Calculus

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CONTENTS

1. Function

Definition (function)**.** Let A and B be nonempty sets of real numbers.

Let f be a rule which associates each element x of the set A with exactly one element y of the set B. The rule f is said to be a *function* defined on A. We write $\boldsymbol{f : A \to B}$. If f associates x with y, we write $\boldsymbol{y = f(x)}$.

The variable x is customary called an *independent variable* and y a *dependent variable*.

The set A is called a *domain* of the function f and denoted by $\boldsymbol{Dom(f)}$.

The set B is a *target set*. The subset of all that elements y of the set B which are generated by the elements from $Dom(f)$ is called an *image* (or *range*) of the function f and denoted by $\boldsymbol{Im(f)}$.

Remark 1.1. In plain words, the function f is a rule which associates a real number x from the domain with another, exactly defined, number y from the image of the function f. This rule can be expressed in the form

$$\text{“}y = \boxed{\text{formula-containing-}x}\text{”},$$

e.g. $y = \dfrac{\sin(x^2+1)}{x}$. Such a function is said to be in its *explicit form*.

The rule which defines the function f can be also expressed in the form

$$\text{“}\boxed{\text{formula-containing-variables-}x\text{-and-}y} = 0\text{”},$$

e.g. $x - y - \ln y = 0$. As a particular example, the value of this function at $x = 2$ is the (unique) solution of the equation $y + \ln y = 2$. Such a function is said to be in its *implicit form.*

Roughly speaking, the explicit form is “effective” and the implicit form “ineffective” for calculating the values of the function.

Remark 1.2 (terminology)**.** If the point a belongs to $Dom(f)$, we say that the function f is defined at the point a or that the value $f(a)$ is well-defined. Usually, a function is given by a formula, without specifying the domain. In this case the domain is understood to be the set of all numbers for which the formula makes any sense. Such a domain is called a *natural domain* of the function (see Remark 1.8).

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Definition (graph). Let f be a function.

A graph of the function f is the set of all of the points in the plane $[x, y] \in \mathbb{R}^2$ with property $y = f(x)$.

If $0 \in Dom(f)$, then the value $f(0)$ is well-defined and the point point $[0, f(0)]$ on the graph is called an *y-intercept* of the graph.

The number $x \in Dom(f)$ which satisfies $f(x) = 0$ is called a *root* or a *zero* of the function f.

If the number x_0 is a zero of the function f, then the point $[x_0, 0]$ on the graph is called an *x-intercept* of the graph.

Remark 1.3. Concerning the function f, we formulate the following important problems:

(i) Given $x \in Dom(f)$, find y which satisfies $y = f(x)$.

(ii) Given $y \in Im(f)$, find all of x which satisfy $y = f(x)$.

(iii) Given a value $c \in \mathbb{R}$, solve the inequalities $f(x) > c$, $f(x) \geq c$, $f(x) \leq c$ and $f(x) < c$.

If the graph of the function f is known, these problems can be very easily solved in a graphical way. (Explain and illustrate on some particular examples.)

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Definition (composite function). Let $f : A \to B$ and $g : B \to C$ be functions. Under a *composite function* $g \circ f$ (read "g of f") we understand the function defined for every $x \in A$ by the relation

$$(g \circ f)(x) = g(f(x)).$$

The function f is said to be an *inside function* and g an *outside function* of the composite function $g \circ f$.

Example 1.1. For $f(x) = x^2 + 1$ and $g(x) = x + 2$ find $(g \circ f)$ and $(f \circ g)$.

Solution:

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = (x^2 + 1) + 2 = x^2 + 3$$
$$(f \circ g)(x) = f(g(x)) = f(x + 2) = (x + 2)^2 + 1 = x^2 + 4x + 5$$

Remark 1.4. Schematically we can illustrate the composite function as follows.

f g

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$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[rr, bend right, "g \circ f"'] & f(x) \arrow[r, bend left, "g"] & g(f(x)) \end{tikzcd}$$

Remark 1.5 (basic elementary functions)**.** Several functions which are common in applications have special names.

(i) *Power function* $y = x^{\alpha}$, $\alpha \in \mathbb{R}$

(ii) *Exponential function* $y = a^x$, $a \in \mathbb{R}^+ \setminus \{1\}$

(iii) *Logarithmic function* $y = \log_a x$, $a \in \mathbb{R}^+ \setminus \{1\}$

(iv) *Trigonometric functions* $y = \sin x$, $y = \cos x$, $y = \operatorname{tg} x$, $y = \operatorname{cotg} x$

(v) *Inverse trigonometric functions* $y = \arcsin x$, $y = \arccos x$, $y = \operatorname{arctg} x$, $y = \operatorname{arccotg} x$

The functions in the preceeding list are called *basic elementary functions*.
A sum of constant multiples of power functions with a nonnegative integer exponent is called a *polynomial*, e.g. $y = 2x^3 - 4x + 1$ is a polynomial. The highest power in the polynomial is called a *degree* of the polynomial.
The quotient of two polynomials is called a *rational function*, e.g. $y = \dfrac{x}{x^3 - 2x + 1}$ is a rational function.

Remark 1.6 (elementary functions)**.** The function f which can be represented by a single formula $y = f(x)$, where the expression on the right-hand side is made up of basic elementary functions and constants by means of a finite number of the operations addition, subtraction, multiplication, division and composition is an *elementary function*. For example the function $y = \dfrac{e^{1-\cos x}}{\sin^2 x} \ln\left(x + x^2 + \operatorname{arctg} x - 4\right)$ is (rather complicated) elementary function.

Another class of non-elementary functions with wide applications in real world problems is a class of *piecewise-defined functions*. This class consists from the functions which are not elementary, but the domain of these functions can be divided into a finite number of subintervals such that the function is elementary in each of these subintervals. For example

$$y = \begin{cases} \sin x & \text{for } x \leq 0 \\ x^2 + 3x - 4 & \text{for } 0 < x \leq 10 \\ e^x & \text{for } x > 10 \end{cases}$$

is a piecewise-defined function consisting from three elementary functions.

Finaly, the infinite series

$$y = x + \frac{x^3}{1 \cdot 3} + \frac{x^5}{2 \cdot 3} + \frac{x^7}{3 \cdot 2 \cdot 1 \cdot 3} + \frac{x^9}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3} + \frac{x^{11}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3} + \cdots$$

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Remark 1.7. In calculus we are interested especially in the composition of the basic elementary functions. Hence $y = \ln(x+1)$, $y = \arcsin\sqrt{x}$ and $y = \sin\frac{x}{2}$ are composite functions. The functions $y = e^x$ and $y = \operatorname{tg} x$ are not composite functions.

Remark 1.8 (natural domain of the composite function)**.** To establish a natural domain of a composite function we usually have to solve a system of (in general nonlinear) inequalities. When writing these inequalities we take care about the functions which yield a restriction on its domain, i.e. which are not defined for all real numbers. We summarize the basic elementary functions with domain different from $\mathbb{R}$ into the following table.

The function ...	is well-defined if ...	
$\frac{f(x)}{g(x)}$	$g(x) \neq 0$	quotient
$\ln f(x)$	$f(x) > 0$	logarithm
$\sqrt[2k]{f(x)}$, $k \in \mathbb{N}$	$f(x) \geq 0$	even root
$\arcsin f(x)$	$-1 \leq f(x) \leq 1$	
$\arccos f(x)$	$-1 \leq f(x) \leq 1$	

Remark 1.9 (shifted and resized graphs of basic elementary functions)**.**

To obtain the graph of ...	shift the graph of $y = f(x)$ ...
$y = f(x) + c$	c units upward
$y = f(x) - c$	c units downward
$y = f(x + c)$	c units to the left
$y = f(x - c)$	c units to the right

To obtain the graph of ...	reflect the graph of $y = f(x)$ ...
$y = -f(x)$	about the x-axis
$y = f(-x)$	about the y-axis

To obtain the graph of ...	do this with the graph of $y = f(x)$ ...
$y = af(x)$	stretch a-times vertically
$y = \frac{1}{a} f(x)$	restrict a-times vertically
$y = f\left(\frac{1}{a} x\right)$	stretch a-times horizontally
$y = f(ax)$	restrict a-times horizontally

2. Basic properties of functions

Definition (odd and even function)**.** Let f be a function with domain $Dom(f)$. Let for every $x \in Dom(f)$ also $-x \in Dom(f)$.

(i) The function f is said to be *even*, if $f(-x) = f(x)$ holds for every $x \in Dom(f)$.

(ii) The function f is said to be *odd*, if $f(-x) = -f(x)$ holds for every $x \in Dom(f)$.

Remark 2.1 (graphical consequence)**.** The graph of an even function is symmetric about the line $x = 0$ (about the y-axis). The graph of an odd function is symmetric about the origin.

Theorem 2.1. *(i) A polynomial function is odd (even) if and only if it contains terms with an odd (even) exponent only.*

(ii) A rational function is odd if and only if it is a quotient of an odd and an even polynomial (in an arbitrary order).

(iii) A rational function is even if and only if it is a quotient of two even or of two odd polynomials.

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(iii) A rational function is even if and only if it is a quotient of two even or of two odd polynomials.

Definition (boundedness)**.** Let f be a function and $M \subseteq Dom(f)$ be a subset of the domain of the function f.

(i) The function f is said to be *bounded from below* on the set M if there exists a real number a with the property $a \leq f(x)$ for all $x \in M$.

(ii) The function f is said to be *bounded from above* on the set M if there exists a real number a with the property $a \geq f(x)$ for all $x \in M$.

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If the set M is not specified, we suppose $M = Dom(f)$.

Example 2.1.

- The functions $y = x^2$, $y = e^x$, $y = e^{-x}$, $y = 2^x$ and in general every exponential function are bounded from below.
- The functions $y = \sin x$, $y = \cos x$, $y = \operatorname{arctg} x$ are bounded functions (both from below and above).

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Motivation. In the following paragraphs we will be interested in the fact, whether an application of the function f on both sides of an equality yields an equivalent equality. We start with an obvious implication which holds for every well-defined function f.

$$x_1 = x_2 \Rightarrow f(x_1) = f(x_2) \tag{2.1}$$

The equations $x_1 = x_2$ and $f(x_1) = f(x_2)$ are equivalent, if the implication in (2.1) can be reversed. It is clear that this implication *cannot* be reversed if two mutually different values $x_1 \neq x_2$ in the domain of f may generate the same value of the function f. A property which excludes this possibility is introduced in the following definition.

Definition (one-to-one function)**.** Let f be a function and let $M \subseteq Dom(f)$ be a subset of the domain of the function f.

The function f is said to be *one-to-one function* on the set M if there exists no pair of different elements x_1, x_2 of the set M which are associated with the same element y of the image $Im(f)$.

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Remark 2.2 (one-to-one functions)**.** Mathematically formulated, for the one-to-one functions the following implication holds

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2. \tag{2.2}$$

This implication means that if the function f is one-to-one, then we can remove (or more precisely un-apply) this function from both sides of the relation $f(x_1) = f(x_2)$ and conclude the equivalent relation $x_1 = x_2$. This property can be used when solving nonlinear equations, since this is an exact description of the fact that the equations $x_1 = x_2$ and $f(x_1) = f(x_2)$ are equivalent.

Remark 2.3. It is easy to see that a graphical criterion for a function to be one-to-one is that every horizontal line crossing the graph of the function must meet it in at most one point.

Definition (inverse function)**.** Let $f : A \to B$ be an one-to-one function. The rule which associates every x with the number y satisfying $f(y) = x$ defines a function on B (really, for every $x \in Im(f) \subseteq B$ there exists only one y satisfying $f(y) = x$). This function is said to be an *inverse function* to the function f. The inverse function to the function f is denoted by f^{-1}.

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Remark 2.4. The inverse function can be considered as a tool which restores information which is lost after application of the function f. If an application of the function f to a number (or quantity or variable) x gives the result y, then for known y we obtain the original x by application of an inverse function to y. ("If the function f is considered to be a kind of poison, then its inverse f^{-1} restores an influence of f, like antidote.")

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

The function $y = f(x)$	The inverse function $y = f^{-1}(x)$
$y = \sqrt{x}$	$y = x^2, x > 0$
$y = x^2, x > 0$	$y = \sqrt{x}$
$y = e^x$	$y = \ln x$
$y = \ln x$	$y = e^x$
$y = a^x$	$y = \log_a x$
$y = \sin x, x \in [-\pi/2, \pi/2]$	$y = \arcsin x$
$y = \cos x, x \in [0, \pi]$	$y = \arccos x$
$y = \operatorname{tg} x, x \in [-\pi/2, \pi/2]$	$y = \operatorname{arctg} x$

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$$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y$$

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Remark 2.5 (nonlinear equations)**.** The inverse function allows an alternative interpretation in terms of solutions of nonlinear equations: If there exists an inverse function f^{-1} of the function f and if the inverse function is defined in some x, then the nonlinear equation with the unknown y

$$f(y) = x \tag{2.3}$$

has exactly one solution. This solution is given by the formula

$$y = f^{-1}(x). \tag{2.4}$$

Example 2.2 (nonlinear equation)**.** Solve the equation

$$e^{\frac{2}{x-1}} = 2. \tag{2.5}$$

Solution: Since the inverse function to the exponential function, e^x, is the natural logarithmic function, $\ln x$, it follows (see the scheme)

$$\frac{2}{x-1} = \ln 2. \qquad \begin{tikzcd} \frac{2}{x-1} \arrow[r, bend left, "\exp(\cdot)"] & 2 \arrow[l, bend left, "\ln(\cdot)"] \end{tikzcd} \tag{2.6}$$

From here it is now easy to find

$$x = \frac{2}{\ln 2} + 1.$$

Motivation. Now we will distinguish the classes of functions which preserve or reverse inequalities.

Definition (monotonicity). Let f be a function and $M \subseteq Dom(f)$ be a subset of its domain.

(i) The function f is said to be *increasing* on the set M if for every pair $x_1, x_2 \in M$ with the property $x_1 < x_2$ the relation $f(x_1) < f(x_2)$ holds.

(ii) The function f is said to be *decreasing* on the set M if for every pair $x_1, x_2 \in M$ with the property $x_1 < x_2$ the relation $f(x_1) > f(x_2)$ holds.

(iii) The function f is said to be *(strictly) monotone* on the set M if it is either increasing or decreasing on M.

If the set M is not specified, we suppose $M = Dom(f)$.

Remark 2.6 (to the monotonicity)**.** If the set M in the preceding definition is an interval, then the monotonicity allows a clear geometrical interpretation on the graph of the function.

Theorem 2.2. *If the function is strictly monotone on the set M, then it is one-to-one on M.*

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Theorem 2.3. *Let f be a function. If the function f is increasing (decreasing, odd), then the inverse function f^{-1} has the same property.*

Remark 2.7 (summing up remark)**.** We summarize the properties of the functions which can be used when solving equations and (or) inequalities.

$$a = b \overset{f \text{ is one-to-one}}{\iff} f(a) = f(b)$$

$$a < b \overset{f \text{ is increasing}}{\iff} f(a) < f(b) \qquad a \leq b \overset{f \text{ is increasing}}{\iff} f(a) \leq f(b)$$

$$a < b \overset{f \text{ is decreasing}}{\iff} f(a) > f(b) \qquad a \leq b \overset{f \text{ is decreasing}}{\iff} f(a) \geq f(b)$$

If the function f is one-to-one, then for every $y \in Im(f)$ there exists a unique solution x of the equation

$$f(x) = y$$

and this solution is given by the formula $x = f^{-1}(y)$.

3. Limit, continuity

In this chapter we introduce mathematical methods which yield more detailed informations about the function than the properties considered in the preceding chapter. We will investigate the manner in which quantities vary and whether they approach some specific values.

We start with the concept of limit and continuity. This concept is one of the fundamental ideas which distinguishes calculus from other areas of mathematics. This concept (like the whole calculus) is based on very natural ideas, but an exact mathematical description is somewhat complicated (for beginners sometimes *very complicated*). Anyway, almost all theorems and results can be illustrated by a simple picture. Always try to illustrate the described situation by some drawing.
We start, as usually, with an introduction to the concept of limits.

Definition (expanded set of the real numbers)**.** Under *an expanded set of the real numbers* $\mathbb{R}^*$ we understand the set $\mathbb{R}$ of the real numbers enriched by the numbers $\pm\infty$ in the following way: We set $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$ and for $a \in \mathbb{R}$ we set:

$$a + \infty = \infty, \qquad a - \infty = -\infty, \qquad \infty + \infty = \infty, \qquad -\infty - \infty = -\infty$$

$$\infty.\infty = -\infty.(-\infty) = \infty, \qquad \infty.(-\infty) = -\infty, \qquad \frac{a}{\infty} = \frac{a}{-\infty} = 0$$

$$-\infty < a < \infty, \qquad |\pm\infty| = \infty.$$

Further, for $a > 0$ we set

$$a.\infty = \infty \qquad a.(-\infty) = -\infty,$$

and for $a < 0$ we set

$$a.\infty = -\infty \qquad a.(-\infty) = \infty.$$

Another operations we define with the commutativity of the operation “$+$” and “$.$”.

Remark 3.1 (indeterminate forms)**.** The operations “$\infty - \infty$”, “$\pm\infty.0$” and “$\dfrac{\pm\infty}{\pm\infty}$” remain undefined. Of course, the division by a zero remains undefined as well.

Definition (neighborhood)**.** Under the *neighborhood* of the point $a \in \mathbb{R}$ we understand any open interval which cotains the point a, we write $\boldsymbol{N(a)}$. Under the *reduced* (also *ring*) *neighborhood* of the point a we understand the set $N(a) \setminus \{a\}$, we write $\boldsymbol{\overline{N}(a)}$.

Under the *neighborhood of the point* ∞ we understand the interval of the type (A, ∞) and under the *neighborhood of the point* $-\infty$ the interval $(-\infty, A)$. Under the reduced neighborhood of the points $\pm\infty$ we understand the same as under the neighborhood of these points.

Definition (limit of the function)**.** Let $a, L \in \mathbb{R}^*$ and $f : \mathbb{R} \to \mathbb{R}$. Let the function f be defined in some reduced neighborhood of the point a. We say that the function $y = f(x)$ approaches to the *limit* L as x approaches to a if for any (arbitrary small) neighborhood $N(L)$ of the number L there exists reduced neighborhood $\overline{N}(a)$ of the point a such that for every $x \in \overline{N}(a)$ the relation $f(x) \in N(L)$ holds.

We write

$$\lim_{x \to a} f(x) = L \tag{3.1}$$

or $f(x) \to L$ for $x \to a$.

Definition (one-sided neighborhood). Under the *right (left) neighborhood* of the point $a \in \mathbb{R}$ we understand the interval of the type $[a, b)$,(or $(b, a]$, for left neighborhood), we write $N^+(a)$ ($N^-(a)$). Under the *reduced right (left) neighborhood* of the point a we understand the corresponding neighborhood wihtout the point a, we write $\overline{N^+}(a)$, ($\overline{N^-}(a)$)

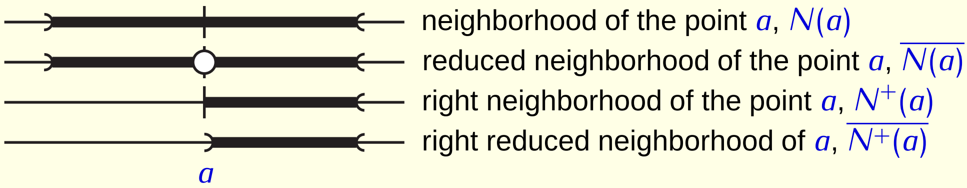


Figure 1: Neighborhood of the point a.

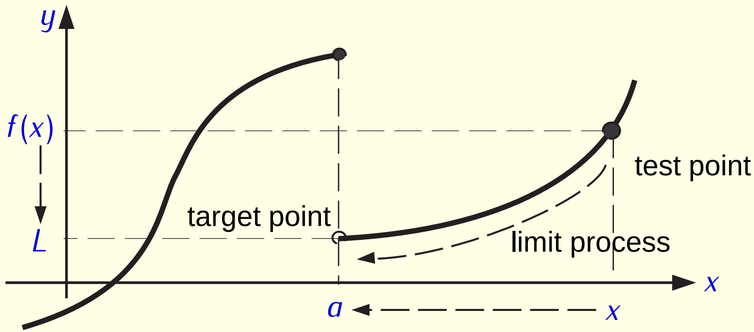
Definition (one-sided limit). If we replace in the definition of the limit the reduced neighborhood of the point a by the reduced right neighborhood of the point a, we obtain a definition of the *limit from the right*. We write $\lim_{x\to a^+} f(x) = L$.

Similarly, we define also the limit from the left. In this case we write $\lim_{x\to a^-} f(x) = L$.

Theorem 3.1 (uniqueness of the limit). *The function f possesses at the point a at most one limit (or one-sided limit).*

Theorem 3.2 (the relationship between the limit and the one-sided limits). *The limit of the function f at the point $a \in \mathbb{R}$ exists if and only if both one-sided limits at the point a exist and are equal. More precisely: If the limits $f(a-)$ and $f(a+)$ exist and $f(a-) = f(a+)$, then the limit $f(a\pm)$ exists as well and $f(a\pm) = f(a-) = f(a+)$. If one of the one-sided limits does not exist or if $f(a-) \neq f(a+)$, then the limit $f(a\pm)$ does not exist.*

Remark 3.2 (How to find the value of the limit on the graph?). The existence and the value of the limit can be estimated from the graph of the function.



Definition (vertical asymptote). Let f be a function and x_0 a real number. The vertical line $x = x_0$ is said to be a *vertical asymptote to the graph of the function* f if at lest one of the one-sided limits of the function f at the point x_0 exists and it is not a finite number.

Definition (horizontal asymptote). The line $y = L$ is said to be a *horizontal asymptote to the graph of the function* $f(x)$ *at* $+\infty$ if the limit of the function f at $+\infty$ exists and

$$\lim_{x\to\infty} f(x) = L$$

holds. In a similar way we define the *horizontal asymptote at* $-\infty$.

Example 3.1. The following limits can be evaluated by an inspection of the graphs of basic elementary functions.

$$\lim_{x\to\infty} \operatorname{arctg} x = \frac{\pi}{2}, \qquad \lim_{x\to 0^+} \operatorname{cotg} x = \infty,$$
$$\lim_{x\to 0^+} \ln x = -\infty, \qquad \lim_{x\to\infty} e^x = \infty,$$
$$\lim_{x\to -\infty} e^x = 0.$$

Example 3.2 (numerical experiment)**.** We estimate the value of the limit $\lim\limits_{x\to 0^+} \dfrac{\sin x}{x}$ by a numerical experiment. Let us evaluate the function $\dfrac{\sin x}{x}$ in the sequence of values approaching to zero. We obtain

x	0.5	0.2	0.1	0.01	0.005	0.00001
$\dfrac{\sin x}{x}$	0.95885	0.99334	0.99833	0.999983	0.9999958	1

From this computation it seems to be reasonable to guess

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

Motivation. As we said before, in general, the value of the function f at a given point a has no influence neither to the existence nor to the value of the limit at this point. Actually this is *not* the case concerning many functions used in the mathematical description of real–world problems. In most cases the value of the limit is exactly *the same* as the value of the function (provided the function is well-defined at the point under consideration). The name for a class of functions with this (nice) property is revealed in following definition.

Definition (continuity at a point). Let f be a function defined at the point $a \in \mathbb{R}$.

The function f is said to be *continuous* at the point a if $\lim_{x\to a} f(x) = f(a)$.

The function f is said to be *continuous from the right* at the point a if $\lim_{x\to a^+} f(x) = f(a)$.

The function f is said to be *continuous from the left* at the point a if $\lim_{x\to a^-} f(x) = f(a)$.

Remark 3.3. According to the definition, the function is continuous at the point a if

- $f(a)$ exists,
- $\lim_{x\to a} f(x)$ exists as finite number,
- $f(a) = \lim_{x\to a} f(x)$ holds.

Remark 3.4. If the function is defined in some (at least one-sided) reduced neighborhood of the point a but at least one of the conditions from the last remark is broken, the point $x = a$ is said to be a point of *discontinuity* of the function f. This covers (among others) the cases when

- the limit at a exists, but $f(a)$ is undefined or not equal to this limit — *removable discontinuity*
- the one-sided limits at a exist but are not equal — *jump discontinuity*
- the limit at a is ∞ or $-\infty$ — *essential discontinuity (blow-up)*

Definition (continuity on an interval). The function is said to be *continuous on the open interval* (a, b) if it is continuous at every point of this interval.

The function is said to be *continuous on the closed interval* $[a, b]$ if it is continuous on (a, b), continuous from the right at the point a and continuous from the left at the point b.

Notation. The class of all functions continuous on the interval I will be denoted by $C(I)$. If $I = (a, b)$ or $I = [a, b]$, then we write shortly $C((a, b))$ or $C([a, b])$, respectively.

Theorem 3.3 (continuity of elementary functions). *Every elementary function is continuous on its domain.*

Example 3.3 (limit by substitution). The function $y = \dfrac{e^x \ln(x)}{x^2 - 1}$ is continuous on $(0, 1)$ and $(1, \infty)$. Hence

$$\lim_{x \to 2} \frac{e^x \ln(x)}{x^2 - 1} = \frac{e^2 \ln 2}{3}.$$

The evaluation of the limit at the points $x = 0$, $x = 1$ and $x = \infty$ is not so easy. The evaluation of the limit in arbitrary negative number has no sense, since the function is defined for positive numbers only, not equal to 1.

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Motivation. As we have seen before, the definition of continuity claims that a continuous function has the same value and the limit at every point from its domain. A natural question arises:

> *How does the definition of continuity from page 27 correspond to the intuitive idea about the continuous functions, that the function is continuous if the graph of this function can be sketched as a single curve without breaks, jumps and holes?*

Surprisingly, this highly expected correspondence is only partial! The Czech mathematician Bernard Bolzano found the function which is continuous but this function *cannot be graphed* by any graphical tool since it has no tangent at every point.[1] This rather extraordinary example shows that the class of continuous functions is more comprehensive than the class of functions whose graphs are "continuous curves" in an intuitive meaning. However, some of the most important properties of "continuous curves in the plane" are preserved for continuous functions.

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[1] A similar example of continuous function which cannot be graphed has been given few years later by C. Weierstrass. This example became to be more familliar to mathematicians, since Bolzano, an "enemy of Austro–Hungarian Monarchy", was not allowed to publish his results.

Theorem 3.4 (Weierstrass)**.** *Let f be a function defined and continuous on $[a, b]$. Then the function f is bounded and takes on an absolute maximum and an absolute minimum on the interval $[a, b]$, i.e. there exist numbers $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.*

Theorem 3.5 (Bolzano, the 1st Bolzano's theorem)**.** *Let f be a function defined and continuous on $[a, b]$. If $f(a).f(b) < 0$ holds (i.e. the values $f(a)$ and $f(b)$ have different signs), then there exists a zero of the function f on the interval (a, b), i.e. there exists $c \in (a, b)$ such that $f(c) = 0$.*

Theorem 3.6 (Bolzano, the 2nd Bolzano's theorem)**.** *Let f be a function defined and continuous on $[a, b]$. Then for every y_0 which is between the maximal and minimal value the function f on the interval $[a, b]$ there exists at least one x_0 with property $f(x_0) = y_0$.*

Algorithm 3.1 (general nonlinear inequality)**.** According to Bolzano's first theorem, a function cannot change its sign on the interval which contains neither a point of discontinuity nor a root of the function. Hence when solving one of the inequalities

$$f(x) > 0, \quad f(x) \geq 0, \quad f(x) \leq 0, \quad \text{and} \quad f(x) < 0,$$

we can proceed in the following steps.

(i) We find all of the solutions of the equation $f(x) = 0$. We mark these points on the real axis.

(ii) We find all points of discontinuity of the function $f(x)$. We mark these points on the real axis.

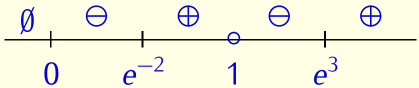
(iii) The real axis is divided into several subintervals now. The function f preserves its sign on each subinterval. We choose arbitrary (convenient) number ξ form each subinterval, evaluate $f(\xi)$ and mark the sign of this value to the subinterval. We do this step for all subintervals.

(iv) Performing the preceding step for all subintervals, we assign the sign of the function $f(x)$ to each subinterval. Now it is clear where $f(x) > 0$ holds and where the inequality is opposite.

Example 3.4. Solve

$$\frac{\ln^2 x - \ln x - 6}{x-1} \geq 0$$

Solution:



The solution of the inequality is the set $[e^{-2}, 1) \cup [e^{3}, \infty)$.

Theorem 3.7 (algebra of limits)**.** *Let $a \in \mathbb{R}^*$, $f, g : \mathbb{R} \to \mathbb{R}$. The following relations hold whenever the limits on the right exist and the formula on the right is well-defined.*

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \tag{3.2}$$

$$\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \tag{3.3}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \tag{3.4}$$

The same holds for one sided limits as well.

Example 3.5 (application of the preceding theorem)**.** Using the preceding theorem we can find the following limits.

(i) $\displaystyle\lim_{x\to\infty}(\operatorname{arctg} x + \operatorname{arccotg} x) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

(ii) $\displaystyle\lim_{x\to 0^-} \frac{1}{x}\cos x = -\infty.1 = -\infty$

(iii) $\displaystyle\lim_{x\to\infty} \frac{1}{xe^x} = \frac{1}{\infty.\infty} = \frac{1}{\infty} = 0$

Theorem 3.8 (limit of the composite function with continuous component)**.** *Let* $\lim_{x\to a} f(x) = b$ *and* $g(x)$ *be a function continuous at* b*. Then* $\lim_{x\to a} g(f(x)) = g(b)$*, i.e.*

$$\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x)).$$

The same holds for one-sided limits as well.

Example 3.6 (application of the preceding theorem)**.**

(i) $\lim_{x\to\infty} \cos(e^{-x}) = \cos 0 = 1$

(ii) $\lim_{x\to-\infty} e^{\operatorname{arctg} x} = e^{-\pi/2}$

Theorem 3.9 (limit of the composite function)**.** *Let* $\lim_{x\to a} f(x) = b$, $\lim_{y\to b} g(y) = L$ *and let there exist a ring neighborhood of the point* $x = a$ *such that* $f(x) \neq b$ *for all* x *in this neighborhood. Then* $\lim_{x\to a} g(f(x)) = L$.

Example 3.7 (application of the preceding theorem)**.**

(i) $\lim_{x\to 0^+} \ln \frac{1}{x} = \text{"} \ln \infty \text{"} = \infty$

(ii) $\lim_{x\to -\infty} \operatorname{arctg}(e^{-x}) = \text{"} \operatorname{arctg} \infty \text{"} = \frac{\pi}{2}$

(iii) $\lim_{x\to 0^+} \ln(\sin x) = \text{"} \ln(0+) \text{"} = -\infty$

The following theorem is applicable (among others) when evaluating a limit of a rational functions in the point of discontinuity (a zero of the denominator), provided the numerator is not equal to zero at this point[2].

---

[2]If both numerator and denominator equal zero at the point a, we can cancel the fraction by the linear factor $(x - a)$ and consider this new limit.

Theorem 3.10 (limit of the type "$\frac{L}{0}$"). *Let $a \in \mathbb{R}^*$, $\lim\limits_{x\to a} g(x) = 0$ and $\lim\limits_{x\to a} f(x) = L \in \mathbb{R}^* \setminus \{0\}$. Suppose that there exists ring neighborhood of the point a such that the function $g(x)$ does not change its sign in this neighborhood. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \begin{cases} +\infty & \text{if } g(x) \text{ and } L \text{ have common sign,} \\ -\infty & \text{if } g(x) \text{ and } L \text{ have an opposite sign,} \end{cases}$$

in the neighborhood under consideration. The same holds for one sided limits as well.

Remark 3.5 (practical)**.** Usually, when using Theorem 3.10 we investigate the corresponding one-sided limits first. From the mutual relationship of these one-sided limits we conclude whether the two sided limit exists or not.

Roughly speaking, Theorem 3.10 states the following: If the denominator of the fraction does not change its sign infinitely many times in a neighborhood of the point a, then the one-sided limits of the type "$\frac{L}{0}$" are infinite, The sign of the result can be established by the "usual" rules – the quotient of two expressions with the same sign is positive and the quotient of two expressions with opposite sign is negative. We will indicate by the symbol "$+0$" the fact that the function tends to zero, but it is positive in some ring neighborhood. With this arrangement we can calculate as follows: $\frac{2}{+0} = \infty$, $\frac{-\infty}{+0} = -\infty$. Similarly, "$-0$" will indicate that the function is negative and tends to zero. With this arrangement we can write e.g. $\frac{-2}{-0} = \infty$.

Example 3.8 (application of the preceding theorem)**.**

(i) $\displaystyle\lim_{x\to 2^+}\frac{1-x}{x^2-4}=\frac{-1}{+0}=-\infty$

(ii) $\displaystyle\lim_{x\to 2^-}\frac{1-x}{x^2-4}=\frac{-1}{-0}=\infty$

(iii) $\displaystyle\lim_{x\to 2}\frac{1-x}{x^2-4}$ does not exist, since both one-sided limits are not equal.

(iv) $\displaystyle\lim_{x\to\infty}\frac{1}{1-e^{\frac{1}{x}}}=\frac{1}{1-e^{\frac{1}{\infty}}}=\frac{1}{1-e^{+0}}=\frac{1}{-0}=-\infty$

(v) $\displaystyle\lim_{x\to 0^+}\frac{1}{1-e^{\frac{1}{x}}}=\frac{1}{1-e^{1/+0}}=\frac{1}{1-e^{\infty}}=\frac{1}{1-\infty}=\frac{1}{-\infty}=0$

The following theorem presents a very fast method for calculating limits of polynomials and rational functions at $\pm\infty$. We can find these limits almost without writing any computations.

Theorem 3.11 (limit of the polynomial or of the rational functions at $\pm\infty$)**.** *It holds*

$$\lim_{x\to\pm\infty}(a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n) = \lim_{x\to\pm\infty} a_0x^n,$$
$$\lim_{x\to\pm\infty}\frac{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m} = \lim_{x\to\pm\infty}\frac{a_0}{b_0}x^{n-m}.$$

Remark 3.6. The preceding theorem is applicable also in the cases when the rules for algebra of limits give undefined expression $\infty-\infty$ for polynomials or $\dfrac{\infty}{\infty}$ for rational functions!

Example 3.9 (application of preceding theorem)**.**

(i) $\lim\limits_{x\to\infty}(6x^3-2x+1) = \lim\limits_{x\to\infty} 6x^3 = 6.(\infty)^3 = \infty,$

(ii) $\lim\limits_{x\to-\infty}(3x^5-2x^2+2) = 3.(-\infty)^5 = -\infty,$

(iii) $\lim\limits_{x\to\infty}\dfrac{x^3-2}{2x^3-4x^2+1} = \lim\limits_{x\to\infty}\dfrac{1}{2} = \dfrac{1}{2},$

(iv) $\lim\limits_{x\to-\infty}\dfrac{x^5-2}{2x^3-4x^2+1} = \lim\limits_{x\to-\infty}\dfrac{1}{2}x^2 = \infty.$

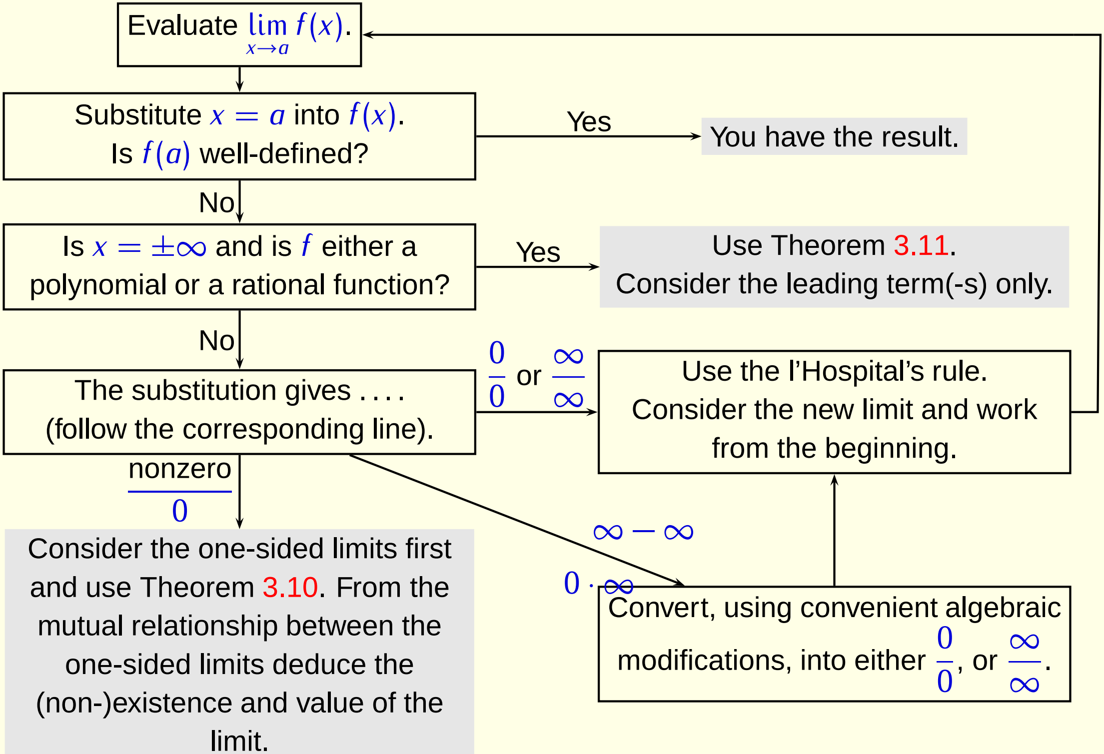


Figure 2: Limits.

- Worked problems on limits are in the file limits.pdf (click the icon on the right).
- Quizzes are in the file tests/lim2.pdf.

4. Derivative

Definition (derivative at a point)**.** Let f be a function and let $x \in Dom(f)$. The function f is said to be *differentiable at the point* x if the finite limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{4.1}$$

exists. The value of this limit is called a *derivative of the function* f *at the point* x.

Remark 4.1 (one-sided derivatives)**.** A similar definition is used for one-sided derivatives from the right and from the left. In this case the corresponding one-sided limit is used in (4.1).

Remark 4.2. Remark that the limit (4.1) is an indeterminate form "$\frac{0}{0}$" and, in general, it is not easy to find the value of this limit.

Definition (derivative as a function)**.** The function is said to be *differentiable on an open interval* I if it is differentiable at every point of this interval. The function which assigns to each point x from the interval the value $f'(x)$ of the derivative is said to be a *derivative of the function* f *on the interval* I and denoted by $\boldsymbol{f'}$.

Definition (higher derivatives)**.** Let $f(x)$ be a function and $f'(x)$ be the derivative of this function. Suppose that there exists derivative $(f'(x))'$ of the function $f'(x)$. Then this derivative is said to be the *second derivative of the function* f and denoted $\boldsymbol{f''(x)}$. By n-times repetition of this process we obtain the n-th derivative $\boldsymbol{f^{(n)}(x)}$ of the function f.

Remark 4.3 (tangent)**.** If the function f is differentiable at the point a, then the point–slope form of the equation of the tangent line in the point a is

$$y = f'(a)(x - a) + f(a). \tag{4.2}$$

Remark 4.4 (practical interpretation of the derivative)**.** Let the quantity x denotes time which is measured in convenient units and suppose that the value of the quantity y changes in the time, i.e. $y = y(x)$. Derivative $y'(x)$ of the function y in the point x denotes the instant rate (velocity) of the change of the function y at the time x. As a practical example consider the following situation.

Let the quantity y denotes the size of population of some species in some bounded area. In this case the derivative $y'(x)$ denotes the rate of the change of the size of this population. This change equals to the number of the individuals which are born in the moment x decreased by the amount of individuals which died in this moment (more precisely in the time interval which starts at given time and has unit length).

Theorem 4.1 (relationship between the differentiability and continuity). *Let f be a function differentiable at the point $x = a$ (on the interval I). Then f is continuous at the point $x = a$ (on the interval I).*

Remark 4.5. The converse statement to the preceding theorem is not true, in general. Really, the function $y = |x|$ is continuous in $\mathbb{R}$ but it is not differentiable at $x = 0$ (the tangent in this point from the left is $y = -x$ and from the right $y = x$).

Notation. The set of all functions with continuous derivative on the interval I is denoted by $C^1(I)$. These functions are called *smooth functions*.

Theorem 4.2 (algebra of derivatives). *Let f, g be functions and $c \in \mathbb{R}$ be a real constant. The following relations hold*

$$\begin{aligned}
[cf(x)]' &= cf'(x), && \textit{the constant multiple rule}\\
[f(x) \pm g(x)]' &= f'(x) \pm g'(x), && \textit{the sum rule}\\
[f(x)g(x)]' &= f(x)g'(x) + f'(x)g(x), && \textit{the product rule}\\
\left[\frac{f(x)}{g(x)}\right]' &= \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}, && \textit{the quotient rule}
\end{aligned}$$

whenever the derivatives on the right-hand side exist and the expression on the right-hand side is well defined.

Example 4.1 (algebra of derivatives). Find $f'(x)$ for $f(x) = \dfrac{xe^x}{x+1}$.

Solution: We start with the quotient rule followed by the product rule, i.e.

$$\left[\frac{xe^x}{x+1}\right]' = \frac{(xe^x)'(x+1) - (x+1)'xe^x}{(x+1)^2}$$

$$= \frac{(e^x + xe^x)(x+1) - 1xe^x}{(x+1)^2} = \frac{e^x(x^2+x+1)}{(x+1)^2}.$$

Remark 4.6 (practical). The sum rule is much simpler than the product rule and the quotient rule. In the following examples the product and the quotient can be rewritten as sums and the sum rule can be used. This is more handy than a blind application of the product rule or the quotient rule.

(i) $[(x+1)(x-2)]' = (x^2 - x - 2)' = 2x - 1$

(ii) $\left(\dfrac{x^3 - x + 1}{4x}\right)' = \dfrac{1}{4}(x^2 - 1 + x^{-1})' = \dfrac{1}{4}(2x - x^{-2})$.

Theorem 4.3 (chain rule). *Let f and g be differentiable functions. The relation*

$$[f(g(x))]' = f'(g(x))g'(x) \tag{4.3}$$

holds in every case in which the right side is defined.

Example 4.2 (chain rule). Find the derivatives of the composite functions $y = \ln \sin x$, $y = \ln(x \sin x)$ and $y = \ln\big(x \sin^2(2x)\big)$.

(i) We use simply the chain rule with $f(x) = \ln x$ and $g(x) = \sin x$. We obtain

$$y' = (\ln(\sin x))' = \frac{1}{\sin x} \cos x.$$

(ii) We use the chain rule followed by the product rule.

$$y' = (\ln(x \sin x))' = \frac{1}{x \sin x}(\sin x + x \cos x).$$

(iii) We use the chain rule. The “inside” function is a product and we use the product rule. The second factor in the product is the composite of three functions – we use two times the chain rule.

$$y' = \Big(\ln\big(x \sin^2(2x)\big)\Big)' = \frac{1}{x \sin^2(2x)}[1.\sin^2(2x) + x.2\sin(2x)\cos(2x)2]$$

Remark 4.7 (some technical tricks)**.** The most effective way how to find the derivative of the function $y = \dfrac{1}{(x^2+1)^5}$ is to write $y = (x^2+1)^{-5}$ and calculate

$$y' = -5(x^2+1)^{-6}(2x) = -\frac{10x}{(x^2+1)^6}.$$

The application of the quotient rule

$$y' = \frac{0(x^2+1)^5 - 5(x^2+1)^4 2x}{(x^2+1)^{10}}$$

is correct, but cumbersome.

In a similar way, the most effective way how to find y' if $y = \dfrac{2x+1}{\sqrt{3}}$ is to write $y = \dfrac{2}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}$ and calculate

$$y' = \frac{2}{\sqrt{3}}.1 + 0 = \frac{2}{\sqrt{3}}.$$

Solved problems on derivatives are in the file derivace.pdf.

Quizzes are in the file tests/der1.pdf.

5. Mathematical applications of derivatives

Remark 5.1 (linear approximation of a function)**.** Let f be a function differentiable at the point $x = a$. Then we can find the equation of its tangent in the point a by using (4.2). From the graph is clear that this tangent is the best linear function which approximates $f(x)$ near the point a. Hence we can write an approximate formula

$$f(x) \approx f(a) + f'(a)(x - a) \tag{5.1}$$

which approximates the function f by a linear function. Remember that this approximation is usually convenient for the points very close to the point a only. If this linear approximation is not sufficient in a particular problem, we can approximate the function f by a higher degree polynomial, as will be explained later. Classical approximation for Einstein's formula
BMI formula approximation

Theorem 5.1 (l'Hospital's rule)**.** *Let $a \in \mathbb{R}^*$ and let f and g be functions defined and differentiable in some ring neighborhood of the point a. Suppose that either $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ or $|\lim\limits_{x\to a} g(x)| = \infty$ holds. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \tag{5.2}$$

holds if the limit on the right-hand side exists (finite or infinite). The same holds for one-sided limits as well.

Remark 5.2 (other indeterminate forms)**.** The l'Hospital's rule can be usually used also to find a limit of the form $0.(\pm\infty)$ and $\infty - \infty$.

- In the case of the indeterminate form $0.\infty$ we write one of the functions as a fraction and convert the product into one fraction.
- In the case of the indeterminate form $\infty - \infty$ we write both terms as fractions and subtract by converting into common denominator.

If this function appears in the limit,	try to write it this form.
$\operatorname{tg} x$	$\dfrac{1}{\operatorname{cotg} x}$ or $\dfrac{\sin x}{\cos x}$
$\operatorname{cotg} x$	$\dfrac{1}{\operatorname{tg} x}$ or $\dfrac{\cos x}{\sin x}$
$e^{f(x)}$	$\dfrac{1}{e^{-f(x)}}$
$f(x)$	$\dfrac{1}{\frac{1}{f(x)}}$

Worked problems are in the file limitsLH.pdf (click the icon on the right).

6. Extremal problems

Remark 6.1 (optimal control)**.** The main problem of the optimal control is to find the conditions under which the quantity, we are interested in, is optimal in some sense. Thus, depending of the context, we would like to gain a maximal profit from producing or selling things, to gain a minimal costs for a production or a minimal amount of produced waste. *Real world applications:* optimal speed for fishes and optimal dimensions of loaded beams

In this section we focus ourselves onto the problem to find the points in which the function takes on its extremal values. More precisely, we will use the following concept.

Definition (local extrema)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in Dom(f)$. The function f is said to take on its *local maximum* at the point x_0 if there exists a neighborhood $N(x_0)$ of the point x_0 such that $f(x_0) \geq f(x)$ for all $x \in N(x_0)$.

The function f is said to take on its *sharp local maximum* at the point x_0 if there exists a neighborhood $N(x_0)$ of the point x_0 such that $f(x_0) > f(x)$ for $x \in N(x_0) \setminus \{x_0\}$.

If the opposite inequalities hold, then the function f is said to take on its *local minimum* or *sharp local minimum* at the point x_0.

A common word for local minimum and maximum is a *local extremum* (pl. *extrema*). A common word for the sharp local maximum and the sharp local minimum is a *sharp local extremum*.

In plain words, the function f takes on its sharp local maximum at the point $x = a$, if there is no greater value than $f(a)$ in a neighborhood of the point a (i.e. close to the point $x = a$). There exists a close relationship between the existence of a local extremum and monotonicity of the function, as the following theorem shows.

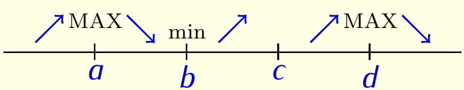
Theorem 6.1 (sufficient conditions for (non-)existence of local extrema)**.** *Let f be a function defined and continuous in some neighborhood of x_0.*

- *If the function f is increasing in some left-hand side neighborhood of the point x_0 and decreasing in some right-hand side neighborhood of the point x_0, then the function f takes on its sharp local maximum at the point x_0.*

- *If the function f is decreasing in some left-hand side neighborhood of the point x_0 and increasing in some right-hand side neighborhood of the point x_0, then the function f takes on its sharp local minimum at the point x_0.*

- *If the function f is either increasing or decreasing in some (two-sided) neighborhood of the point x_0, then there is no local extremum of the function f has no local extremum at x_0.*

Remark 6.2. Graphically the situation can be represented in the following scheme.



Remark 6.3 (absolute extremum)**.** Consider a function f defined and continuous on the closed interval $[a,b]$. According to Weierstrass Theorem 3.4, page 31 the function f takes on its absolute maximum and its absolute minimum on the interval $[a,b]$. It is clear (explain!) that this absolute extremum can occur only at the local extremum or at one of the end points of the interval $[a,b]$.

The following definition concerns points with a very close relationship to local extrema.

Definition (stationary point)**.** The point x_0 is said to be a *stationary point* of the function f if $f'(x_0) = 0$.

Remark 6.4 (geometric interpretation)**.** Tangent to the function in the stationary point is a horizontal line (explain!).

Example 6.1.

- The functions $y = e^x$ and $y = \ln x$ have no stationary point. Really, neither of the equations $e^x = 0$ and $\frac{1}{x} = 0$ possesses a solution in $\mathbb{R}$.
- The functions $y = x^2$ and $y = x^3$ have its stationary point at the point $x = 0$. Really, both equations $2x = 0$ and $3x^2 = 0$ possess a unique solution $x = 0$.

- The function $y = \dfrac{x^2}{x^3+1}$ has the derivative $y' = -\dfrac{x(x^3-2)}{(x^3+1)^2}$ and the stationary points are the solutions of equation

$$-\frac{x(x^3-2)}{(x^3+1)^2} = 0,$$

i.e. $x = 0$ and $x = \sqrt[3]{2}$.

Theorem 6.2 (relationship between stationary point and local extremum)**.** *Let f be a function defined in x_0. If the function f takes on a local extremum at $x = x_0$, then the derivative of the function f at the point x_0 either does not exist or equals zero and hence $x = x_0$ is a stationary point of the function f.*

Example 6.2. All the following observations are in harmony with Theorem 6.2.

- The point $x = 0$ is a stationary point and a local minimum of the function $y = x^2$.
- The point $x = 0$ is a stationary point of the function $y = x^3$. However, this function takes on no local extremum at this point.
- Both functions $y = |x|$ and $y = \sqrt[3]{x^2}$ take on its local minimum at the point $x = 0$. However, none of them has a derivative at $x = 0$.

Theorem 6.3 (relationship between derivative and monotonicity)**.** *Let f be a function. Suppose that f is differentiable on the open interval I.*

- *If $f'(x) > 0$ on I, then the function f is increasing on I.*
- *If $f'(x) < 0$ on I, then the function f is decreasing on I.*

Algorithm 6.1 (first derivative test).

(i) We find a natural domain of the function f

(ii) We find the derivative $f'(x)$ of the function $f(x)$.

(iii) We solve the equation $f'(x) = 0$.

(iv) We specify the domain of the derivative $f'(x)$.

(v) Suppose that we found all stationary points and all points of discontinuity of the derivative. We mark these points on the real line.

(vi) Now, the real line is divided into several subintervals. On each subinterval the derivative preserves its sign.

(vii) We record the type of monotonicity in a simple graphical scheme and for all subintervals.

(viii) We find the points where the function is continuous and the type of monotonicity changes. At these points the function $f(x)$ takes on a local extremum.

(ix) To recognize the type of an extremum, we consider the type of monotonicity on the right and on the left.

Worked problems are in the file minmax.pdf.
Quizzes are in the file tests/loc11.pdf.

7. Concavity

Another property of functions which possesses a clear interpretation on the graph is concavity.

Definition (concavity)**.** Let f be a function differentiable at x_0.

The function f is said to be *concave up* (*concave down*) at x_0 if there exists ring neighborhood $\overline{N}(x_0)$ of the point x_0 such that for all $x \in \overline{N}(x_0)$ the points of the graph of the function f are above (below) the tangent to the graph in the point x_0, i.e. if

$$f(x) > f(x_0) + f'(x_0)(x - x_0) \qquad \Big(f(x) < f(x_0) + f'(x_0)(x - x_0)\Big) \tag{7.1}$$

holds.

The function is said to be *concave up (concave down) on the interval I* if it has this property in each point of the interval I.

Example 7.1.

- The natural exponential function $y = e^x$ is concave up on $\mathbb{R}$ and its inverse, the natural logarithmic function $y = \ln x$, is concave down on $\mathbb{R}^+$.
- The cubic function $y = x^3$ is concave down for $x < 0$ and concave up for $x > 0$.
- The function $y = \sin x$ is concave down on the interval $(0, \pi)$.

Remark 7.1. Consider a function f defined on the interval I. Suppose, for brevity, that the function f is increasing on I.

- If the function f is concave up on I, then the rate of the increase of the function f is increasing; the increase of the function f speeds up.
- If the function f is concave down on I, then the rate of the increase slows down.

At the points where the type of concavity changes, the velocity of the increase is either highest or smallest comparing with the points in the neighborhood. From this reason, these points are of high interest when we investigate the function f and these points have a special name, introduced by the following definition.

Definition (point of inflection). The point x_0 in which the type concavity changes is said to be a *point of inflection* of the function f.

Example 7.2.

- The function $y = x^3$ has the point of inflection at $x = 0$.
- The function $y = \sin x$ has the points of inflection at $x = k\pi$, where k is an arbitrary integer.

The concavity contains an information about the rate of change of the increase or decrease. Since the information about the increase or decrease is contained in the first derivative and the rate of change can be recorded also by the derivative, it is natural to expect that the concavity of the function is closely related with the second derivative. The following theorem introduces this relationship.

Theorem 7.1 (relationship between the 2nd derivative and concavity)**.** *Let f be a function and f'' be the second derivative of the function f on the open interval I.*

- *If $f''(x) > 0$ on I, then the function f is concave up on I.*
- *If $f''(x) < 0$ on I, then the function f is concave down on I.*

The concavity can also indicate the type of local extremum at a stationary point. Roughly speaking, a concave-up function cannot take on a local maximum and a concave-down function cannot take on a local minimum.

Theorem 7.2 (2nd derivative test, concavity and local extrema)**.** *Let f be a function and x_0 a stationary point of this function.*

- *If $f''(x_0) > 0$, then the function f has its local minimum at the point x_0.*
- *If $f''(x_0) < 0$, then the function f has its local maximum at the point x_0.*
- *If $f''(x_0) = 0$, then the 2nd derivative test fails. A local extremum may or may not occur. Both cases are possible.*

To find the intervals on which the function is concave up/down we use the same method as we used to find the intervals of monotonicity of the function. The only difference is that we work with the second derivative $f''(x)$ instead of $f'(x)$ and we interpret the signs of $f''(x)$ in terms of concavity of $f(x)$.

8. Behavior of the function near infinity, asymptotes

In the remaining part of this file we investigate functions near the points $+\infty$ and $-\infty$. We will be interested in the fact, whether the graph approaches a line or not.

Definition (inclined asymptote)**.** Let f be a function defined in some neighborhood of $+\infty$. The line $y = kx + q$ is said to be an *inclined asymptote at $+\infty$ to the graph of the function $y = f(x)$* if

$$\lim_{x\to\infty} |kx + q - f(x)| = 0$$

holds.

Similarly, if we consider the point $-\infty$ instead of $+\infty$, we obtain the definition of the inclined asymptote at $-\infty$.

Remark 8.1. As a special case of the preceding definition we obtain for $k = 0$ the horizontal asymptote, introduced on the page 25.

Theorem 8.1 (inclined asymptote)**.** *Let f be a function defined in some neighborhood of the point $+\infty$. The line $y = kx + q$ is an inclined asymptote at $+\infty$ to the graph of the function $f(x)$ if and only if the following limits exist as finite numbers*

$$k := \lim_{x\to\infty} \frac{f(x)}{x} \qquad \text{and} \qquad q := \lim_{x\to\infty} (f(x) - kx). \tag{8.1}$$

Similarly, if we consider $-\infty$ instead of $+\infty$, we obtain the asymptote at $-\infty$.

The following theorem presents an equivalent definition of the inclined (or horizontal) asymptote. The meaning is the same – the vertical distance of the points on the graph and the points on the asymptotic line approaches zero.

Theorem 8.2. *Let f be a function defined in some neighborhood of $+\infty$ The line $y = kx+q$ is an asymptote to the graph $y = f(x)$ if and only if the function f can be written in the form*

$$f(x) = kx + q + g(x),$$

where $g(x)$ tends to zero as x tend to infinity, i.e. $\lim_{x\to\infty} g(x) = 0$.
A similar statement holds also for $-\infty$.

Figure 3 illustrates this obvious statement.

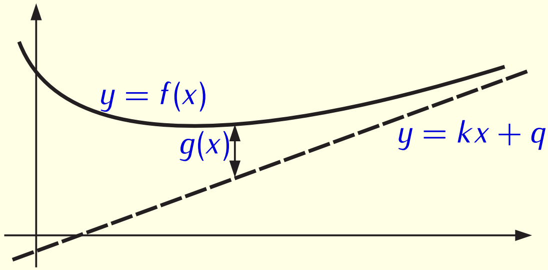


Figure 3: Inclined asymptote at $+\infty$.

Example 8.1. • The long division of polynomials shows that the function $f(x) = \dfrac{x^3}{x^2+1}$ can be written in an equivalent form

$$f(x) = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}.$$

The first part of this sum is a linear function and the second one tends to zero as x approaches to $\pm\infty$. Hence the line $y = x$ is an asymptote to the graph of the function f at both $\pm\infty$.

- The function $f(x) = 2x + \pi - \operatorname{arctg} x$ can be written in an equivalent form

$$f(x) = \left(2x + \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - \operatorname{arctg} x\right).$$

The expression in the first parentheses is linear and the expression inside the second parentheses tends to a zero as x approaches to ∞ since $\lim\limits_{x\to\infty} \operatorname{arctg} x = \dfrac{\pi}{2}$. Hence the line $y = 2x + \dfrac{\pi}{2}$ is an asymptote to the graph of the function f at $+\infty$. In a similar way we can show that the asymptote to the graph of the function f at $-\infty$ is $y = 2x + \dfrac{3}{2}\pi$.

- Concerning the last two examples, an application of Theorem 8.1 based on the evaluation of the limits (8.1) is also possible, but cumbersome.

In general, there exists no relationship between the asymptote at $+\infty$ and $-\infty$. Only in a very special case, the case of rational functions, we can use the following theorem.

Theorem 8.3 (inclined asymptotes of rational functions). *Inclined asymptotes to the graph of a rational function at $\pm\infty$ exist simultaneously (either both exist or both do not exist) and are equal.*

Worked problems on investigating functions are in the file prubeh.pdf.