Multivariable Calculus

Robert Mařík

May 9, 2006

Contents

1 Introduction

Definition (function of two variables)**.** The rule f which to each ordered pair of the real numbers (x, y) from the given set $A \subseteq \mathbb{R}^2$ assigns exactly one another real number is called a *function of two variables* defined on A. We write $f : A \to \mathbb{R}$, particularly if $A = \mathbb{R}^2$ we write $f : \mathbb{R}^2 \to \mathbb{R}$.

- For the function of two variables we define a *domain* and an *image* of the function in the same manner as for the function of one variable.
- Under a *graph* of the function f of two variables we understand the set $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \mathrm{Dom}(f)$ and $z = f(x, y)\}$. Usually the graph can be considered as a surface in the 3-dimensional space.
- Let z_0 be a real number. An intersection of the graph of the function $f(x, y)$ with the horizontal plane $z = z_0$ is called a *level curve at the level* z_0. (Such a level curves you know from the geography.)

1 Introduction

Definition (function of two variables). The rule f which to each ordered pair of the real numbers (x, y) from the given set $A \subseteq \mathbb{R}^2$ assigns exactly one another real number is called a *function of two variables* defined on A. We write $f : A \to \mathbb{R}$, particularly if $A = \mathbb{R}^2$ we write $f : \mathbb{R}^2 \to \mathbb{R}$.

- For the function of two variables we define a *domain* and an *image* of the function in the same manner as for the function of one variable.

- Under a *graph* of the function f of two variables we understand the set $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \mathrm{Dom}(f)$ and $z = f(x, y)\}$. Usually the graph can be considered as a surface in the 3-dimensional space.

- Let z_0 be a real number. An intersection of the graph of the function $f(x, y)$ with the horizontal plane $z = z_0$ is called a *level curve at the level* z_0. (Such a level curves you know from the geography.)

1 Introduction

Definition (function of two variables)**.** The rule f which to each ordered pair of the real numbers (x, y) from the given set $A \subseteq \mathbb{R}^2$ assigns exactly one another real number is called a *function of two variables* defined on A. We write $f : A \to \mathbb{R}$, particularly if $A = \mathbb{R}^2$ we write $f : \mathbb{R}^2 \to \mathbb{R}$.

- For the function of two variables we define a *domain* and an *image* of the function in the same manner as for the function of one variable.
- Under a *graph* of the function f of two variables we understand the set $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \mathrm{Dom}(f)$ and $z = f(x, y)\}$. Usually the graph can be considered as a surface in the 3-dimensional space.
- Let z_0 be a real number. An intersection of the graph of the function $f(x, y)$ with the horizontal plane $z = z_0$ is called a *level curve at the level* z_0. (Such a level curves you know from the geography.)

1 Introduction

Definition (function of two variables)**.** The rule f which to each ordered pair of the real numbers (x, y) from the given set $A \subseteq \mathbb{R}^2$ assigns exactly one another real number is called a *function of two variables* defined on A. We write $f : A \to \mathbb{R}$, particularly if $A = \mathbb{R}^2$ we write $f : \mathbb{R}^2 \to \mathbb{R}$.

- For the function of two variables we define a *domain* and an *image* of the function in the same manner as for the function of one variable.
- Under a *graph* of the function f of two variables we understand the set $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \mathrm{Dom}(f)$ and $z = f(x, y)\}$. Usually the graph can be considered as a surface in the 3-dimensional space.
- Let z_0 be a real number. An intersection of the graph of the function $f(x, y)$ with the horizontal plane $z = z_0$ is called a *level curve at the level z_0*. (Such a level curves you know from the geography.)

2 Continuity

Definition (neighborhood in $\mathbb{R}^2$). Let $X = (x, y) \in \mathbb{R}^2$ be a point in $\mathbb{R}^2$ and $\varepsilon > 0$ be a positive real number. Under an ε-*neighborhood of the point* X we understand the set

$$N_\varepsilon(X) = \{X_0 = (x_0, y_0) \in \mathbb{R}^2 : \rho(X, X_0) < \varepsilon\},$$

where $\rho(X, X_0)$ is the *Euclidean distance* of the points X and Y defined in a usual way, by the relation

$$\rho(X, X_0) = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Under the *reduced* ε-*neighborhood of the point* X we understand the set

$$\overline{N}_\varepsilon(X) = N_\varepsilon(X) \setminus \{X\}.$$

Definition (continuity)**.** Let f be a function of two variables and $X_0 = (x_0, y_0)$ be a point from its domain. The function f is said to be *continuous* at $X_0 = (x_0, y_0)$ if for every real number ε, $\varepsilon > 0$, there exists a neighborhood $N(X_0)$ of the point X_0 such that $|f(x, y) - f(x_0, y_0)| < \varepsilon$ for every $X = (x, y) \in N(X_0)$.

Theorem 1 (continuity of elementary functions)**.** Every elementary function is continuous in every point in which it is defined.

Definition (continuity). Let f be a function of two variables and $X_0 = (x_0, y_0)$ be a point from its domain. The function f is said to be *continuous* at $X_0 = (x_0, y_0)$ if for every real number ε, $\varepsilon > 0$, there exists a neighborhood $N(X_0)$ of the point X_0 such that $|f(x, y) - f(x_0, y_0)| < \varepsilon$ for every $X = (x, y) \in N(X_0)$.

Theorem 1 (continuity of elementary functions). Every elementary function is continuous in every point in which it is defined.

3 Derivative

We fix the value of the variable y and investigate the function as a function in one variable x and vice versa.

Definition (partial derivative). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. Suppose that a finite limit

$$f'_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

exists. The value of this limit is called a *partial derivative with respect to* x *of the function* f *in the point* (x, y).

The rule which associates every $(x, y) \in M$ with the value of $f'_x(x, y)$ is called a *partial derivative of the function* f *with respect to* x.

Similarly, a *partial derivative with respect to* y is defined through the limit

$$f'_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

By differentiating the first derivatives we obtain higher derivatives.

3 Derivative

We fix the value of the variable y and investigate the function as a function in one variable x and vice versa.

Definition (partial derivative). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. Suppose that a finite limit

$$f'_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

exists. The value of this limit is called a *partial derivative with respect to* x *of the function* f *in the point* (x, y).

The rule which associates every $(x, y) \in M$ with the value of $f'_x(x, y)$ is called a *partial derivative of the function* f *with respect to* x.

Similarly, a *partial derivative with respect to* y is defined through the limit

$$f'_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

By differentiating the first derivatives we obtain higher derivatives.

3 Derivative

We fix the value of the variable y and investigate the function as a function in one variable x and vice versa.

Definition (partial derivative)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. Suppose that a finite limit

$$f'_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

exists. The value of this limit is called a *partial derivative with respect to* x *of the function* f *in the point* (x,y).

The rule which associates every $(x,y) \in M$ with the value of $f'_x(x,y)$ is called a *partial derivative of the function* f *with respect to* x.

Similarly, a *partial derivative with respect to* y is defined through the limit

$$f'_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}.$$

By differentiating the first derivatives we obtain higher derivatives.

3 Derivative

We fix the value of the variable y and investigate the function as a function in one variable x and vice versa.

Definition (partial derivative). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. Suppose that a finite limit

$$f'_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

exists. The value of this limit is called a *partial derivative with respect to* x *of the function* f *in the point* (x,y).

The rule which associates every $(x,y) \in M$ with the value of $f'_x(x,y)$ is called a *partial derivative of the function* f *with respect to* x.

Similarly, a *partial derivative with respect to* y is defined through the limit

$$f'_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}.$$

By differentiating the first derivatives we obtain higher derivatives.

Remark 1 (practical evaluation of the partial derivatives)**.** According to the preceding definition, the partial derivative with respect to a given variable is the "usual" derivative of the function of one variable, where only the given variable remains under consideration and the other variables are considered to be constant (like parameters). When differentiating, the usual formulas for differentiation of basic elementary functions and the usual rules for differentiation serve as a main tool.

Remark 2 (tangent plane, linear approximation)**.** If the graph of the function $z = f(x, y)$ is considered as a surface in $\mathbb{R}^3$, then the tangent plane to this surface in the point (x_0, y_0) is

$$z = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0).$$

The tangent plane is (under some additional conditions) the best linear approximation of the graph. Hence the approximate formula

$$f(x, y) \approx f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0)$$

holds for (x, y) close to (x_0, y_0).

Remark 3 (partial derivatives and the rate of change)**.** From a practical point of view $f_x(x_0, y_0)$ gives the instantaneous rate of change of f with respect to x when the remaining independent variable y is fixed. Thus, when y is fixed at y_0 and x is close to x_0, a small change h of the variable x will lead to the change approximately $hf_x(x_0, y_0)$ in f.

Remark 2 (tangent plane, linear approximation)**.** If the graph of the function $z = f(x, y)$ is considered as a surface in $\mathbb{R}^3$, then the tangent plane to this surface in the point (x_0, y_0) is

$$z = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0).$$

The tangent plane is (under some additional conditions) the best linear approximation of the graph. Hence the approximate formula

$$f(x, y) \approx f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0)$$

holds for (x, y) close to (x_0, y_0).

Remark 3 (partial derivatives and the rate of change)**.** From a practical point of view $f_x(x_0, y_0)$ gives the instantaneous rate of change of f with respect to x when the remaining independent variable y is fixed. Thus, when y is fixed at y_0 and x is close to x_0, a small change h of the variable x will lead to the change approximately $hf_x(x_0, y_0)$ in f.

Remark 2 (tangent plane, linear approximation)**.** If the graph of the function $z = f(x, y)$ is considered as a surface in $\mathbb{R}^3$, then the tangent plane to this surface in the point (x_0, y_0) is

$$z = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0).$$

The tangent plane is (under some additional conditions) the best linear approximation of the graph. Hence the approximate formula

$$f(x, y) \approx f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0)$$

holds for (x, y) close to (x_0, y_0).

Remark 3 (partial derivatives and the rate of change)**.** From a practical point of view $f_x(x_0, y_0)$ gives the instantaneous rate of change of f with respect to x when the remaining independent variable y is fixed. Thus, when y is fixed at y_0 and x is close to x_0, a small change h of the variable x will lead to the change approximately $hf_x(x_0, y_0)$ in f.

Remark 4 (another notation). Another notation for the derivative of the function $z = f(x, y)$ with respect to x is f_x, z'_x, z_x, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial z}{\partial x}$ and similarly for the derivative with respect to y, e.g., z'_y or $\dfrac{\partial z}{\partial y}$. The second derivatives are denoted in one of the following way: z''_{xx}, f''_{yy}, z''_{xy}, z_{xx}, f_{yy}, z_{xy}, $\dfrac{\partial^2 z}{\partial x \partial y}$, $\dfrac{\partial^2 z}{\partial y^2}$.

Theorem 2 (Schwarz). If the partial derivatives f''_{xy} a f''_{yx} exist and are continuous on the open set M, then they equal on the set M, i.e.

$$f''_{xy}(x, y) = f''_{yx}(x, y)$$

hold for every $(x, y) \in M$.

Remark 4 (another notation)**.** Another notation for the derivative of the function $z = f(x, y)$ with respect to x is f_x, z'_x, z_x, $\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial x}$ and similarly for the derivative with respect to y, e.g., z'_y or $\frac{\partial z}{\partial y}$. The second derivatives are denoted in one of the following way: z''_{xx}, f''_{yy}, z''_{xy}, z_{xx}, f_{yy}, z_{xy}, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$.

Theorem 2 (Schwarz)**.** If the partial derivatives f''_{xy} a f''_{yx} exist and are continuous on the open set M, then they equal on the set M, i.e.

$$f''_{xy}(x, y) = f''_{yx}(x, y)$$

hold for every $(x, y) \in M$.

4 Extremal problems

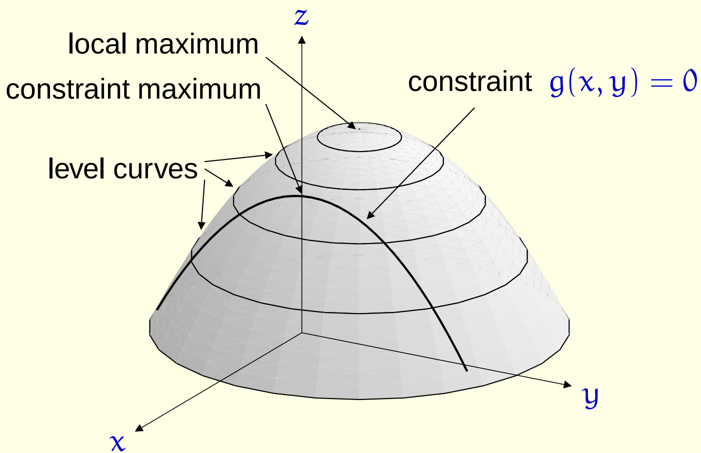

Figure 1: Extrema of the functions of two variables on the graph

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined and continuous in some neighborhood of the point (x_0, y_0). We are interested in the problem, whether or not the values of the function in the point (x_0, y_0) are maximal when comparing to the values of the function in “other points”, i.e. whether

$$f(x_0, y_0) > f(x, y) \quad \text{for } (x, y) \neq (x_0, y_0), \tag{1}$$

or

$$f(x_0, y_0) \geq f(x, y) \tag{2}$$

holds. First, we have to specify, what does exactly mean the phrase “other points”. Giving a specific meaning to this phrase, we obtain three important types of extrema.

Definition (maxima of the functions of two variables)**.**

- Suppose that there exists a neighborhood $N((x_0, y_0))$ of the point (x_0, y_0) such that (1) holds for every $(x, y) \in N((x_0, y_0))$. Then the function f is said to gain its *sharp local maximum* at the point (x_0, y_0).

- Suppose that another function of two variables $g(x, y)$ is given and $g(x_0, y_0) = 0$. If there exists a neighborhood $N((x_0, y_0))$ of the point (x_0, y_0) such that inequality (1) holds for every $(x, y) \in N((x_0, y_0))$ which satisfies the relation

$$g(x, y) = 0, \tag{3}$$

  then function f is said to gain its *constrained sharp local maximum with respect to the constraint condition* (3) at the point (x_0, y_0).

- Suppose that the set $M \subseteq \mathrm{Dom}(f)$ is given. If inequality (1) holds for every $(x, y) \in M$, the function f is said to gain its *sharp global maximum of the function f on the set M* at the point (x_0, y_0).

Definition (other local extrema).

- If inequality (1) is replaced by (2) in the preceding definition, we omit the word "sharp".
- If the direction of the inequality sign in (1) and (2) is changed in the preceding definition, we obtain a definition of *a local minimum, a constrained local minimum and a global minimum* of the function f.

Remark 5. Even in the case when we speak about the sharp extrema is the word "sharp" sometimes omitted.

Theorem 3 (Fermat). Let $f(x,y)$ be a function of two variables. Suppose that the function $f(x,y)$ takes on a local extremum in the point (x_0,y_0). Then either at least one of the partial derivatives at the point (x_0,y_0) does not exist, or both partial derivatives at this point vanish, i.e.

$$f'_x(x_0,y_0) = 0 = f'_y(x_0,y_0). \tag{4}$$

Remark 6. The point with vanishing both partial derivatives is called a *stationary* point. We usually use the following test to recognize whether a local extremum at the stationary point exists.

The stationary point where local maximujm or minimum fails to exist is called a *saddle point*.

Theorem 3 (Fermat). Let $f(x, y)$ be a function of two variables. Suppose that the function $f(x, y)$ takes on a local extremum in the point (x_0, y_0). Then either at least one of the partial derivatives at the point (x_0, y_0) does not exist, or both partial derivatives at this point vanish, i.e.

$$f'_x(x_0, y_0) = 0 = f'_y(x_0, y_0). \tag{4}$$

Remark 6. The point with vanishing both partial derivatives is called a *stationary* point. We usually use the following test to recognize whether a local extremum at the stationary point exists.

The stationary point where local maximujm or minimum fails to exist is called a *saddle point*.

Theorem 4 (2nd derivative test). Let f be a function of two variables and (x_0, y_0) a stationary point of this function, i.e. suppose that (4) holds. Further suppose that the function f has continuous derivatives up to the second order in some neighborhood of this stationary point. Denote by D the following determinant[1]

$$D(x_0, y_0) = \begin{vmatrix} f''_{xx}(x_0, y_0) & f''_{xy}(x_0, y_0) \\ f''_{xy}(x_0, y_0) & f''_{yy}(x_0, y_0) \end{vmatrix} = f''_{xx}(x_0, y_0) f''_{yy}(x_0, y_0) - \left[f''_{xy}(x_0, y_0)\right]^2.$$

One of the following cases occurs

- If $D > 0$ and $f''_{xx} > 0$, then the function f has a sharp local minimum in the point (x_0, y_0).
- If $D > 0$ and $f''_{xx} < 0$, then the function f has a sharp local maximum in the point (x_0, y_0).
- If $D < 0$, then there is no local extremum of the function f in the point (x_0, y_0).
- If $D = 0$, then the test fails. Any of the preceding situations may occur.

[1] called Hessian