Algebraic equations
Basic numerical methods

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Contents

1 Algebraic equations – general theory

We are able to solve linear and quadratic equations:

- If $a \neq 0$ and $ax + b = 0$, then $x = -\dfrac{b}{a}$.
- If $a \neq 0$ and $ax^2 + bx + c = 0$, then $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Now we wish to be able to solve higher order equations.

Definition (polynomial)**.** Let n be a nonnegative integer and

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n \tag{1}$$

be an n-degree polynomial with real coefficients a_0, a_1, $\dots a_n$, where $a_0 \neq 0$ (i.e., the highest power really appears in the polynomial).

The coefficient a_0 is called a *leading coefficient of the polynomial* $P_n(x)$ and the coefficient a_n is an *absolute term of the polynomial* $P_n(x)$. The term a_0x^n is called a *leading term of the polynomial* $P_n(x)$.

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Definition (algebraic equation). Under an *n-degree algebraic equation* we understand the equation of the form $P_n(x) = 0$, i.e.

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0 \tag{2}$$

Definition (zero of an algebraic equation). Under a *zero of equation* (2) (or a *zero of polynomial* (1)) we understand the number c which satisfies $P_n(c) = 0$, i.e., which substituted for x converts (2) into a valid relation.

A zero of an algebraic equation is sometimes called also a *root* of this equation or simply a *solution*.

Example 2. The numbers $x = 1$ and $x = -2$ are zeros of the polynomial

$$P(x) = x^3 + 2x^2 - x - 2. \tag{3}$$

Really, a quick evaluation shows $P(1) = 0$ and $P(-2) = 0$. The number $x = 3$ is not a zero of this polynomial, since $P(3) = 40$.

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Theorem 1 (the basic theorem of algebra). Every algebraic equation has a solution in the set of complex numbers.

Theorem 2 (Bezout). The number c is a zero of polynomial (1) (of equation (2)) if and only if the linear polynomial $(x-c)$ is a factor of this polynomial, i.e., if and only if there exists an $(n-1)$ degree polynomial $Q_{n-1}(x)$ with property

$$P_n(x) = (x-c)Q_{n-1}(x). \tag{4}$$

Definition (linear factor corresponding to the zero of algebraic equation). If c is a zero of polynomial (1) (of equation (2)), then the linear polynomial $(x-c)$ with an independent variable x is said to be a *linear factor of the polynomial* (1) *corresponding to the zero* $x=c$.

Example 3. Polynomial (3) can be written in each of the following equivalent forms

$$y = (x-1)(x^2+3x+2), \quad y = (x+2)(x^2-1), \quad y = (x-1)(x+1)(x+2).$$

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Remark 1 (Horner's scheme)**.** The Horner's scheme is a numerical method for easy

- calculation of the value of the polynomial in given number $x = a$,
- division of a polynomial by a linear polynomial $(x - a)$.

Consider an n-degree polynomial

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n$$

and a real number a. Let us write the following scheme (an example follows)

	a_0	a_1	a_2	$\dots$	a_{i-1}	a_i	$\dots$	a_{n-1}	a_n
a	b_0	b_1	b_2	$\dots$	b_{i-1}	b_i	$\dots$	b_n	

where $a_0 = b_0$, $b_1 = a \cdot b_0 + a_1$, $b_2 = a \cdot b_1 + a_2$, $\dots$, $b_i = a \cdot b_{i-1} + a_i$, $\dots$.
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$$Q_{n-1} = b_0x^{n-1} + b_1x^{n-2} + b_2n^{n-3} + \cdots + +b_{n-2}x + b_{n-1},$$

then

$$\boxed{P_n(a) = b_n}, \qquad \frac{P_n(x)}{x-a} = Q_{n-1}(x) + \frac{P_n(a)}{x-a}, \qquad \boxed{P_n(x) = (x-a)Q_{n-1}(x) + P_n(a)}.$$

If $b_n = 0$, then $x = a$ is a zero of the polynomial $P_n(x)$.

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Example 4. Divide $P(x) = x^6 + 3x^5 - 12x^4 - 38x^3 + 21x^2 + 99x + 54$

1. by the linear polynomial $(x-1)$
2. by the linear polynomial $(x+1)$

and establish the values $P(1)$ and $P(-1)$.
Solution: For $x = 1$ we have by the Horner's scheme

$$\begin{array}{c|ccccccc} & 1 & 3 & -12 & -38 & 21 & 99 & 54 \\ \hline 1 & 1 & 4 & -8 & -46 & -25 & 74 & 128 \end{array}$$

and hence

$$P(x) = (x^5 + 4x^4 - 8x^3 - 46x^2 - 25x + 74)(x-1) + 128.$$

In a similar way, for $x = -1$ we get

$$\begin{array}{c|ccccccc} & 1 & 3 & -12 & -38 & 21 & 99 & 54 \\ \hline -1 & 1 & 2 & -14 & -24 & 45 & 54 & 0 \end{array}$$

and hence

$$P(x) = (x^5 + 2x^4 - 14x^3 - 24x^2 + 45x + 54)(x+1). \tag{5}$$

The values $P(1)$ and $P(-1)$ are the last numbers in the second row of Horner's scheme, hence $P(1) = 128$ and $P(-1) = 0$.

Definition (multiplicity of zero)**.** Let c be a zero of the polynomial (1). The zero c is said to be of the *multiplicity* k if there exists $(n-k)$-degree polynomial $Q_{n-k}(x)$ such that,

$$P_n(x) = (x-c)^k Q_{n-k}(x) \qquad \text{and} \qquad Q_{n-k}(c) \neq 0. \tag{6}$$

Remark 2. A zero of multiplicity one is called a simple zero, a zero of multiplicity two is called a double zero.

Theorem 3. Let $x = c$ be a zero of the polynomial $P_n(x)$ and let $Q_{n-k}(x)$ be polynomial from the relation (6). Then the polynomials $P_n(x)$ and $Q_{n-k}(x)$ have common zeros, including multiplicity of these zeros, with exception of the zero $x = c$.

Theorem 4 (multiplicity of zeros and the derivatives)**.** The number c is a zero of the polynomial (1) (equation (2)) of the multiplicity k if and only if

$$P_n(c) = P_n'(c) = P_n''(c) = \cdots = P_n^{(k-1)}(c) = 0 \qquad \text{and} \qquad P_n^{(k)}(c) \neq 0,$$

i.e. if and only if c is a zero of the polynomial and all its derivatives up to the order $(k-1)$ (including $(k-1)$) and it is not a zero of the derivative of the order k.

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Remark 3 (a geometric consequence)**.** The zeros of a polynomial are the x-intercepts of the graph of this polynomial. In a neighbourhood of this zero the polynomial either changes its sign or not. Which of these two possibilities actually occurs depends on the multiplicity of this zero. More precisely, the following holds:

- At the zero of an *odd* multiplicity the polynomial changes the sign. If the multiplicity is at least 3, the zero is also a stationary point of the polynomial (tangent in this point is horizontal) and also a point of inflection.

- At the zero of an *even* multiplicity the polynomial does not change the sign. Hence there is a local extremum at this point; a local maximum, if the sign of the polynomial is negative in a neighbourhood of the zero and a local minimum, if the sign of the polynomial is positive.

Theorem 5 (the number of zeros of polynomial)**.** Every n-degree polynomial has exactly n zeros in the field of complex numbers, counted with multiplicity.

Theorem 6 (the number of real zeros)**.** Every n-degree polynomial has in the field of real numbers either exactly n zeros or $n-l$ zeros, where l is an even number. Each zero is in this amount counted with multiplicity.

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2 Basic numerical methods for algebraic equations

The problem to find all zeros of a given polynomial is not solvable in general. From this reason it is sometimes is necessary to use approximation methods. Some of the methods available for working with polynomials are presented in the following paragraphs. Our aim is

- to find the interval which contains all zeros of the polynomial,
- to find the system of intervals with the property that each interval contains exactly one zero (separation of zeros),
- to approximate zeros of the polynomial with error not higher than a given number.

Theorem 7 (estimate for zeros of algebraic equation)**.** Let x_i (for $i = 1..n$) be zeros (real or complex) of equation (2). The following estimate holds for all of these zeros

$$|x_i| < 1 + \frac{A}{|a_0|}, \tag{7}$$

where $A = \max\{|a_i|, i = 1..n\}$.

Theorem 8 (estimate for the number of real zeros, Descartes)**.** Let p be the number of positive zeros of the polynomial (1) and s the number of the changes of the sign in the sequence of its coefficients a_0, a_1, a_2, ..., a_n, (the coefficients which are equal to zero are not considered in this sequence). Then either $p = s$ or $p < s$ and in the latter case the number $s - p$ is and even number.

Example 5. The polynomial $P(x) = x^8 - x^5 + x^3 + x^2 - x + 1$ has either 4 or 2 or no positive zero. These zeros (if there are at least two) are in the interval $(0, 2)$.

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Remark 4 (number of negative zeros – a modification of Descarte's rule)**.** To estimate the number of negative zeros we use the simple fact that

$$P(x) = P(-(-x)) = \overline{P}(-x), \quad \text{where} \quad \overline{P}(x) := P(-x).$$

Hence to find negative zeros of the polynomial $P(x)$, suffices to find the positive zeros of $\overline{P}(x)$. Practically, we work in the following steps.

- We write the polynomial $\overline{P}(x) := P(-x)$ substituting $-x$ for x in the polynomial $P(x)$. This gives a new polynomial which differs form the polynomial $P(x)$ i the signs of the coefficients at the odd powers of x.
- We establish the number of the sign changes in the sequence of the coefficients in the polynomial $\overline{P}(x)$.
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Let us study the polynomials (algebraic equations) with integer coefficients first. In this case it is easy to find all its integer zeros (if exists any).

Theorem 9 (necessary condition for polynomial with integer coefficients)**.** Suppose that all coefficients of a given polynomial (1) are integers. Suppose that $c \in \mathbb{Z}$ is an integer zero of this polynomial. Then the absolute term must be divisible by the number c, i.e. $c|a_n$.

Remark 5 (practical)**.** From Theorem 9 it follows that it is sufficient to look for integer zeros of a polynomial with integer coefficients in the set of all integers factors of the term a_0. An importance of this idea is in the fact that the set of all factors of a_0 contains only a finite amount of number. We can test in a finite time each of these candidates whether it is actually a zero and what is a multiplicity of this zero. If we find a zero c of multiplicity k, we divide k-times the polynomial with linear factor $(x - c)$ and consider the obtained new polynomial in the sequel.

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We find al factors of the absolute term 36 (both positive and negative).

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The number 36 possesses the following factors: ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18 and ± 36.

$$1 \quad 1 \quad -5 \quad -9 \quad -24 \quad -36$$

We will use the Horner's scheme for evaluation of the polynomial at the test numbers and for division by the linear factor.

Solve the equation $x^5 + x^4 - 5x^3 - 9x^2 - 24x - 36 = 0.$

The number 36 possesses the following factors: ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18 and ± 36.

$$\begin{array}{c|cccccc} & 1 & 1 & -5 & -9 & -24 & -36 \\ \hline 1 & 1 & 2 & -3 & -12 & -36 & -72 \end{array}$$

We substitute $x = 1$. We see that $P(1) = -72$ and the number $x = 1$ is not a zero of the polynomial.

Solve the equation $x^5 + x^4 - 5x^3 - 9x^2 - 24x - 36 = 0.$

The number 36 possesses the following factors: ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18 and ± 36.

	1	1	−5	−9	−24	−36
1	1	2	−3	−12	−36	−72
−1	1	0	−5	−4	−20	−16

Similarly, the number $x = -1$ is not a zero.

Solve the equation $x^5 + x^4 - 5x^3 - 9x^2 - 24x - 36 = 0$.

The number 36 possesses the following factors: ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18 and ± 36.

	1	1	−5	−9	−24	−36
1	1	2	−3	−12	−36	−72
−1	1	0	−5	−4	−20	−16
2	1	3	1	−7	−38	$\neq 0$

$x = 2$ is not a zero.

Solve the equation $x^5 + x^4 - 5x^3 - 9x^2 - 24x - 36 = 0$.

The number 36 possesses the following factors: ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18 and ±36.

	1	1	−5	−9	−24	−36
1	1	2	−3	−12	−36	−72
−1	1	0	−5	−4	−20	−16
2	1	3	1	−7	−38	$\neq 0$
$\boxed{-2}$	1	−1	−3	−3	−18	$\parallel 0$

The number $x = -2$ **is** a zero of the polynomial under consideration. The left hand side of the equation possesses factorization

$$(x+2)(x^4 - x^3 - 3x^2 - 3x - 18) = 0.$$

In the following steps we investigate the second polynomial in this product only.

Solve the equation $x^5 + x^4 - 5x^3 - 9x^2 - 24x - 36 = 0$.

The number 36 possesses the following factors: ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18 and ± 36.

	1	1	−5	−9	−24	−36
1	1	2	−3	−12	−36	−72
−1	1	0	−5	−4	−20	−16
2	1	3	1	−7	−38	$\neq 0$
$\boxed{-2}$	1	−1	−3	−3	−18	$\parallel 0$
$\boxed{-2}$	1	−3	3	−9	$\parallel 0$	

We substitute again $x = -2$. This number is again a zero. It is a multiple zero (at least double) of the original polynomial.

Solve the equation $x^5 + x^4 - 5x^3 - 9x^2 - 24x - 36 = 0$.

The number 36 possesses the following factors: ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18 and ± 36.

	1	1	-5	-9	-24	-36
1	1	2	-3	-12	-36	-72
-1	1	0	-5	-4	-20	-16
2	1	3	1	-7	-38	$\neq 0$
$\boxed{-2}$	1	-1	-3	-3	-18	$\parallel 0$
$\boxed{-2}$	1	-3	3	-9	$\parallel 0$	
-2	1	-5	13	-35		

- We substitute again $x = -2$. This number is no more zero (it is a double zero of the original polynomial).
- In fact, we focus our attention to the factors of the absolute term 9.

Solve the equation $x^5 + x^4 - 5x^3 - 9x^2 - 24x - 36 = 0.$

The number 36 possesses the following factors: ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18 and ± 36.

	1	1	−5	−9	−24	−36
1	1	2	−3	−12	−36	−72
−1	1	0	−5	−4	−20	−16
2	1	3	1	−7	−38	$\neq 0$
$\boxed{-2}$	1	−1	−3	−3	−18	$\parallel 0$
$\boxed{-2}$	1	−3	3	−9	$\parallel 0$	
−2	1	−5	13	−35		
$\boxed{3}$	1	0	3	$\parallel 0$		

- We remove the numbers which are not divisors of the number 9 and continue with $x = 3$.
- The number $x = 3$ is a zero.

Solve the equation $x^5 + x^4 - 5x^3 - 9x^2 - 24x - 36 = 0.$

The number 36 possesses the following factors: ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18 and ± 36.

	1	1	−5	−9	−24	−36
1	1	2	−3	−12	−36	−72
−1	1	0	−5	−4	−20	−16
2	1	3	1	−7	−38	$\neq 0$
$\boxed{-2}$	1	−1	−3	−3	−18	$\parallel 0$
$\boxed{-2}$	1	−3	3	−9	$\parallel 0$	
−2	1	−5	13	−35		
$\boxed{3}$	1	0	3	$\parallel 0$		
−3	1	−3	12			

We explore the divisors of the absolute term — the number 3. The sequence of coefficients does not contain any change of sign and by the Descart's rule of signs there is no positive zero. It is sufficient to test the number $x = -3$. This number is not a zero.

Solve the equation $x^5 + x^4 - 5x^3 - 9x^2 - 24x - 36 = 0$.

The number 36 possesses the following factors: ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18 and ±36.

	1	1	-5	-9	-24	-36
1	1	2	-3	-12	-36	-72
-1	1	0	-5	-4	-20	-16
2	1	3	1	-7	-38	$\neq 0$
$\boxed{-2}$	1	-1	-3	-3	-18	$\Vert\ 0$
$\boxed{-2}$	1	-3	3	-9	$\Vert\ 0$	
-2	1	-5	13	-35		
$\boxed{3}$	1	0	3	$\Vert\ 0$		
-3	1	-3	12			

Factorization of the polynomial is $(x+2)^2(x-3)(x^2+3) = 0$.

- The polynomial possesses double zero $x = -2$, simple zero $x = 3$ and the remaining two zeros are not integers.
- The polynomial with coefficients $1, 0, 3$ (i.e. the polynomial $x^2 + 0x + 3$) remains when we divide by all linear factors corresponding to these zeros.

Arrangement. The number c is said to be a zero of polynomial (1) *with an error less than* ε if it differs from the actual zero at most by ε, i.e. if the actual zero is in the interval $(c-\varepsilon, c+\varepsilon)$.

It is clear that if the interval $[a,b]$ contains[1] a zero of the polynomial $P(x)$, then the centre $c = \dfrac{a+b}{2}$ of this interval is a zero of the polynomial with error less than the half of the length of this interval, i.e. $\varepsilon = \dfrac{b-a}{2}$.

The method of bisection. Given a polynomial $P(x)$ and real numbers $a, b \in \mathbb{R}$, suppose that $P(a)P(b) < 0$ holds. Consider the point $c = \dfrac{b+a}{2}$ and the value $P(c)$ of the polynomial in this point. One of the relations

$$P(a)P(c) < 0 \quad \text{or} \quad P(c)P(b) < 0 \quad \text{or} \quad P(c) = 0$$

holds. Omitting the last possibility (in this case $x = c$ is an exact zero of the polynomial $P(x)$), we see that one of the intervals (a,c) and (b,c) contains a zero of the polynomial $P(x)$ When seeking a zero, we can focus our attention to the appropriate left half or right half of the interval (a,b). Following this idea we can, after a finite number of steps, obtain the value of the zero with an arbitrary precision.

[1] This is guaranteed especially if $P(a)$ and $P(b)$ have different signs.

Arrangement. The number c is said to be a zero of polynomial (1) *with an error less than* ε if it differs from the actual zero at most by ε, i.e. if the actual zero is in the interval $(c-\varepsilon, c+\varepsilon)$.

It is clear that if the interval $[a,b]$ contains[1] a zero of the polynomial $P(x)$, then the centre $c = \dfrac{a+b}{2}$ of this interval is a zero of the polynomial with error less than the half of the length of this interval, i.e. $\varepsilon = \dfrac{b-a}{2}$.

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holds. Omitting the last possibility (in this case $x = c$ is an exact zero of the polynomial $P(x)$), we see that one of the intervals (a,c) and (b,c) contains a zero of the polynomial $P(x)$ When seeking a zero, we can focus our attention to the appropriate left half or right half of the interval (a,b). Following this idea we can, after a finite number of steps, obtain the value of the zero with an arbitrary precision.

[1] This is guaranteed especially if $P(a)$ and $P(b)$ have different signs.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

All zeros are in the interval $(-2, 2)$.

- All coefficients are ± 1.
- The largest coefficient (in absolute value) is 1.
- All zeros satisfy the estimate

$$|x_i| < 1 + \frac{1}{1} = 2.$$

$\boxed{\text{Solve } P(x) = x^3 + x - 1 = 0 \text{ with error less than } 0.03.}$

All zeros are in the interval $(-2, 2)$.

$+ \quad + \quad -$, one positive zero

- We write the sequence of signs.
- The sequence contains one change of sign.
- The equation possesses one positive solution.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

All zeros are in the interval $(-2, 2)$.

$+ \ + \ -$, one positive zero

$P(-x) = (-x)^3 + (-x) + 1 = -x^3 - x - 1$

- We look for the number of negative zeros.
- We write the auxiliary polynomial $P(-x)$ and determine the number of changes of sign.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

All zeros are in the interval $(-2, 2)$.

$+ \quad + \quad -$, one positive zero

$P(-x) = (-x)^3 + (-x) + 1 = -x^3 - x - 1$

$- \quad - \quad -$, no negative zero

There is no change of sign. The equation possesses no negative solution (no solution is on $(-\infty, 0)$).

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

All zeros are in the interval $(-2, 2)$.

$+ \quad + \quad -$, one positive zero

$P(-x) = (-x)^3 + (-x) + 1 = -x^3 - x - 1$

$- \quad - \quad -$, no negative zero

$P(0) = -1;$

$P(1) = 1 + 1 - 1 = 1;$

$P(2) = 8 + 2 - 1 = 9;$

- The zero is in the interval $(0, 2)$.
- Evaluating the polynomial at the integer values, we restrict this interval to the interval of unit length.

$\boxed{\text{Solve } P(x) = x^3 + x - 1 = 0 \text{ with error less than } 0.03.}$

All zeros are in the interval $(-2, 2)$.

$+ \quad + \quad -$, one positive zero

$P(-x) = (-x)^3 + (-x) + 1 = -x^3 - x - 1$

$- \quad - \quad -$, no negative zero

$P(0) = -1$;

$P(1) = 1 + 1 - 1 = 1$;

$P(2) = 8 + 2 - 1 = 9$; The zero is between 0 and 1.

- The solution is in $(0, 1)$.
- We use the bisection to meet the required precision. (The current precision is 0.5.)

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0		1	$-$		$+$	

- We write our computations in the simple table.
- When writing the values of the polynomial we are interested in signs only.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$		$+$	

We find the half of the interval $[a, b]$.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	

$(0.5)^3 + 0.5 - 1 = -0.375$

We evaluate polynomial at this number.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5		1	$-$		$+$	

$(0.5)^3 + 0.5 - 1 = -0.375$

- We determine the half of the interval $[a, b]$ where the polynomial changes its sign (in red).
- The boundaries of this interval are our new approximation of the zero.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$		$+$	

$(0.5)^3 + 0.5 - 1 = -0.375$

We bisect interval again. The middle of the interval is a solution with error less than

$$\varepsilon = \frac{1-0.5}{2} = 0.25.$$

This is still too much.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	

$(0.5)^3 + 0.5 - 1 = -0.375$

$(0.75)^3 + 0.75 - 1 = 0.171875$

We evaluate the polynomial at the middle of the interval.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5		0.75	$-$		$+$	

$(0.5)^3 + 0.5 - 1 = -0.375$

$(0.75)^3 + 0.75 - 1 = 0.171875$

- We determine the half of the interval $[a, b]$ where the polynomial changes its sign (in red).
- The boundaries of this interval are our new approximation of the zero.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$		$+$	

$(0.5)^3 + 0.5 - 1 = -0.375$

$(0.75)^3 + 0.75 - 1 = 0.171875$

We bisect the interval.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	

$(0.5)^3 + 0.5 - 1 = -0.375$

$(0.75)^3 + 0.75 - 1 = 0.171875$

$(0.625)^3 + 0.625 - 1 = -0.130859$

We evaluate the polynomial at the middle of the interval.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \dfrac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \dfrac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625		0.75	$-$		$+$	

$(0.5)^3 + 0.5 - 1 = -0.375$
$(0.75)^3 + 0.75 - 1 = 0.171875$
$(0.625)^3 + 0.625 - 1 = -0.130859$

- We determine the half of the interval $[a, b]$ where the polynomial changes its sign (in red).
- The boundaries of this interval are our new approximation of the zero.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625		0.75	$-$		$+$	0.62

$(0.5)^3 + 0.5 - 1 = -0.375$

$(0.75)^3 + 0.75 - 1 = 0.171875$

$(0.625)^3 + 0.625 - 1 = -0.130859$

We compute the error.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \dfrac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \dfrac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625	0.6875	0.75	$-$		$+$	0.62

$(0.5)^3 + 0.5 - 1 = -0.375$
$(0.75)^3 + 0.75 - 1 = 0.171875$
$(0.625)^3 + 0.625 - 1 = -0.130859$

We bisect the interval.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625	0.6875	0.75	$-$	$+0.01$	$+$	0.62

$(0.5)^3 + 0.5 - 1 = -0.375$

$(0.75)^3 + 0.75 - 1 = 0.171875$

$(0.625)^3 + 0.625 - 1 = -0.130859$

$(0.6875)^3 + 0.6875 - 1 = 0.0124511$

We evaluate the polynomial at the middle of the interval.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \dfrac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \dfrac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625	0.6875	0.75	$-$	$+0.01$	$+$	0.62
0.625		0.6875	$-$		$+$	

$(0.5)^3 + 0.5 - 1 = -0.375$
$(0.75)^3 + 0.75 - 1 = 0.171875$
$(0.625)^3 + 0.625 - 1 = -0.130859$
$(0.6875)^3 + 0.6875 - 1 = 0.0124511$

- We determine the half of the interval $[a, b]$ where the polynomial changes its sign (in red).
- The boundaries of this interval are our new approximation of the zero.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625	0.6875	0.75	$-$	$+0.01$	$+$	0.62
0.625	0.6563	0.6875	$-$		$+$	0.0312

$(0.5)^3 + 0.5 - 1 = -0.375$
$(0.75)^3 + 0.75 - 1 = 0.171875$
$(0.625)^3 + 0.625 - 1 = -0.130859$
$(0.6875)^3 + 0.6875 - 1 = 0.0124511$

We bisect the interval. We compute the error.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625	0.6875	0.75	$-$	$+0.01$	$+$	0.62
0.625	0.6563	0.6875	$-$	-0.06	$+$	0.0312

$(0.5)^3 + 0.5 - 1 = -0.375$
$(0.75)^3 + 0.75 - 1 = 0.171875$
$(0.625)^3 + 0.625 - 1 = -0.130859$
$(0.6875)^3 + 0.6875 - 1 = 0.0124511$

We evaluate the polynomial at the middle of the interval.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625	0.6875	0.75	$-$	$+0.01$	$+$	0.62
0.625	0.6563	0.6875	$-$	-0.06	$+$	0.0312
0.6563		0.6875				

$(0.5)^3 + 0.5 - 1 = -0.375$
$(0.75)^3 + 0.75 - 1 = 0.171875$
$(0.625)^3 + 0.625 - 1 = -0.130859$
$(0.6875)^3 + 0.6875 - 1 = 0.0124511$

- We determine the half of the interval $[a, b]$ where the polynomial changes its sign (in red).
- The boundaries of this interval are our new approximation of the zero.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \frac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \frac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625	0.6875	0.75	$-$	$+0.01$	$+$	0.62
0.625	0.6563	0.6875	$-$	-0.06	$+$	0.0312
0.6563	0.6719	0.6875				0.0156

$(0.5)^3 + 0.5 - 1 = -0.375$

$(0.75)^3 + 0.75 - 1 = 0.171875$

$(0.625)^3 + 0.625 - 1 = -0.130859$

$(0.6875)^3 + 0.6875 - 1 = 0.0124511$

- The error is sufficiently small.
- Now it is sufficient to bisect the interval.

Solve $P(x) = x^3 + x - 1 = 0$ with error less than 0.03.

The zero is between 0 and 1.

a	$c = \dfrac{a+b}{2}$	b	$P(a)$	$P(c)$	$P(b)$	$\varepsilon = \dfrac{b-a}{2}$
0	0.5	1	$-$	-0.37	$+$	
0.5	0.75	1	$-$	$+0.17$	$+$	
0.5	0.625	0.75	$-$	-0.13	$+$	
0.625	0.6875	0.75	$-$	$+0.01$	$+$	0.62
0.625	0.6563	0.6875	$-$	-0.06	$+$	0.0312
0.6563	0.6719	0.6875				0.0156

$(0.5)^3 + 0.5 - 1 = -0.375$
$(0.75)^3 + 0.75 - 1 = 0.171875$
$(0.625)^3 + 0.625 - 1 = -0.130859$
$(0.6875)^3 + 0.6875 - 1 = 0.0124511$

The solution is $x = 0.67 \pm 0.02$. It is in the interval $(0.65, 0.69)$.

- We round the error to one nonzero digit. We round the solution to the same number of decimal digits.
- We check that after this rounding both last estimates a and b are inside the interval, where we claim the existence of solution.

3 Lagrange interpolation formula

Motivation. An n-degree polynomial is given uniquely by $(n+1)$ independent conditions.[2] The values of the polynomial at the given $(n+1)$ mutually different points can play the role of such conditions.
It can be proved: For a given set of $(n+1)$ pairs $[x_i, y_i]$ of the real numbers there exists a unique n-degree polynomial $P(x)$ which satisfies $P(x_i) = y_i$ for all x_i.
The main problem of this section is to find an analytic formula for this polynomial. There are several possibilities. One of these possibilities is introduced in the following Theorem.

Theorem 10 (Lagrange). Let us consider the set of $(n+1)$ ordered pairs $[x_i, y_i]$, $i \in \mathbb{N}_0$, $0 \leq i \leq n$. Polynomial

$$L(x) = y_0 l_0(x) + y_1 l_1(x) + \cdots + y_n l_n(x) \tag{8}$$

where

$$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

satisfies $L(x_i) = y_i$ for all $i \in \mathbb{N}_0$, $0 \leq i \leq n$.

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[2]This follows from the fact that this polynomial contains $(n+1)$ coefficients. As a particular case, two different points define a line.

Definition (Lagrange polynomial). The polynomial $L(x)$ from Theorem 10 is called a *Lagrange polynomial*.

Polynomials $l_i(x)$ from Theorem 10 are called *small Lagrange polynomials*.

Formula (8) is called a *Lagrange interpolation formula*.

Example 8. Find a 3-degree polynomial $P(x)$ which satisfies $P(1) = 3$, $P(2) = -2$, $P(-1) = 0$ and $P(0) = 1$.

Solution: We write the values into the following table.

i	0	1	2	3
x_i	1	2	-1	0
y_i	3	-2	0	1

By the Lagrange interpolation formula (8) we have

$$P(x) = 3l_0(x) + (-2)l_1(x) + 0l_2(x) + 1l_3(x) = 3l_0(x) - 2l_1(x) + l_3(x).$$

The small Lagrange polynomials are

$$l_0(x) = \frac{(x-2)(x+1)x}{(1-2)(1+1)1} = \frac{x^3 - x^2 - 2x}{-2}$$
$$= -\frac{1}{2}(x^3 - x^2 - 2x),$$
$$l_1(x) = \frac{(x-1)(x+1)x}{(2-1)(2+1)2} = \frac{x^3 - 1}{6}$$
$$= \frac{1}{6}(x^3 - x),$$
$$l_3(x) = \frac{(x-1)(x-2)(x+1)}{(0-1)(0-2)(0+1)} = \frac{(x^2-1)(x-2)}{2}$$
$$= \frac{1}{2}(x^3 - 2x^2 - x + 2).$$

Note that it is not necessary to write the polynomial $l_2(x)$, since $y_2 = 0$ and $l_2(x)$ is multiplied by a zero. The Lagrange interpolation formula gives

$$P(x) = -\frac{3}{2}(x^3 - x^2 - 2x) - 2\frac{1}{6}(x^3 - x) + \frac{1}{2}(x^3 - 2x^2 - x - 2)$$
$$= x^3\left(-\frac{3}{2} - \frac{1}{3} + \frac{1}{2}\right) + x^2\left(\frac{3}{2} - 1\right) + x\left(3 + \frac{1}{3} - \frac{1}{2}\right) + 1$$
$$= -\frac{4}{3}x^3 + \frac{1}{2}x^2 + \frac{17}{6}x + 1.$$

Further reading:

- http://mathworld.wolfram.com/PolynomialRoots.html
- http://mathworld.wolfram.com/PolynomialFactorTheorem.html
- http://www.sosmath.com/algebra/factor/fac02/fac02.html