\ln 0.1 &= \sin 0 + C \\
C &= \ln 0.1
\end{aligned}$$

We find C.

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \ \ \mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

$$\ln y = \sin x + \ln 0.1$$

We substitute C into the equation which decribes all solutions and find a particular solution of the initial value problem. This solution is written in the implicit form.

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x\ \ \mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

$$\ln y = \sin x + \ln 0.1$$

$$\ln y - \ln 0.1 = \sin x$$

We convert logarithms into one side.

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \ \ \mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

$$\ln y = \sin x + \ln 0.1$$

$$\ln y - \ln 0.1 = \sin x$$

$$\ln \frac{y}{0.1} = \sin x$$

We substract logarithms.

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$
$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \ \mathrm{d}x$$
$$\ln y = \sin x + C$$
$$\ln 0.1 = \sin 0 + C$$
$$C = \ln 0.1$$
$$\ln y = \sin x + \ln 0.1$$

$$\ln y - \ln 0.1 = \sin x$$
$$\ln \frac{y}{0.1} = \sin x$$
$$\frac{y}{0.1} = e^{\sin x}$$

We use the inverse function to remove logarithms.

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\begin{aligned}\frac{dy}{dx} &= y \cdot \cos x\\ \int \frac{1}{y}\,dy &= \int \cos x\ \ dx\\ \ln y &= \sin x + C\\ \ln 0.1 &= \sin 0 + C\\ C &= \ \ln 0.1\\ \ln y &= \sin x + \ln 0.1\end{aligned}$$

$$\begin{aligned}\ln y - \ln 0.1 &= \sin x\\ \ln \frac{y}{0.1} &= \sin x\\ \frac{y}{0.1} &= e^{\sin x}\\ y &= 0.1 \cdot e^{\sin x}\end{aligned}$$

We isolate y. This is an explicit form of the solution of the initial value problem.

Find the function $y(x)$ satistfying $\boxed{y' = y\cos x}$ and $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \;\; \mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

$$\ln y = \sin x + \ln 0.1$$

$$\ln y - \ln 0.1 = \sin x$$

$$\ln \frac{y}{0.1} = \sin x$$

$$\frac{y}{0.1} = e^{\sin x}$$

$$y = 0.1 \cdot e^{\sin x}$$

Notation:

$\boxed{\text{differential equation}}$ + $\boxed{\text{initial condition}}$ = *initial value problem*,

general solution, particular solution (solution of IVP)

Differential equation with separated variables

$$y' = f(x)g(y)$$

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$$y' = f(x)g(y)$$

The equation possesses a family of constant solutions of the type $y = y_i$, where y_i is a number which is a solution of the equation $g(y_i) = 0$.

We look for constant solutions first. Since the derivative of a constant function is zero, the constant solution produces zero on both sides for any admissible x.

Differential equation with separated variables

$$y' = f(x)g(y)$$

The equation possesses a family of constant solutions of the type $y = y_i$, where y_i is a number which is a solution of the equation $g(y_i) = 0$. In the following steps we are intrested in nonconstant solutions only.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

We write the derivativey' as quotient $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Differential equation with separated variables

$$y' = f(x)g(y)$$

The equation possesses a family of constant solutions of the type $y = y_i$, where y_i is a number which is a solution of the equation $g(y_i) = 0$. In the following steps we are intrested in nonconstant solutions only.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

$$\frac{\mathrm{d}y}{g(y)} = f(x)\,\mathrm{d}x$$

We cross-multiply the equation in order to convert the expressions in variable x on one of the sides and expressions in variable y on the other side.

Differential equation with separated variables

$$y' = f(x)g(y)$$

The equation possesses a family of constant solutions of the type $y = y_i$, where y_i is a number which is a solution of the equation $g(y_i) = 0$. In the following steps we are intrested in nonconstant solutions only.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

$$\frac{\mathrm{d}y}{g(y)} = f(x)\,\mathrm{d}x$$

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x + C$$

We integrate both sides and consider only one of the constants of integration. This gives the general solution of the equation.

Differential equation with separated variables

$$y' = f(x)g(y)$$

The equation possesses a family of constant solutions of the type $y = y_i$, where y_i is a number which is a solution of the equation $g(y_i) = 0$. In the following steps we are intrested in nonconstant solutions only.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

$$\frac{\mathrm{d}y}{g(y)} = f(x)\,\mathrm{d}x$$

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x + C$$

If the initial condition is given, we find the value of the constant C for which the initival condition is satisfied.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

First of all we isolate y'.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$y' = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

From this relation it is clear that the equation has separated variables. The equation is meaningful for $y \neq 0$ and $x \neq \pm 1$.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

We write the derivative as the quotient of differentials.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \, \frac{1}{1 - x^2}$$

$$\frac{y}{y^2 - 1} \, \mathrm{d}y = \frac{1}{1 - x^2} \, \mathrm{d}x$$

We separate the variables. We multiply the equation by the factor $\frac{y}{y^2 - 1}$. This is possible if $y \neq \pm 1$. This is guaranteed by the initial condition.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{dy}{dx} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\, dy = \int \frac{1}{1 - x^2}\, dx$$

We add the integrals. . .

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y}\,\frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2 - 1| =$$

...and evaluate. The first integral is (up to the constant multiple) the integral of the type $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$.

Solve the IVP $y^2-1+yy'(x^2-1)=0, \quad y(0)=2.$

$$yy'(1-x^2)=y^2-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int\frac{y}{y^2-1}\,\mathrm{d}y=\int\frac{1}{1-x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2-1|=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+c$$

The second integral is either the rule, or we can expand into partial fractions $\frac{1}{1-x^2}=\frac{1}{2}\frac{1}{1-x}+\frac{1}{2}\frac{1}{1+x}$ and from here $\int\frac{1}{1-x^2}\,\mathrm{d}x=-\frac{1}{2}\ln|1-x|+\frac{1}{2}\ln|1+x|$, which is equivalent to the blue expression.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y}\,\frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2 - 1| = \frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right| + c$$

$$\ln(y^2 - 1) = \ln\frac{1 + x}{1 - x} + 2c$$

According to the init. cond. , we can omit both absolute values and multiply by the number 2.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2} \ln |y^2 - 1| = \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| + c$$

$$\ln(y^2 - 1) = \ln \frac{1 + x}{1 - x} + 2c$$

$$\ln(y^2 - 1) = \ln \frac{1 + x}{1 - x} + \ln e^{2c}$$

We write $2c$ in the logarithmic form $\ln e^{2c}$...

Solve the IVP $y^2-1+yy'(x^2-1)=0, \quad y(0)=2.$

$$yy'(1-x^2)=y^2-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int\frac{y}{y^2-1}\,\mathrm{d}y=\int\frac{1}{1-x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2-1|=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+c$$

$$\ln(y^2-1)=\ln\frac{1+x}{1-x}+2c$$

$$\ln(y^2-1)=\ln\frac{1+x}{1-x}+\ln e^{2c}$$

$$\ln(y^2-1)=\ln\left(\frac{1+x}{1-x}e^{2c}\right)$$

...and add the logarithms.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y}\,\frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2 - 1| = \frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right| + c$$

$$\ln(y^2 - 1) = \ln\frac{1 + x}{1 - x} + 2c$$

$$\ln(y^2 - 1) = \ln\frac{1 + x}{1 - x} + \ln e^{2c}$$

$$\cancel{\ln}(y^2 - 1) = \cancel{\ln}\left(\frac{1 + x}{1 - x}e^{2c}\right)$$

$$y^2 - 1 = \frac{1 + x}{1 - x}e^{2c}$$

Logarithm is one-to-one function and can be omitted from both sides of equation.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y}\,\frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2 - 1| = \frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right| + c$$

$$\ln(y^2 - 1) = \ln\frac{1 + x}{1 - x} + 2c$$

$$\ln(y^2 - 1) = \ln\frac{1 + x}{1 - x} + \ln e^{2c}$$

$$\ln(y^2 - 1) = \ln\left(\frac{1 + x}{1 - x}e^{2c}\right)$$

$$y^2 - 1 = \frac{1 + x}{1 - x}e^{2c}$$

$$y^2 = 1 + C \cdot \frac{1 + x}{1 - x}$$

The general solution. Here $C = e^{2c}$ is a new constant.

Solve the IVP $y^2-1+yy'(x^2-1)=0$, $y(0)=2$.

$$yy'(1-x^2)=y^2-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int\frac{y}{y^2-1}\,\mathrm{d}y=\int\frac{1}{1-x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2-1|=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+c$$

$$\ln(y^2-1)=\ln\frac{1+x}{1-x}+2c$$

$$\ln(y^2-1)=\ln\frac{1+x}{1-x}+\ln e^{2c}$$

$\boxed{y=2}$ $\boxed{x=0}$

$$\ln(y^2-1) \quad \frac{1+x}{-x}e^{2c}\Big)$$

$$y^2-1=\frac{1+x}{1-x}e^{2c}$$

$$y^2=1+C\cdot\frac{1+x}{1-x}$$

$$2^2=1+C\frac{1+0}{1-0}$$

We substitute from the initial condition...

Solve the IVP $y^2 - 1 + yy'(x^2-1) = 0, \quad y(0) = 2.$

$$yy'(1-x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int \frac{y}{y^2-1}\,\mathrm{d}y = \int \frac{1}{1-x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2-1| = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + 2c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + \ln e^{2c}$$

$$\ln(y^2-1) = \ln\left(\frac{1+x}{1-x}e^{2c}\right)$$

$$y^2 - 1 = \frac{1+x}{1-x}e^{2c}$$

$$y^2 = 1 + C\cdot\frac{1+x}{1-x}$$

$$2^2 = 1 + C\frac{1+0}{1-0}$$

$$C = 3$$

…and solve for C.

Solve the IVP $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2} \ln |y^2 - 1| = \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| + c$$

$$\ln(y^2 - 1) = \ln \frac{1 + x}{1 - x} + 2c$$

$$\ln(y^2 - 1) = \ln \frac{1 + x}{1 - x} + \ln e^{2c}$$

$$\ln(y^2 - 1) = \ln \left(\frac{1 + x}{1 - x} e^{2c} \right)$$

$$y^2 - 1 = \frac{1 + x}{1 - x} e^{2c}$$

$$y^2 = 1 + C \cdot \frac{1 + x}{1 - x}$$

$$2^2 = 1 + C \frac{1 + 0}{1 - 0}$$

$$C = 3$$

$$y^2 = 1 + 3 \frac{1 + x}{1 - x}$$

We use that C in the general solution.

Solve the IVP $y^2-1+yy'(x^2-1)=0, \quad y(0)=2.$

$$yy'(1-x^2)=y^2-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int\frac{y}{y^2-1}\,\mathrm{d}y=\int\frac{1}{1-x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2-1|=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+c$$

$$\ln(y^2-1)=\ln\frac{1+x}{1-x}+2c$$

$$\ln(y^2-1)=\ln\frac{1+x}{1-x}+\ln e^{2c}$$

$$\ln(y^2-1)=\ln\left(\frac{1+x}{1-x}e^{2c}\right)$$

$$y^2-1=\frac{1+x}{1-x}e^{2c}$$

$$y^2=1+C\cdot\frac{1+x}{1-x}$$

$$2^2=1+C\frac{1+0}{1-0}$$

$$C=3$$

$$y^2=1+3\frac{1+x}{1-x}$$

$$y^2=\frac{4+2x}{1-x}$$

We simplify. The problem has been solved.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

We start with the equation.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$(2y-2)\,\mathrm{d}y = \quad (2x+1)\,\mathrm{d}x$$

The equation has separated variables and is meaningful for $y \neq 1$. To find the solution we multiply the equation by $2(y-1)$

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

We add integrals

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

We integrate both sides of the equation. We have $\boxed{\int 2y - 2\,\mathrm{d}y = y^2 - 2y}$ and $\boxed{\int 2x + 1\,\mathrm{d}x = x^2 + x}$. We use the constant of integration on the right-hand side and get the general solution of the equation.

Solve the IVP $\quad y' = \frac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

We complete square on the left...

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

$$(y-1)^2 = x^2 + x + C + 1$$

...and solve for y.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$
\begin{aligned}
y' &= \frac{2x+1}{2(y-1)} \\
\int (2y-2)\,\mathrm{d}y &= \int (2x+1)\,\mathrm{d}x \\
y^2 - 2y &= x^2 + x + C \\
(y-1)^2 - 1 &= x^2 + x + C \\
(y-1)^2 &= x^2 + x + C + 1 \\
y - 1 &= \pm\sqrt{x^2 + x + K}
\end{aligned}
$$

Let $K = C + 1$ be new constant. We take the second root of both sides of equation...

Solve the IVP $\qquad y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$\begin{aligned} y' &= \frac{2x+1}{2(y-1)} \\ \int (2y-2)\,\mathrm{d}y &= \int (2x+1)\,\mathrm{d}x \\ y^2 - 2y &= x^2 + x + C \\ (y-1)^2 - 1 &= x^2 + x + C \\ (y-1)^2 &= x^2 + x + C + 1 \\ y - 1 &= \pm\sqrt{x^2 + x + K} \\ y &= 1 \pm \sqrt{x^2 + x + K} \end{aligned}$$

. . . and solve for y.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$
$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$
$$y^2 - 2y = x^2 + x + C$$
$$(y-1)^2 - 1 = x^2 + x + C$$
$$(y-1)^2 = x^2 + x + C + 1$$
$$y - 1 = \pm\sqrt{x^2 + x + K}$$
$$y = 1 \pm \sqrt{x^2 + x + K}$$

$$y_1 = 1 + \sqrt{x^2 + x + K}$$
$$y_2 = 1 - \sqrt{x^2 + x + K}$$

This shows that there are two explicit formulas for general solution. Since $y_1(x) \geq 1$ and $y_2(x) \leq 1$ for all x, we consider the solution y_2 only (see the initial condition).

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$
$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$
$$y^2 - 2y = x^2 + x + C$$
$$(y-1)^2 - 1 = x^2 + x + C$$
$$(y-1)^2 = x^2 + x + C + 1$$
$$y - 1 = \pm\sqrt{x^2 + x + K}$$
$$y = 1 \pm \sqrt{x^2 + x + K}$$

$\boxed{y = 0}$

$\boxed{x = 2}$

~~$y_1 = 1 + \sqrt{x^2 + x + K}$~~

$$y_2 = 1 - \sqrt{x^2 + x + K}$$
$$0 = 1 - \sqrt{4 + 2 + K}$$

We substitute the initial condition into y_2.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

$$(y-1)^2 = x^2 + x + C + 1$$

$$y - 1 = \pm\sqrt{x^2 + x + K}$$

$$y = 1 \pm \sqrt{x^2 + x + K}$$

$$\xcancel{y_1 = 1 + \sqrt{x^2 + x + K}}$$

$$y_2 = 1 - \sqrt{x^2 + x + K}$$

$$0 = 1 - \sqrt{4 + 2 + K}$$

$$K = -5$$

The solution of $0 = 1 - \sqrt{4 + 2 + K}$ is $K = -5$.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

$$(y-1)^2 = x^2 + x + C + 1$$

$$y - 1 = \pm\sqrt{x^2 + x + K}$$

$$y = 1 \pm \sqrt{x^2 + x + K}$$

$$\xcancel{y_1 = 1 + \sqrt{x^2 + x + K}}$$

$$y_2 = 1 - \sqrt{x^2 + x + K}$$

$$0 = 1 - \sqrt{4 + 2 + K}$$

$$K = -5$$

$$y_p = 1 - \sqrt{x^2 + x - 5}$$

We use the obtained value of K in the formula for y_2. The initial value problem has been solved.

Solve DE $\quad 3xy^2y' = (y^3 - 1)(x^3 - 1).$

Solve DE $3xy^2y' = (y^3-1)(x^3-1).$

$$y' = \frac{y^3-1}{3y^2} \cdot \frac{x^3-1}{x}$$

- We solve the equation for y'.
- This shows that the equation has separated variables and is meaningful for $x \neq 0$ and $y \neq 0$.

Solve DE $3xy^2y' = (y^3 - 1)(x^3 - 1)$.

$$y' = \frac{y^3 - 1}{3y^2} \cdot \frac{x^3 - 1}{x}$$

The function $y \equiv 1$ is a solution.

The right-hand side equals zero for $y = 1$. Hence the constant function $y(x) = 1$ is a solution. This can be verified by direct substitution.

Solve DE $3xy^2y' = (y^3 - 1)(x^3 - 1)$.

$$y' = \frac{y^3 - 1}{3y^2} \cdot \frac{x^3 - 1}{x}$$

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

Let us continue with the cases in which $y \not\equiv 1$. In this case we can multiply the equation by the factor $\frac{3y^2}{y^3 - 1}$. This separates the variables.

Solve DE $3xy^2y' = (y^3-1)(x^3-1).$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y' = \frac{y^3-1}{3y^2}\cdot\frac{x^3-1}{x}$$

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\frac{3y^2}{y^3-1}\,\mathrm{d}y = \frac{x^3-1}{x}\,\mathrm{d}x$$

The variable y is on the left and x on the right.

Solve DE $3xy^2y' = (y^3-1)(x^3-1).$

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

We add integrals . . .

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

...and evaluate. The integral on the left is of the type $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$ and the integral on the right can be written as the integral

$$\int \frac{x^3-1}{x}\,\mathrm{d}x = \int x^2 - \frac{1}{x}\,\mathrm{d}x = \frac{x^3}{3} - \ln|x|.$$

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

We write the expressions $\dfrac{x^3}{3}$ and c in logarithmic forms $\ln e^{x^3/3}$ and $\ln e^c$ and add (subtract) logarithms.

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\cancel{\ln}|y^3-1| = \cancel{\ln}\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

$$|y^3-1| = e^{x^3/3}\,\frac{1}{|x|}\,e^c$$

Logarithm is one-to-one function and can be removed from both sides of equation.

Solve DE $3xy^2y' = (y^3-1)(x^3-1).$

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

$$|y^3-1| = e^{x^3/3}\,\frac{1}{|x|}\,e^c$$

$$y^3-1 = (\pm e^c)\,e^{x^3/3}\,\frac{1}{x}$$

If we omit the absolute values, the right and left side can differ by the sign. We add this sign to the constant factor e^c...

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

$$|y^3-1| = e^{x^3/3}\,\frac{1}{|x|}\,e^c$$

$$y^3-1 = (\pm e^c)\,e^{x^3/3}\,\frac{1}{x} \qquad C = \pm e^c \in \mathbb{R}\setminus\{0\}$$

…and introduce new constant $C = \pm e^c$. Since c can take arbitrary real value, the expression e^c can take arbitrary positive value and $\pm e^c$ can take arbitrary real nonzero value.

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

$$|y^3-1| = e^{x^3/3}\,\frac{1}{|x|}\,e^c$$

$$y^3-1 = (\pm e^c)\,e^{x^3/3}\,\frac{1}{x} \qquad C = \pm e^c \in \mathbb{R} \setminus \{0\}$$

$$y^3-1 = \frac{C}{x}e^{x^3/3} \qquad C \in \mathbb{R}$$

If we allow $C = 0$, the general solution gives $y = 1$ which is also a solution. Hence C can be arbitrary real value.

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

$$|y^3-1| = e^{x^3/3}\,\frac{1}{|x|}\,e^c$$

$$y^3-1 = (\pm e^c)\,e^{x^3/3}\,\frac{1}{x} \qquad C = \pm e^c \in \mathbb{R}$$

$$y^3-1 = \frac{C}{x}e^{x^3/3} \qquad C \in \mathbb{R}$$

The equation has been solved.

Solve DE $(1+e^x)y' + e^x y = 0$

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$

We start with the equation.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$

$$y' = -\frac{e^x}{e^x+1}y$$

We solve the equation for y'.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$

$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \not\equiv 0$.

The right-hand side is zero for $y = 0$.

Solve DE $\quad (1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$

$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \not\equiv 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$

We substitute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for y'.

Solve DE $(1+e^x)y' + e^x y = 0$

$$\begin{aligned}(1+e^x)y' &= -e^x y\\ y' &= -\frac{e^x}{e^x+1}y\end{aligned}$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$

$$\frac{\mathrm{d}y}{y} = - \quad \frac{e^x}{1+e^x}\,\mathrm{d}x$$

We multiply by $\mathrm{d}x$ and divide by y. Since $y \neq 0$, we can do the division.

Solve DE $\quad (1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \not\equiv 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$

We write integral signs.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \not\equiv 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$
$$\ln|y| = -\ln(1+e^x) + c$$

We evaluate the integrals. In the integral on the right we have the derivative of denominator in numerator.

Solve DE $(1+e^x)y'+e^xy=0$

$$(1+e^x)y' = -e^xy$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \not\equiv 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$
$$\ln|y| = -\ln(1+e^x)+c$$

$$\ln\Big[|y|\big(1+e^x\big)\Big] = \ln e^c$$

We convert logarithms to the left–hand side and add. Further we convert the number c into logarithmic form.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$

$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \not\equiv 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$

$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$

$$\ln|y| = -\ln(1+e^x) + c$$

$$\cancel{\ln}\Big[|y|\big(1+e^x\big)\Big] = \cancel{\ln} e^c$$

$$|y|(1+e^x) = e^c$$

Logarithmic function is one-to-one and can be removed from both sides on equation.

Solve DE $(1+e^x)y'+e^xy=0$

$$(1+e^x)y'=-e^xy$$

$$y'=-\frac{e^x}{e^x+1}y$$

The function $y\equiv 0$ is a solution. In the following suppose $y\not\equiv 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{e^x}{e^x+1}y$$

$$\int\frac{\mathrm{d}y}{y}=-\int\frac{e^x}{1+e^x}\,\mathrm{d}x$$

$$\ln|y|=-\ln(1+e^x)+c$$

$$\ln\Big[|y|\big(1+e^x\big)\Big]=\ln e^c$$

$$|y|(1+e^x)=e^c$$

$$y(1+e^x)=K \qquad K=\pm e^c$$

We remove the absolute value. This yields ± sign on the right. We join this sign to the number e^c which gives a new constant K.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \not= 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$
$$\ln|y| = -\ln(1+e^x) + c$$

$$\ln\Big[|y|\big(1+e^x\big)\Big] = \ln e^c$$
$$|y|(1+e^x) = e^c$$
$$y(1+e^x) = K \qquad K = \pm e^c$$
$$y = \frac{K}{1+e^x} \qquad K \in \mathbb{R}\backslash\{0\}$$

We solve the obtained relation for y.

Solve DE $(1+e^x)y'+e^xy=0$

$$(1+e^x)y'=-e^xy$$

$$y'=-\frac{e^x}{e^x+1}y$$

The function $y\equiv 0$ is a solution. In the following suppose $y\neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{e^x}{e^x+1}y$$

$$\int\frac{\mathrm{d}y}{y}=-\int\frac{e^x}{1+e^x}\,\mathrm{d}x$$

$$\ln|y|=-\ln(1+e^x)+c$$

$$\ln\Bigl[|y|\bigl(1+e^x\bigr)\Bigr]=\ln e^c$$

$$|y|(1+e^x)=e^c$$

$$y(1+e^x)=K \qquad K=\pm e^c$$

$$y=\frac{K}{1+e^x} \qquad K\in\mathbb{R}\setminus\{0\}$$

The choice $K=0$ gives $y\equiv 0$, which gives the constant solution.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$

$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \not= 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$

$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$

$$\ln|y| = -\ln(1+e^x) + c$$

$$\ln\Big[|y|\big(1+e^x\big)\Big] = \ln e^c$$

$$|y|(1+e^x) = e^c$$

$$y(1+e^x) = K \qquad K = \pm e^c$$

$$y = \frac{K}{1+e^x} \qquad K \in \mathbb{R}$$

The problem is resolved.

Solve DE $\qquad y'e^{x^2+y} = -\dfrac{x}{y}$

Solve DE $\quad y' e^{x^2+y} = -\dfrac{x}{y}$

$$y' e^{x^2} e^{y} = -x\frac{1}{y}$$

We factor the exponential function e^{x^2+y}. This separates the variables in the exponent.

Solve DE $\quad y' e^{x^2+y} = -\frac{x}{y}$

$$y' e^{x^2} e^{y} = -x \frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} e^{x^2} e^{y} = -x \frac{1}{y}$$

We substitute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for y'.

Solve DE $\quad y' e^{x^2+y} = -\frac{x}{y}$

$$y' e^{x^2} e^{y} = -x \frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} e^{x^2} e^{y} = -x \frac{1}{y}$$

$$y e^{y} \,\mathrm{d}y = - \quad x e^{-x^2} \,\mathrm{d}x$$

We multiply by y and divide by e^{x^2}. The latter is equivalent to the multiplication by e^{-x^2}.

Solve DE $y'e^{x^2+y} = -\frac{x}{y}$

$$y'e^{x^2}e^y = -x\frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}e^{x^2}e^y = -x\frac{1}{y}$$

$$\int ye^y\,\mathrm{d}y = -\int xe^{-x^2}\,\mathrm{d}x$$

We write integral sings.

Solve DE $y'e^{x^2+y} = -\dfrac{x}{y}$

$$y'e^{x^2}e^y = -x\frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}e^{x^2}e^y = -x\frac{1}{y}$$

$$\int ye^y\,\mathrm{d}y = -\int xe^{-x^2}\,\mathrm{d}x$$

$$ye^y - e^y =$$

On the left we integrate by parts:

$$\int ye^y\,\mathrm{d}y \quad \boxed{\begin{matrix} u = y & u' = 1 \\ v' = e^y & v = e^y \end{matrix}} = ye^y - \int e^y\,\mathrm{d}y = ye^y - e^y$$

Solve DE $\quad y'e^{x^2+y} = -\frac{x}{y}$

$$y'e^{x^2}e^y = -x\frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}e^{x^2}e^y = -x\frac{1}{y}$$

$$\int ye^y\,\mathrm{d}y = -\int xe^{-x^2}\,\mathrm{d}x$$

$$ye^y - e^y = \frac{1}{2}e^{-x^2} + C$$

On the right we use a the substitution suggested by the inside function. Hence

$$-\int xe^{-x^2}\,\mathrm{d}x \quad \boxed{\begin{array}{c} -x^2 = t \\ -2x\,\mathrm{d}x = \mathrm{d}t \\ -x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}t \end{array}} = \frac{1}{2}\int e^t\,\mathrm{d}t = \frac{1}{2}e^t = \frac{1}{2}e^{-x^2}$$

Solve DE $\quad y' e^{x^2+y} = -\dfrac{x}{y}$

$$y' e^{x^2} e^y = -x\frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} e^{x^2} e^y = -x\frac{1}{y}$$

$$\int y e^y \,\mathrm{d}y = -\int x e^{-x^2} \,\mathrm{d}x$$

$$y e^y - e^y = \frac{1}{2} e^{-x^2} + C$$

$$2y e^y - 2e^y = e^{-x^2} + C \qquad C \in \mathbb{R}$$

We multiply the equation by the number 2. This gives the general solution in its implicit form. Unfortunately, we cannot solve explicitly this relation with respect to y. We keep the solution in its implicit form.

Solve DE $y' e^{x^2+y} = -\frac{x}{y}$

$$y' e^{x^2} e^y = -x\frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} e^{x^2} e^y = -x\frac{1}{y}$$

$$\int y e^y \,\mathrm{d}y = -\int x e^{-x^2} \,\mathrm{d}x$$

$$y e^y - e^y = \frac{1}{2} e^{-x^2} + C$$

$$2y e^y - 2e^y = e^{-x^2} + C \qquad C \in \mathbb{R}$$

The problem is resolved.

3 Linear equation

Definition (first order linear ODE). Let a, $b \in C(I)$. The equation

$$y' + a(x)y = b(x) \tag{6}$$

is said to be the *first order linear ordinary differential equation* (shortly *LDE*). If $b(x) \equiv 0$ on I, then the equation is called *homogeneous* and *nonhomogeneous* otherwise.

Definition (associated homogeneous equation). The equation

$$y' + a(x)y = 0 \tag{7}$$

is said to be a *homogeneous equation associated with the nonhomogeneous equation (6).*

Remark 2. Every IVP for the linear equation possesses a unique solution defined on the interval I.

Example 3. Equations

$$y' - 2y\ln(x) = \frac{\sin x}{x} \quad \text{and} \quad y' = y + x$$

are linear. Equations

$$y' - \boxed{y^2} = x^2 \quad \text{and} \quad \boxed{yy'} = x^2$$

are not linear. The linearity is broken because of the presence of the square y^2 in the first equation and the product yy' in the second one.

Remark 3 (trivial solution of homogeneous LDE). Homogeneous LDE

$$y' + a(x)y = 0$$

possesses the constant solution $y(x) = 0$ for an arbitrary coefficient $a(x)$. The solution $y(x) = 0$ is called a *trivial solution* and can be obtained by the initial condition $y(\alpha) = 0$ for α arbitrary.

Remark 4 (operator $L[\cdot]$). Let $L[\cdot]$ be an operator defined on the set of smooth functions by the relation

$$L[y](x) = y'(x) + a(x)y(x).$$

Then the linear equation (6) and the corresponding homogeneous equation take the form $L[y] = b(x)$ and $L[y] = 0$.

Remark 5 (linearity of $L[\cdot]$). The operator $L[\cdot]$ satisfies

$$L[C_1y_1 + C_2y_2] = C_1L[y_1] + C_2L[y_2]$$

for all real numbers C_1, C_2 and all differentiable functions $y_1(x)$, $y_2(x)$. Really

$$\begin{aligned} L[C_1y_1 + C_2y_2](x) &= \Big(C_1y_1(x) + C_2y_2(x)\Big)' + a(x)\Big(C_1y_1(x) + C_2y_2(x)\Big) \\ &= C_1y_1'(x) + C_2y_2'(x) + a(x)C_1y_1(x) + a(x)C_2y_2(x) \\ &= C_1\Big(y_1'(x) + a(x)y_1(x)\Big) + C_2\Big(y_2'(x) + a(x)y_2(x)\Big) \\ &= C_1L[y_1](x) + C_2L[y_2](x). \end{aligned}$$

Remark 4 (operator $L[\cdot]$). Let $L[\cdot]$ be an operator defined on the set of smooth functions by the relation

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Remark 5 (linearity of $L[\cdot]$). The operator $L[\cdot]$ satisfies

$$L[C_1 y_1 + C_2 y_2] = C_1 L[y_1] + C_2 L[y_2]$$

for all real numbers C_1, C_2 and all differentiable functions $y_1(x)$, $y_2(x)$. Really

$$\begin{aligned} L[C_1 y_1 + C_2 y_2](x) &= \Big(C_1 y_1(x) + C_2 y_2(x)\Big)' + a(x)\Big(C_1 y_1(x) + C_2 y_2(x)\Big) \\ &= C_1 y_1'(x) + C_2 y_2'(x) + a(x) C_1 y_1(x) + a(x) C_2 y_2(x) \\ &= C_1 \Big(y_1'(x) + a(x) y_1(x)\Big) + C_2 \Big(y_2'(x) + a(x) y_2(x)\Big) \\ &= C_1 L[y_1](x) + C_2 L[y_2](x). \end{aligned}$$

$$L[C_1 y_1 + C_2 y_2] = C_1 L[y_1] + C_2 L[y_2].$$

In plain words: If the function y_1 solves equation $y' + a(x)y = b_1(x)$ and y_2 solves $y' + a(x)y = b_2(x)$, then the function $y = C_1 y_1 + C_2 y_2$ is a solution of the equation

$$y' + a(x)y = C_1 b_1(x) + C_2 b_2(x).$$

Theorem 1 (superposition principle). Let y, y_1 and y_2 be differentiable functions and C be real number

$$\begin{aligned} &L[y_1] = 0 && \Rightarrow && L[C \cdot y_1] = C \cdot 0 = 0, \\ &L[y_1] = 0 \text{ a } L[y_2] = b(x) && \Rightarrow && L[C \cdot y_1 + y_2] = C \cdot 0 + b(x) = b(x), \\ &L[y_1] = L[y_2] = b(x) && \Rightarrow && L[y_1 - y_2] = b(x) - b(x) = 0. \end{aligned}$$

$$L[C_1 y_1 + C_2 y_2] = C_1 L[y_1] + C_2 L[y_2].$$

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Theorem 1 (superposition principle). Let y, y_1 and y_2 be differentiable functions and C be real number

$$\begin{aligned}
&L[y_1] = 0 && \Rightarrow && L[C \cdot y_1] = C \cdot 0 = 0,\\
&L[y_1] = 0 \text{ a } L[y_2] = b(x) && \Rightarrow && L[C \cdot y_1 + y_2] = C \cdot 0 + b(x) = b(x),\\
&L[y_1] = L[y_2] = b(x) && \Rightarrow && L[y_1 - y_2] = b(x) - b(x) = 0.
\end{aligned}$$

Theorem 2 (general solution of homogeneous equation). If $y_{PH}(x)$ is a nontrivial solition of homogeneous LDE, then the function

$$y_{GH}(x) = Cy_{PH}(x), \quad C \in \mathbb{R}$$

is a general solution of this equation.

Theorem 3 (general solution of LDE). Consider nonhomogeneous LDE (6) and associated homogeneous LDE (7).

- If $y_{PN}(x)$ is a particular solution of nonhomogeneous LDE and $y_{GN}(x, C)$ is a general solution of the associated homogeneous LDE, then the function
$$y_{GN}(x, C) = y_{PN}(x) + y_{GH}(x, C) \tag{8}$$
is a general solution of nonhomogeneous LDE.
- If $y_{PN}(x)$ is a particular solution of nonhomogeneous LDE and $y_{PH}(x)$ a nontrivial particular solution of the associated homogeneous LDE, then the function
$$y_{GN}(x, C) = y_{PN}(x) + Cy_{GH}(x) \tag{9}$$
is a general solution of nonhomogeneous LDE.

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$$y_{GN}(x, C) = y_{PN}(x) + Cy_{GH}(x) \tag{9}$$

  is a general solution of nonhomogeneous LDE.

Homogeneous LDE $y' + a(x)y = 0$ via separation of variables.

$y(x) = 0$ is a solution and in the sequel we consider $y \neq 0$.

The homogeneous linear differential equation can be solved by separation of variables.

Homogeneous LDE $y' + a(x)y = 0$ via separation of variables.

$y(x) = 0$ is a solution and in the sequel we consider $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -a(x)y$$

We isolate y' and substitute y' by the quotient $\dfrac{\mathrm{d}x}{\mathrm{d}y}$.

Homogeneous LDE $y' + a(x)y = 0$ via separation of variables.

$y(x) = 0$ is a solution and in the sequel we consider $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -a(x)y$$

$$\frac{1}{y}\,\mathrm{d}y = - \quad a(x)\,\mathrm{d}x,$$

We separate variables.

Homogeneous LDE $y' + a(x)y = 0$ via separation of variables.

$y(x) = 0$ is a solution and in the sequel we consider $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -a(x)y$$

$$\int \frac{1}{y}\,\mathrm{d}y = -\int a(x)\,\mathrm{d}x,$$

$$\ln|y| = -\int a(x)\,\mathrm{d}x + c, \qquad c \in \mathbb{R}$$

We integrate both sides of the equation.

Homogeneous LDE $y' + a(x)y = 0$ via separation of variables.

$y(x) = 0$ is a solution and in the sequel we consider $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -a(x)y$$

$$\int \frac{1}{y}\,\mathrm{d}y = -\int a(x)\,\mathrm{d}x,$$

$$\ln|y| = -\int a(x)\,\mathrm{d}x + c, \qquad c \in \mathbb{R}$$

$$\ln|y| = \ln\left(e^{-\int a(x)\,\mathrm{d}x} \cdot e^{c}\right)$$

We convert both side into logarithmic form.

Homogeneous LDE $y' + a(x)y = 0$ via separation of variables.

$y(x) = 0$ is a solution and in the sequel we consider $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -a(x)y$$

$$\int \frac{1}{y}\,\mathrm{d}y = -\int a(x)\,\mathrm{d}x,$$

$$\ln|y| = -\int a(x)\,\mathrm{d}x + c, \qquad c \in \mathbb{R}$$

$$\cancel{\ln}|y| = \cancel{\ln}\left(e^{-\int a(x)\,\mathrm{d}x} \cdot e^{c}\right)$$

$$y = Ce^{-\int a(x)\,\mathrm{d}x}, \quad C = \pm e^{c} \in \mathbb{R} \setminus \{0\}$$

- The logarithm is one-to-one function and can be removed from both sides.
- We remove the absolute value and rename the constant.

Homogeneous LDE $y' + a(x)y = 0$ via separation of variables.

$y(x) = 0$ is a solution and in the sequel we consider $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -a(x)y$$

$$\int \frac{1}{y}\,\mathrm{d}y = -\int a(x)\,\mathrm{d}x,$$

$$\ln|y| = -\int a(x)\,\mathrm{d}x + c, \qquad c \in \mathbb{R}$$

$$\ln|y| = \ln\left(e^{-\int a(x)\,\mathrm{d}x} \cdot e^{c}\right)$$

$$y = Ce^{-\int a(x)\,\mathrm{d}x}, \quad C = \pm e^{c} \in \mathbb{R} \setminus \{0\}$$

$C = 0$ is also possible

$$\boxed{y = Ce^{-\int a(x)\,\mathrm{d}x}, \quad C \in \mathbb{R}}$$ . . . the general solution

For $C = 0$ we obtain the trivial solution. Hence C can take arbitrary real values.

Homogeneous LDE $y' + a(x)y = 0$ via educated guessing.

$$\begin{aligned}\left(e^{f(x)}\right)' &= e^{f(x)} \cdot f'(x)\\ y' &= -a(x) \cdot y\end{aligned}$$

We compare the chain rule for the exponential function and the homogeneous differential equation.

Homogeneous LDE $y' + a(x)y = 0$ via educated guessing.

$$\begin{aligned} \left(e^{f(x)}\right)' &= e^{f(x)} \cdot f'(x) \\ y' &= -a(x) \cdot y \\ y &= e^{-\int a(x)\,\mathrm{d}x}, \end{aligned}$$

- Both formulas are equivalent, provided $y = e^{f(x)}$ and $f' = -a(x)$. Hence $f(x) = -\int a(x)\,\mathrm{d}x$.
- This is a *particular solution* of the equation.

Homogeneous LDE $y' + a(x)y = 0$ via educated guessing.

$$\begin{aligned}
\left(e^{f(x)}\right)' &= e^{f(x)} \cdot f'(x)\\
y' &= -a(x) \cdot y\\
y &= C \cdot e^{-\int a(x)\,\mathrm{d}x},\ C \in \mathbb{R}
\end{aligned}$$

The general solution is an arbitrary constant multiple of any nontrivial particular solution.

Homogeneous LDE $y' + a(x)y = 0$ via educated guessing.

$$\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x)$$

$$y' = -a(x) \cdot y$$

$$y = C \cdot e^{-\int a(x)\,\mathrm{d}x}, \; C \in \mathbb{R}$$

$$\boxed{y_{GH} = Ce^{-\int a(x)\,\mathrm{d}x}, \quad C \in \mathbb{R}}$$ . . . the general solution

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

- Now consider the nonhomogeneous equation.
- Recall that if $y_P(x)$ is a particular solution of nonhomogeneous LDE and $y_{GH}(x)$ is a general solution of the associated homogeneous LDE, then the function

$$y(x, C) = y_P(x) + y_{GH}(x)$$

is a general solution of nonhomogeneous LDE.

Nonhomogeneous LDE $y' + a(x) \cdot y = \cancel{b(x)}$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

- We consider the associated homogeneous equation first.
- This equation can be solved by separation of variables, as has been shown before.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$y_P(x) = K(x) \cdot y_{PH}(x)$

variation of constant

- Now it is sufficient to find any particular solution of the nonhomogenous equation.
- We replace the constant C in the general solution of the associated homogeneous equation by a function $K(x)$ and try to find condition on the function $K(x)$ which guarantee that this function is a (particular) solution of the nonhomogenous equation.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y_P'(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y_{PH}'(x)$$

- We have to find the function $K(x)$.
- We have to evaluate the derivative y' and substitute into the equation.
- We use the chain rule $(uv)' = u' \cdot v + u \cdot v'$

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y_P'(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y_{PH}'(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y_{PH}'(x)}^{y'}$$

We substitute into the equation.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y_P'(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y_{PH}'(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y_{PH}'(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

We substitute into the equation.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$y_P(x) = K(x) \cdot y_{PH}(x)$

$y'_P(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y'_{PH}(x)$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y'_{PH}(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

$$K'(x)y_{PH}(x) + K(x)\big[y'_{PH}(x) + a(x)y_{PH}(x)\big] = b(x)$$

We factor out the last two terms on the left hand side.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y_P'(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y_{PH}'(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y_{PH}'(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

$$K'(x)y_{PH}(x) + K(x)\left[y_{PH}'(x) + a(x)y_{PH}(x)\right] = b(x)$$

$$K'(x)y_{PH}(x) = b(x)$$

The highlighted expression equals to zero.

Nonhomogeneous LDE $y' + a(x)\cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C\cdot y_{PH}(x)}$

$$y_P(x) = K(x)\cdot y_{PH}(x)$$

$$y_P'(x) = K'(x)\cdot y_{PH}(x) + K(x)\cdot y_{PH}'(x)$$

$$\begin{aligned}
\overbrace{K'(x)y_{PH}(x) + K(x)y_{PH}'(x)}^{y'} + a(x)\cdot\overbrace{K(x)y_{PH}(x)}^{y} &= b(x)\\
K'(x)y_{PH}(x) + K(x)\big[y_{PH}'(x) + a(x)y_{PH}(x)\big] &= b(x)\\
K'(x)y_{PH}(x) &= b(x)\\
K'(x) = \quad &\frac{b(x)}{y_{PH}(x)}
\end{aligned}$$

The resulting equation does not contain the function $K(x)$, but only its derivative $K'(x)$. We isolate $K'(x)$.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y'_P(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y'_{PH}(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y'_{PH}(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

$$K'(x)y_{PH}(x) + K(x)\left[y'_{PH}(x) + a(x)y_{PH}(x)\right] = b(x)$$

$$K'(x)y_{PH}(x) = b(x)$$

$$K\,(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

Integrating we get $K(x)$. The constant of integration can be arbitrary.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y_P'(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y_{PH}'(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y_{PH}'(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

$$K'(x)y_{PH}(x) + K(x)\big[y_{PH}'(x) + a(x)y_{PH}(x)\big] = b(x)$$

$$K'(x)y_{PH}(x) = b(x)$$

$$K\ (x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

We keep the important informations only.

Nonhomogeneous LDE $\boxed{y' + a(x) \cdot y = b(x)}$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$y_P(x) = K(x) \cdot y_{PH}(x)$

$$K(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

We keep the important informations only.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom eq. $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$y_P(x) = K(x) \cdot y_{PH}(x)$

$$K(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

$$y_P(x) = y_{PH}(x) \cdot \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

We use the function $K(x)$ to get the particular solution of the equation.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$K(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

$$y_P(x) = y_{PH}(x) \cdot \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

$$y(x) = y_P(x) + y_{GH}(x)$$

We add the particular solution of the nonhomogenous equation and the general solution of the homogeneous equation. The equation has been solved.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via variation of constant.

The asociated hom. eq. $\boxed{y' + a(x)y = 0}$

The gen. sol. of the assoc. hom $\boxed{y_{GH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH}(x)}$

$y_P(x) = K(x) \cdot y_{PH}(x)$

$$K(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

$$y_P(x) = y_{PH}(x) \cdot \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

$$y(x) = y_P(x) + y_{GH}(x) = e^{-\int a(x)\,\mathrm{d}x} \int b(x) e^{\int a(x)\,\mathrm{d}x}\,\mathrm{d}x + Ce^{-\int a(x)\,\mathrm{d}x}$$

Substituting for y_{GH} and y_P we obtain an explicit formula for the solution of the (nonhomogeneous) linear equation.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via integrating factor.

$$y' + a(x)y = b(x)$$

We derive the explicit formula for the solution of LDE using the so called *integrating factor*.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via integrating factor.

$$y' + a(x)y = b(x)$$

$$y' e^{\int a(x)\,\mathrm{d}x} + a(x) y e^{\int a(x)\,\mathrm{d}x} = b(x) e^{\int a(x)\,\mathrm{d}x}$$

We multiply both sides of the equation by the *integrating factor* $e^{\int a(x)\,\mathrm{d}x}$.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via integrating factor.

$$\begin{aligned} y' + a(x)y &= b(x) \\ y'e^{\int a(x)\,\mathrm{d}x} + a(x)ye^{\int a(x)\,\mathrm{d}x} &= b(x)e^{\int a(x)\,\mathrm{d}x} \\ \left(ye^{\int a(x)\,\mathrm{d}x}\right)' &= b(x)e^{\int a(x)\,\mathrm{d}x} \end{aligned}$$

The left hand side can be written as derivative of product. Really

$$\begin{aligned} \left(ye^{\int a(x)\,\mathrm{d}x}\right)' &= y'e^{\int a(x)\,\mathrm{d}x} + y\left(e^{\int a(x)\,\mathrm{d}x}\right)' \\ &= y'e^{\int a(x)\,\mathrm{d}x} + ye^{\int a(x)\,\mathrm{d}x}a(x) \end{aligned}$$

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via integrating factor.

$$\begin{aligned} y' + a(x)y &= b(x) \\ y'e^{\int a(x)\,\mathrm{d}x} + a(x)ye^{\int a(x)\,\mathrm{d}x} &= b(x)e^{\int a(x)\,\mathrm{d}x} \\ \left(ye^{\int a(x)\,\mathrm{d}x}\right)' &= b(x)e^{\int a(x)\,\mathrm{d}x} \\ ye^{\int a(x)\,\mathrm{d}x} &= \int b(x)e^{\int a(x)\,\mathrm{d}x}\,\mathrm{d}x + C \end{aligned}$$

Integrating we remove the derivative.

Nonhomogeneous LDE $y' + a(x) \cdot y = b(x)$ via integrating factor.

$$\begin{aligned}
y' + a(x)y &= b(x)\\
y' e^{\int a(x)\,\mathrm{d}x} + a(x) y e^{\int a(x)\,\mathrm{d}x} &= b(x) e^{\int a(x)\,\mathrm{d}x}\\
\left(y e^{\int a(x)\,\mathrm{d}x}\right)' &= b(x) e^{\int a(x)\,\mathrm{d}x}\\
y e^{\int a(x)\,\mathrm{d}x} &= \int b(x) e^{\int a(x)\,\mathrm{d}x}\,\mathrm{d}x + C\\
y &= e^{-\int a(x)\,\mathrm{d}x}\left[\int b(x) e^{\int a(x)\,\mathrm{d}x}\,\mathrm{d}x + C\right]
\end{aligned}$$

We isolate y. Finished.

Solve DE $\quad y' + \frac{2}{x}\, y = \frac{1}{x+1}$

The equation is a linear differential equation.

Solve DE $y' + \frac{2}{x} y = \xcancel{\frac{1}{x+1}}$

$$y' + \frac{2}{x} y = 0$$

- We write the corresponding homogeneous equation.
- We replace the function on the right hand side by zero.

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y' + \frac{2}{x} y = 0$$

$$y_{GH}(x) = Ke^{-\int \frac{2}{x}\,\mathrm{d}x}$$

- The general solution of $y + a(x)y = 0$ is given by formula $y = Ke^{-\int a(x)\,\mathrm{d}x}$.
- In our problem we have $a(x) = \frac{2}{x}$.

Solve DE $y' + \frac{2}{x}\, y = \frac{1}{x+1}$

$$y' + \frac{2}{x} y = 0$$

$$y_{GH}(x) = K e^{-\int \frac{2}{x}\,\mathrm{d}x} = K e^{-2\ln|x|}$$

We integrate. . .

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y' + \frac{2}{x} y = 0$$

$$y_{GH}(x) = Ke^{-\int \frac{2}{x}\,\mathrm{d}x} = Ke^{-2\ln|x|} = Ke^{\ln x^{-2}}$$

…simplify …

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$y_{GH}(x) = Kx^{-2}$

$$y' + \frac{2}{x}y = 0$$

$$y_{GH}(x) = Ke^{-\int \frac{2}{x}\,\mathrm{d}x} = Ke^{-2\ln|x|} = Ke^{\ln x^{-2}} = Kx^{-2}$$

and simplify even more. Remember that exponential function is inverse to logarithmic function and hence the composition of these functions is identity.

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$y_{GH}(x) = Kx^{-2}$

$y_{PN}(x) = K(x) \cdot x^{-2}$

- Now we look for the particular solution of nonhomogeneous equation.
- Suppose that K is a function in $y_{GH}(x)$.

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

- We find the derivative $y'_{PN}(x)$.
- We use the product rule for derivatives: $(uv)' = u'v + uv'$.

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

We substitute into nonhomogeneous equation.

$$\boxed{\text{Solve DE} \quad y' + \frac{2}{x} y = \frac{1}{x+1}}$$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

- We solve the last equation with respect to K'.
- The expressions with K cancel. Really: $(-2)Kx^{-3}+\frac{2}{x}Kx^{-2} = 0$.

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

$$K'(x) = x - 1 + \frac{1}{x+1}$$

- The function K can be obtained from K' by integration.
- We would like to integrate the improper rational function. We have to use the long division first.

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

$$K'(x) = x - 1 + \frac{1}{x+1}$$

$$K(x) = \int x - 1 + \frac{1}{x+1}\,\mathrm{d}x$$

We integrate. . .

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

$$K'(x) = x - 1 + \frac{1}{x+1}$$

$$K(x) = \int x - 1 + \frac{1}{x+1}\,\mathrm{d}x$$

$$= \frac{x^2}{2} - x + \ln|x+1|$$

…and $K(x) = \frac{x^2}{2} + x + \ln|x+1|$.

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

$$K'(x) = x - 1 + \frac{1}{x+1}$$

$$K(x) = \int x - 1 + \frac{1}{x+1}\,\mathrm{d}x$$

$$= \frac{x^2}{2} - x + \ln|x+1|$$

We keep the important computations only.

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$K(x) = \frac{x^2}{2} - x + \ln|x+1|$$

The function $K(x)$ is known and it can be used in the formula for $y_{PN}(x)$.

Solve DE $y' + \frac{2}{x}\, y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$K(x) = \frac{x^2}{2} - x + \ln|x+1|$$

$$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2}$$

We use this $K(x)$...

Solve DE $y' + \frac{2}{x}\, y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$K(x) = \frac{x^2}{2} - x + \ln|x+1|$$

$$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2} = \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$$

. . . and simplify

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$K(x) = \frac{x^2}{2} - x + \ln|x+1|$$

$$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2} = \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$$

$$y_{GN}(x) = y_{GH}(x) + y_{PN}(x)$$

The general solution $y_{GN}(x)$ of nonhomogeneous equation is a sum of $y_{GH}(x)$ and $y_{PN}(x)$.

Solve DE $y' + \frac{2}{x}\, y = \frac{1}{x+1}$

$$y_{GH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$K(x) = \frac{x^2}{2} - x + \ln|x+1|$$

$$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2} = \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$$

$$y_{GN}(x) = y_{GH}(x) + y_{PN}(x) = \frac{K}{x^2} + \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$$

We substitute for y_{PN} and y_{GH}.

Solve DE $y' + \frac{2}{x} y = \frac{1}{x+1}$

$y_{GH}(x) = Kx^{-2}$

$y_{PN}(x) = K(x) \cdot x^{-2}$

$K(x) = \frac{x^2}{2} - x + \ln|x+1|$

$$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2} = \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$$

$$y_{GN}(x) = y_{GH}(x) + y_{PN}(x) = \frac{K}{x^2} + \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}, \quad K \in \mathbb{R}$$

The problem is resolved.

Solve DE $\quad y' = 1 + 3y \operatorname{tg} x.$

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$y' - 3y \operatorname{tg} x = 1 \ldots$ original equation

We convert the linear equation into the form

$$y' - a(x)y = b(x).$$

Hence $a(x) = -3 \operatorname{tg} x$ and $b(x) = 1$.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$y' - 3y \operatorname{tg} x = \cancel{1}$... original equation

$y' - 3y \operatorname{tg} x = 0$... associated homogeneous equation

We write the corresponding homogeneous equation. We replace the right-hand side by zero.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx}$$

The general solution of

$$y' + a(x)y = 0$$

is given by the formula $y = Ce^{-\int a(x)\, dx}$.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx} = Ce^{-3 \ln \cos x}$$

We evaluate the integral as follows:

$$\int -3 \operatorname{tg} x \, dx = \int 3 \frac{-\sin x}{\cos x} \, dx = 3 \ln |\cos x|.$$

Here we used the formula $\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)|$. In the following we will suppose that we work on the interval, where $\cos x > 0$. In this case we omit the absolute value.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \dots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \dots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \,\mathrm{d}x} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x}$$

We convert the function into the form in which the exponential function follows the logarithmic function.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, \mathrm{d}x} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

The functions $\ln(x)$ and e^x are mutually inverse function and the composition $e^{\ln x}$ is identity.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \dots \text{original equation}$$

$$y' - 3y \operatorname{tg} x = 0 \dots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

$$y_{PN}(x) = K(x) \cos^{-3} x$$

- Now we have the general solution of homogeneous equation.
- We look for the particular solution of nonhomogeneous equation in the form, in which the constant from y_{GH} is replaced by the function $K(x)$.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

$$y_{PN}(x) = K(x) \cos^{-3} x$$
$$y'_{PN}(x) = K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)$$

- When evaluating the derivative of $y'_{PN}(x)$ we use the product rule $(uv)' = u'v + uv'$.
- The derivative of $\cos^{-3} x$ is evaluated by chain rule.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, \mathrm{d}x} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

$$y_{PN}(x) = K(x) \cos^{-3} x$$
$$y'_{PN}(x) = K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)$$

$$\overbrace{K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x) \cos^{-3} x}^{y} \operatorname{tg} x = 1$$

We substitute for y and y' into the original equation.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$y' - 3y \operatorname{tg} x = 1$… original equation

$y' - 3y \operatorname{tg} x = 0$… associated homogeneous equation

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

$$y_{PN}(x) = K(x) \cos^{-3} x$$

$$y'_{PN}(x) = K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)$$

$$\overbrace{K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x) \cos^{-3} x}^{y} \operatorname{tg} x = 1$$

We clean the informations which are no more important.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C\cos^{-3} x$ $\qquad$ $y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

The term with $K(x)$ disappear, since

$$K(x)(-3)\cos^{-4} x(-\sin x) - 3K(x)\cos^{-3} x \operatorname{tg} x = 0.$$

We obtain the equation for $K'(x)$.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C \cos^{-3} x$ $\qquad$ $y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

We solve that equation for $K'(x)$...

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C\cos^{-3} x$ $\qquad$ $y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x \, \mathrm{d}x$$

…and integrate. This gives $K(x)$.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C \cos^{-3} x$ $\qquad$ $y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x \,\mathrm{d}x = \int (1 - \sin^2 x)\cos x \,\mathrm{d}x$$

We write $\cos^3 x$ in the form

$$\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x)\cos x.$$

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C \cos^{-3} x$ $\qquad y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x \, \mathrm{d}x = \int (1 - \sin^2 x)\cos x \, \mathrm{d}x$$

$$= \sin x - \frac{\sin^3 x}{3}$$

The integral is ready for substitution $\sin x = t$, $\cos x \, \mathrm{d}x = \mathrm{d}t$. This converts the integral into

$$\int (1 - t^2)\,\mathrm{d}t = t - \frac{t^3}{3} = \sin x - \frac{\sin^3 x}{3}.$$

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C\cos^{-3} x$ $\qquad$ $y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y}\operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^{3} x$$

$$K(x) = \int \cos^{3} x \,\mathrm{d}x = \int (1 - \sin^{2} x)\cos x \,\mathrm{d}x$$

$$= \sin x - \frac{\sin^{3} x}{3}$$

$$y_{PN}(x) = \left(\sin x - \frac{\sin^{3} x}{3}\right) \cdot \cos^{-3} x = \frac{\sin x}{\cos^{3} x} - \frac{\sin^{3} x}{3\cos^{3} x}$$

We use the function K in the formula for y_{PN}.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C \cos^{-3} x$ $\qquad$ $y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^{3} x$$

$$K(x) = \int \cos^{3} x \, dx = \int (1 - \sin^{2} x)\cos x \, dx$$

The general solution of nonhomogeneous equation is a sum of particular solution of that equation and general solution of homogeneous equation.

$$y_{PN}(x) = \left(\sin x - \frac{\sin^3 x}{3}\right) \cdot \cos^{-3} x = \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}$$

$$y_{GN}(x) = y_{GH}(x) + y_{PN}(x)$$

Solve DE $y' = 1 + 3y\operatorname{tg} x$.

Summary: $y_{GH}(x) = C\cos^{-3} x$ $\qquad y_{PN}(x) = K(x)\cdot\cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y}\operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x\,\mathrm{d}x = \int (1-\sin^2 x)\cos x\,\mathrm{d}x$$

Both y_{GH} and y_{PN} are known and we can substitute.

$$y_{PN}(x) = \left(\sin x - \frac{\sin^3 x}{3}\right)\cdot\cos^{-3} x \quad \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}$$

$$y_{GN}(x) = y_{GH}(x) + y_{PN}(x) = \frac{C}{\cos^3 x} + \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}, \qquad C\in\mathbb{R}$$

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C \cos^{-3} x$ $\qquad$ $y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^{3} x$$

$$K(x) = \int \cos^{3} x \,\mathrm{d}x = \int (1 - \sin^{2} x)\cos x \,\mathrm{d}x$$

The problem is resolved.

$$y_{PN}(x) = \left(\sin x - \frac{\sin^3 x}{3}\right) \cdot \cos^{-3} x = \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}$$

$$y_{GN}(x) = y_{GH}(x) + y_{PN}(x) = \frac{C}{\cos^3 x} + \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}, \qquad C \in \mathbb{R}$$

Solve DE $\quad xy' + y = x\ln(x+1)$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1)$$

- We write the equation in its normal form $y' + a(x)y = b$.
- We divide by x. Hence we lok for the solution either on $(-1, 0)$ (see the logarithmic function) or on $(0, \infty)$.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \xcancel{\ln(x+1)} \qquad\qquad y' + \frac{1}{x}y = 0$$

We write the corresponding homogeneous equation.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x}$$

The general solution of the homogeneous equation

$$y' + a(x)y = 0$$

is

$$y_{GH} = Ce^{-\int a(x)\,\mathrm{d}x}.$$

In our case we have $a(x) = \dfrac{1}{x}$.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x} = Ce^{-\ln|x|}$$

We evaluate the integral. . .

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x} = Ce^{-\ln|x|} = Ce^{\ln|x|^{-1}}$$

…and simplify.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x} = Ce^{-\ln|x|} = Ce^{\ln|x|^{-1}} = C|x|^{-1} = \frac{C}{|x|}$$

The composition $e^{\ln x}$ is identity.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x} = Ce^{-\ln|x|} = Ce^{\ln|x|^{-1}} = C|x|^{-1} = \frac{C}{|x|} = \frac{K}{x}$$

If we introduce the new constant $C = \pm K$, we can write the general solution of homogeneous equation in the form $y_{GH} = \dfrac{K}{x}$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x}$$

- Now let us look for the solution of nonhomogeneous equation.
- We replace the constant K in the formula for y_{GH} by the function $K(x)$.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

We evaluate the derivative of the function y_{PN} by the product rule

$$(uv)' = u'v + uv'.$$

We differentiate the function $\frac{1}{x}$ as a power function x^{-1}. Hence

$$\left(\frac{1}{x}\right)' = (x^{-1})' = (-1)x^{-2}.$$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$\overbrace{K'(x)\frac{1}{x} + K(x)(-1)x^{-2}}^{y'} + \frac{1}{x}\overbrace{K(x)\frac{1}{x}}^{y} = \ln(x+1)$$

We substitute for y' and y into original equation

$$y' + \frac{1}{x}y = \ln(x+1).$$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$\overbrace{K'(x)\frac{1}{x} + K(x)(-1)x^{-2}}^{y'} + \frac{1}{x}\overbrace{K(x)\frac{1}{x}}^{y} = \ln(x+1)$$

$$K'(x)\frac{1}{x} = \ln(x+1)$$

The terms with $K(x)$ cancel and only $K'(x)$ remains.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$\overbrace{K'(x)\frac{1}{x} + K(x)(-1)x^{-2}}^{y'} + \frac{1}{x}\overbrace{K(x)\frac{1}{x}}^{y} = \ln(x+1)$$

$$K'(x)\frac{1}{x} = \ln(x+1) \qquad K'(x) = x\ln(x+1)$$

We solve the equation for $K'(x)$ ...

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x$$

…and integrate.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

We use integration by parts with $\boxed{\begin{array}{ll} u = \ln(x+1) & u' = \dfrac{1}{x+1} \\ v' = x & v = \dfrac{x^2}{2} \end{array}}$. This gives

$$\begin{aligned}\int x\ln(x+1)\,\mathrm{d}x &= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\,\mathrm{d}x \\ &= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int x - 1 + \frac{1}{x+1}\,\mathrm{d}x\end{aligned}$$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

$$y_{PN}(x) = K(x)\frac{1}{x} = \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)$$

We substitute for $K(x)$ into the relation for $y_{PN}(x)$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

$$y_{PN}(x) = K(x)\frac{1}{x} = \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)$$

$$y_{GN} = y_{GH} + y_{PN}$$

The general solution of nonhomogeneous equation is a sum of general solution of homogeneous equation and the particular solution of nonhomogeneous equation.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

$$y_{PN}(x) = K(x)\frac{1}{x} = \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)$$

$$y_{GN} = y_{GH} + y_{PN} = \frac{K}{x} + \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)\,, \qquad K \in \mathbb{R}$$

We use that solutions. . .

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

$$y_{PN}(x) = K(x)\frac{1}{x} = \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)$$

$$y_{GN} = y_{GH} + y_{PN} = \frac{K}{x} + \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)\ , \qquad K\in\mathbb{R}$$

. . . and the problem is resolved.

Solve IVP $\quad xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x}y = x$$

We write the equation in the standard form

$$y' + a(x)y = b(x).$$

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x}y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

The comparison with

$$y' + a(x)y = b(x)$$

gives $a(x)$ and $b(x)$.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x$$

The solution of

$$y' + a(x)y = b(x)$$

is

$$y(x) = \frac{C + \int b(x)A\,\mathrm{d}x}{A}, \quad \text{where} \quad A = e^{\int a(x)\,\mathrm{d}x}.$$

We evaluate $\int a(x)\,\mathrm{d}x$.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x}y = x \qquad a(x) = \frac{x+1}{x} \qquad \text{and} \qquad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = \int \frac{x+1}{x}\,\mathrm{d}x$$

This is $\int a(x)\,\mathrm{d}x$.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = \int \frac{x+1}{x}\,\mathrm{d}x = \int \left(1 + \frac{1}{x}\right)\,\mathrm{d}x$$

We divide the numerator...

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = \int \frac{x+1}{x}\,\mathrm{d}x = \int \left(1 + \frac{1}{x}\right)\,\mathrm{d}x = x + \ln|x|$$

…and integrate.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = \int \frac{x+1}{x}\,\mathrm{d}x = \int \left(1 + \frac{1}{x}\right)\,\mathrm{d}x = x + \ln|x|$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{x+\ln x}$$

Since the initial condition is given in $x = 1$, the absolute value can be omitted. Now we evaluate $A = e^{\int a(x)\,\mathrm{d}x}$.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = \int \frac{x+1}{x}\,\mathrm{d}x = \int \left(1 + \frac{1}{x}\right)\,\mathrm{d}x = x + \ln|x|$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{x+\ln x} = e^{\ln x} e^{x}$$

We use the rule $e^{a+b} = e^{a} e^{b}$...

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = \int \frac{x+1}{x}\,\mathrm{d}x = \int \left(1 + \frac{1}{x}\right)\,\mathrm{d}x = x + \ln|x|$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{x+\ln x} = e^{\ln x} e^{x} = xe^{x}$$

. . . and the fact that the composition of exponential and logarithm is identity.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x) A\,\mathrm{d}x = \int b(x) e^{\int a(x)\,\mathrm{d}x}$$

Now we evaluate $\int b(x) A\,\mathrm{d}x$, i.e. $\int b(x) e^{\int a(x)\,\mathrm{d}x}\,\mathrm{d}x$.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x) A\,\mathrm{d}x = \int b(x) e^{\int a(x)\,\mathrm{d}x} = \int x^2 e^x\,\mathrm{d}x$$

Substitution for $b(x)$ and $e^{\int a(x)\,\mathrm{d}x}$ gives this integral.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x)A\,\mathrm{d}x = \int b(x)e^{\int a(x)\,\mathrm{d}x} = \int x^2e^x\,\mathrm{d}x = e^x(x^2 - 2x + 2)$$

We integrate two times by parts.

$$\begin{aligned}\int x^2e^x\,\mathrm{d}x &= x^2e^x - 2\int xe^x\,\mathrm{d}x\\ &= x^2e^x - 2(xe^x - \int e^x\,\mathrm{d}x)\\ &= x^2e^x - 2(xe^x - e^x)\end{aligned}$$

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x)A\,\mathrm{d}x = \int b(x)e^{\int a(x)\,\mathrm{d}x} = \int x^2 e^x\,\mathrm{d}x = e^x(x^2 - 2x + 2)$$

$$y = \frac{C + e^x(x^2 - 2x + 2)}{xe^x}$$

The formula for the general solution is

$$y(x) = \frac{C + \int b(x)A\,\mathrm{d}x}{A}.$$

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x)A\,\mathrm{d}x = \int b(x)e^{\int a(x)\,\mathrm{d}x} = \int x^2e^x\,\mathrm{d}x = e^x(x^2 - 2x + 2)$$

$$y = \frac{C + e^x(x^2 - 2x + 2)}{xe^x} = \frac{C}{xe^x} + x - 2 + \frac{2}{x}$$

After simplification we obtain the general solution in the explicit form.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x) A\,\mathrm{d}x = \int b(x) e^{\int a(x)\,\mathrm{d}x} = \int x^2 e^x\,\mathrm{d}x = e^x(x^2 - 2x + 2)$$

$$y = \frac{C + e^x(x^2 - 2x + 2)}{xe^x} = \frac{C}{xe^x} + x - 2 + \frac{2}{x}$$

$y(1) = -1$

We have to solve the initial value problem.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x}y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x)A\,\mathrm{d}x = \int b(x)e^{\int a(x)\,\mathrm{d}x} = \int x^2e^x\,\mathrm{d}x = e^x(x^2 - 2x + 2)$$

$$y = \frac{C + e^x(x^2 - 2x + 2)}{xe^x} = \frac{C}{xe^x} + x - 2 + \frac{2}{x}$$

$$y(1) = -1 \qquad \Rightarrow \qquad -1 = \frac{C}{e} + 1 - 2 + 2$$

We substitute from the initial condition to the general solution. We put $x = 1$ and $y = -1$.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x)A\,\mathrm{d}x = \int b(x)e^{\int a(x)\,\mathrm{d}x} = \int x^2e^x\,\mathrm{d}x = e^x(x^2 - 2x + 2)$$

$$y = \frac{C + e^x(x^2 - 2x + 2)}{xe^x} = \frac{C}{xe^x} + x - 2 + \frac{2}{x}$$

$$y(1) = -1 \quad \Rightarrow \quad -1 = \frac{C}{e} + 1 - 2 + 2 \quad \Rightarrow \quad C = -2e$$

The solution of this equation is $C = -2e$.

Solve IVP $xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x) A\,\mathrm{d}x = \int b(x) e^{\int a(x)\,\mathrm{d}x} = \int x^2 e^x\,\mathrm{d}x = e^x(x^2 - 2x + 2)$$

$$y = \frac{C + e^x(x^2 - 2x + 2)}{xe^x} = \frac{C}{xe^x} + x - 2 + \frac{2}{x}$$

$$y(1) = -1 \qquad \Rightarrow \qquad -1 = \frac{C}{e} + 1 - 2 + 2 \qquad \Rightarrow \qquad C = -2e$$

$$y = x - 2 + \frac{2}{x} - \frac{2e}{xe^x}$$

We use this constant in the general solution. . .

Solve IVP $\quad xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x) A \,\mathrm{d}x = \int b(x) e^{\int a(x)\,\mathrm{d}x} = \int x^2 e^x \,\mathrm{d}x = e^x(x^2 - 2x + 2)$$

$$y = \frac{C + e^x(x^2 - 2x + 2)}{xe^x} = \frac{C}{xe^x} + x - 2 + \frac{2}{x}$$

$$y(1) = -1 \qquad \Rightarrow \qquad -1 = \frac{C}{e} + 1 - 2 + 2 \qquad \Rightarrow \qquad C = -2e$$

$$y = x - 2 + \frac{2}{x} - \frac{2e}{xe^x} = x - 2 + \frac{2}{x} - \frac{2}{x} e^{1-x}$$

. . . and simplify.

Solve IVP $\quad xy' + xy + y - x^2 = 0, \quad y(1) = -1.$

$$y' + \frac{x+1}{x} y = x \qquad a(x) = \frac{x+1}{x} \quad \text{and} \quad b(x) = x$$

$$\int a(x)\,\mathrm{d}x = x + \ln|x| \qquad A = e^{\int a(x)\,\mathrm{d}x} = xe^x$$

$$\int b(x) A\,\mathrm{d}x = \int b(x) e^{\int a(x)\,\mathrm{d}x} = \int x^2 e^x\,\mathrm{d}x = e^x(x^2 - 2x + 2)$$

$$y = \frac{C + e^x(x^2 - 2x + 2)}{xe^x} = \frac{C}{xe^x} + x - 2 + \frac{2}{x}$$

$$y(1) = -1 \qquad \Rightarrow \qquad -1 = \frac{C}{e} + 1 - 2 + 2 \qquad \Rightarrow \qquad C = -2e$$

$$y = x - 2 + \frac{2}{x} - \frac{2e}{xe^x} = x - 2 + \frac{2}{x} - \frac{2}{x} e^{1-x}$$

The initial value problem is resolved.

Solve DE $\quad xy' + 2y = e^{-x^2}$.

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x}$$

We divide the equation by x. This removes the term x which multiplies the derivative y'.

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

Equation is now in its standard form $y' + a(x)y = b(x)$. We can indetify the functions $a(x)$ and $b(x)$.

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x$$

The formula for the general solution is

$$y(x) = \frac{C + \int b(x)A\,\mathrm{d}x}{A}, \quad \text{where} \quad A = e^{\int a(x)\,\mathrm{d}x}.$$

We evaluate $\int a(x)\,\mathrm{d}x$.

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x|$$

This integral is simple.

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

We convert the result into the form of logarithm.

Solve DE $xy' + 2y = e^{-x^2}.$

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x}$$

We continue with evaluation of the quantity A from the general formula.

Solve DE $\quad xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{\ln x^2}$$

We substitute for the integral in the exponent.

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{\ln x^2} = x^2$$

Exponential function and logarithm are mutually inverse functions. The composition of the function and its inverse gives identiy function. Hence the argument of logarithm remains only.

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{\ln x^2} = x^2$$

$$\int b(x)A\,\mathrm{d}x$$

The last expression from general formula was this expression.

Solve DE $xy' + 2y = e^{-x^2}.$

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{\ln x^2} = x^2$$

$$\int b(x)A\,\mathrm{d}x = \int \frac{e^{-x^2}}{x}x^2\,\mathrm{d}x$$

We substitute for A and $b(x)$.

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{\ln x^2} = x^2$$

$$\int b(x)A\,\mathrm{d}x = \int \frac{e^{-x^2}}{x}x^2\,\mathrm{d}x = \int xe^{-x^2}\,\mathrm{d}x$$

We simpify.

Solve DE $\quad xy' + 2y = e^{-x^2}.$

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{\ln x^2} = x^2$$

$$\int b(x)A\,\mathrm{d}x = \int \frac{e^{-x^2}}{x}x^2\,\mathrm{d}x = \int xe^{-x^2}\,\mathrm{d}x = -\frac{1}{2}e^{-x^2}$$

The composite function e^{-x^2} suggests substitution $(-x^2) = t$, $-2x\,\mathrm{d}x = \mathrm{d}t$. This substitution gives

$$\int xe^{-x^2}\,\mathrm{d}x = -\frac{1}{2}\int e^t\,\mathrm{d}t = -\frac{1}{2}e^t = -\frac{1}{2}e^{-x^2}.$$

Solve DE $xy' + 2y = e^{-x^2}.$

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{\ln x^2} = x^2$$

$$\int b(x)A\,\mathrm{d}x = \int \frac{e^{-x^2}}{x}x^2\,\mathrm{d}x = \int xe^{-x^2}\,\mathrm{d}x = -\frac{1}{2}e^{-x^2}$$

$$y = \frac{1}{x^2}\left[C - \frac{1}{2}e^{-x^2}\right]$$

Now we use our results in the general formula for solution.

$$y = \frac{1}{A}\left[C + \int b(x)A\,\mathrm{d}x\right]$$

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{\ln x^2} = x^2$$

$$\int b(x)A\,\mathrm{d}x = \int \frac{e^{-x^2}}{x}x^2\,\mathrm{d}x = \int xe^{-x^2}\,\mathrm{d}x = -\frac{1}{2}e^{-x^2}$$

$$y = \frac{1}{x^2}\left[C - \frac{1}{2}e^{-x^2}\right] = \frac{1}{2x^2}\left[K - e^{-x^2}\right]$$

We take out the factor $\frac{1}{2}$ from the brackets. This yields the constant $2C$ inside. We replace this constant $2C$ by new constant K

Solve DE $xy' + 2y = e^{-x^2}$.

$$y' + \frac{2}{x}y = \frac{e^{-x^2}}{x} \qquad a(x) = \frac{2}{x} \qquad b(x) = \frac{e^{-x^2}}{x}$$

$$\int a(x)\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x = 2\ln|x| = \ln x^2$$

$$A = e^{\int a(x)\,\mathrm{d}x} = e^{\ln x^2} = x^2$$

$$\int b(x)A\,\mathrm{d}x = \int \frac{e^{-x^2}}{x}x^2\,\mathrm{d}x = \int xe^{-x^2}\,\mathrm{d}x = -\frac{1}{2}e^{-x^2}$$

$$y = \frac{1}{x^2}\left[C - \frac{1}{2}e^{-x^2}\right] = \frac{1}{2x^2}\left[K - e^{-x^2}\right] \qquad K \in \mathbb{R}$$

The problem is resolved.

Real world applications (local links)

- Suspenstion bridges
- Sociologic diffusion

Further reading

- `http://eqworld.ipmnet.ru/en/solutions/ode/ode-toc1.htm`
- `http://en.wikipedia.org/wiki/Examples_of_differential_equations`
- `http://www.sosmath.com/diffeq/first/separable/separable.html`
- `http://www.sosmath.com/diffeq/first/lineareq/lineareq.html`
- `http://www.sosmath.com/diffeq/slope/slope1.html`
- `http://www.sosmath.com/diffeq/second/linear/secondlinear.html`
- `http://www.sosmath.com/diffeq/second/homolinear/homolinear.html`
- `http://www.sosmath.com/diffeq/second/nonhomo/nonhomo.html`
- `http://www.sosmath.com/diffeq/slope/slope1.html`
- `http://www.chass.utoronto.ca/~osborne/MathTutorial/`