Conjugacy criteria for half-linear ODE
in theory of PDE
with generalized \( p \)-Laplacian
and mixed powers

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\[
\text{div} \left( A(x) \| \nabla y \|^{p-2} \nabla y \right) + \langle \vec{b}(x), \| \nabla y \|^{p-2} \nabla y \rangle \\
+ c(x) |y|^{p-2} y + \sum_{i=1}^{m} c_i(x) |y|^{p_i-2} y = e(x),
\]

where:

- \( x = (x_1, \ldots, x_n)_{i=1}^{n} \in \mathbb{R}^n, \ p > 1, \ p_i > 1, \)
- \( A(x) \) is an elliptic \( n \times n \) matrix with differentiable components, \( c(x) \) and \( c_i(x) \) are H"older continuous functions, \( \vec{b}(x) = (b_1(x), \ldots, b_n(x)) \) is a continuous \( n \)-vector function,
- \( \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)_{i=1}^{n} \) and \( \text{div} = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n} \) are the usual nabla and divergence operators,
- \( q \) is a conjugate number to the number \( p \), i.e., \( q = \frac{p}{p-1} \),
- \( \langle \cdot, \cdot \rangle \) is the usual scalar product in \( \mathbb{R}^n \), \( \| \cdot \| \) is the usual norm in \( \mathbb{R}^n \), \( \| A \| = \sup \{ \| Ax \| : x \in \mathbb{R}^n \text{ with } \| x \| = 1 \} = \lambda_{\text{max}} \) is the spectral norm
- The solution of \( (E) \) in \( \Omega \subset \mathbb{R}^n \) is a differentiable function \( u(x) \) such that \( A(x) \| \nabla u(x) \|^{p-2} \nabla u(x) \) is also differentiable and \( u \) satisfies \( (E) \) in \( \Omega \)
- \( S(a) = \{ x \in \mathbb{R}^n : \| x \| = a \} \), \( \Omega(a) = \{ x \in \mathbb{R}^n : a \leq \| x \| \} \), \( \Omega(a,b) = \{ x \in \mathbb{R}^n : a \leq \| x \| \leq b \} \).
Concept of oscillation for ODE

\[ u'' + c(x)u = 0 \]  \hspace{1cm} (1)

- Equation (1) is oscillatory if each solution has infinitely many zeros in \([x_0, \infty)\).
- Equation (1) is oscillatory if each solution has a zero \([a, \infty)\) for each \(a\).
- Equation (1) is oscillatory if each solution has conjugate points on the interval \([a, \infty)\) for each \(a\).
- All definition are equivalent (no accumulation of zeros and Sturm separation theorem).
- Equation is oscillatory if \(c(x)\) is large enough. Many oscillation criteria are expressed in terms of the integral \(\int_\infty^\infty c(x) \, dx\) (Hille and Nehari type)
- There are oscillation criteria which can detect oscillation even if \(\int_\infty^\infty c(x) \, dx\) is extremely small. These criteria are in fact series of conjugacy criteria.
Equation with mixed powers

\[ (p(t)u')' + c(t)u + \sum_{i=1}^{m} c_i(t)|u|^{\alpha_i} \text{sgn} u = e(t) \]  

(2)

where \( \alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0 \).

**Theorem A** (Sun, Wong (2007)). If for any \( T \geq 0 \) there exists \( a_1, b_1, a_2, b_2 \) such that \( T \leq a_1 < b_1 \leq a_2 < b_2 \) and

\[
\begin{align*}
  c_i(t) &\geq 0 & t \in [a_1, b_1] \cup [a_2, b_2], \ i = 1, 2, \ldots, n \\
  e(x) &\leq 0 & t \in [a_1, b_1] \\
  e(x) &\geq 0 & t \in [a_2, b_2]
\end{align*}
\]

and there exists a continuously differentiable function \( u(t) \) satisfying \( u(a_i) = u(b_i) = 0, \ u(t) \neq 0 \) on \( (a_i, b_i) \) and

\[
\int_{a_i}^{b_i} \left\{ p(t)u'^2(t) - Q(t)u^2(t) \right\} \ dt \leq 0
\]

(3)

for \( i = 1, 2, \) where

\[
Q(t) = k_0 |e(t)|^{\eta_0} \prod_{i=1}^{m} \left( c_i^{\eta_i}(t) \right) + c(t),
\]

\[ k_0 = \prod_{i=0}^{m} \eta_i^{-\eta_i} \] and \( \eta_i, \ i = 0, \ldots, n \) are positive constants satisfying \( \sum_{i=1}^{m} \alpha_i\eta_i = 1 \) and \( \sum_{i=0}^{m} \eta_i = 1 \),

then all solutions of (2) are oscillatory.
**Concept of oscillation for linear PDE**

\[ \Delta u + c(x)u = 0 \]  

(4)

- Equation (4) is *oscillatory* if every solution has a zero on \( \{ x \in \mathbb{R}^n : \| x \| \geq a \} \) for each \( a \).

- Equation (4) is *nodally oscillatory* if every solution has a nodal domain on \( \{ x \in \mathbb{R}^n : \| x \| \geq a \} \) for each \( a \).

- Both definition are equivalent (Moss+Piepenbrink).

**Concept of oscillation for half-linear PDE**

\[ \text{div} \left( \| \nabla u \|^{p-2} \nabla u \right) + c(x)|u|^{p-2}u = 0 \]  

(5)

- Essentially the same approach to oscillation as in linear case

- The equivalence between two oscillations is open problem.
\[
div \left( A(x) \| \nabla y \|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \| \nabla y \|^{p-2} \nabla y \right\rangle \\
+ c(x) |y|^{p-2}y + \sum_{i=1}^{m} c_i(x) |y|^{p_i-2}y = e(x),
\]

Detection of oscillation from ODE

**Theorem B** (O. Došlý (2001)). *Equation*

\[
\text{div}(\| \nabla u \|^{p-2} \nabla u) + c(x) |u|^{p-2}u = 0
\]

is oscillatory, if the ordinary differential equation

\[
\left( r^{n-1} |u'|^{p-2} u' \right)' + r^{n-1} \left( \frac{1}{\omega_n r^{n-1}} \int_{S(r)} c(x) \, dx \right) |u|^{p-2}u = 0
\]

is oscillatory. *The number \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \).*

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for \( A(x) = a(\| x \|)I \), \( a(\cdot) \) differentiable).

**Our aim**

- Extend method used in Theorem A to (E). Derive a general result, like Theorem B.
- Derive a result which does depend on more general expression, than the mean value of \( c(x) \) over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2009) excluded the possibility \( p_i > p \) for every \( i \)).
\[
\text{div} \left( A(x) \| \nabla y \|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \| \nabla y \|^{p-2} \nabla y \right\rangle \\
+ c(x) |y|^{p-2}y + \sum_{i=1}^{m} c_i(x) |y|^{p_i-2}y = e(x),
\]

(E)

**Modus operandi**

- Get rid of terms \( \sum_{i=1}^{m} c_i(x) |y|^{p_i-2}y \) and \( e(x) \) (join with \( c(x) |y|^{p-2}y \)) and convert the problem into

\[
\text{div} \left( A(x) \| \nabla y \|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \| \nabla y \|^{p-2} \nabla y \right\rangle + C(x) |y|^{p-2}y = 0.
\]

- Derive Riccati type inequality in \( n \) variables.
- Derive Riccati type inequality in 1 variable.
- Use this inequality as a tool which transforms results from ODE to PDE.
Using generalized AG inequality \( \sum \alpha_i \geq \prod \left( \frac{\alpha_i}{\eta_i} \right)^{\eta_i} \), if \( \alpha_i \geq 0, \eta_i > 0 \) and \( \sum \eta_i = 1 \) we eliminate the right-hand side and terms with mixed powers.

**Lemma 1.** Let either \( y > 0 \) and \( e(x) \leq 0 \) or \( y < 0 \) and \( e(x) \geq 0 \). Let \( \eta_i > 0 \) be numbers satisfying \( \sum \eta_i = 1 \) and \( \eta_0 + \sum_{i=1}^{m} p_i \eta_i = p \) and let \( c_i(x) \geq 0 \) for every \( i \). Then

\[
\frac{1}{|y|^{p-2}y} \left( -e(x) + \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y \right) \geq C_1(x),
\]

where

\[
C_1(x) := \left| \frac{e(x)}{\eta_0} \right| \prod_{i=1}^{m} \left( \frac{c_i(x)}{\eta_i} \right)^{\eta_i}.
\] (8)

**Remark:** The numbers \( \eta_i \) from Lemma 1 exist, if \( p_i > p \) for some \( i \).

**Lemma 2.** Suppose \( c_i(x) \geq 0 \). Let \( \eta_i > 0 \) be numbers satisfying \( \sum \eta_i = 1 \) and \( \sum_{i=1}^{m} p_i \eta_i = p \). Then

\[
\frac{1}{|y|^{p-2}y} \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y \geq C_2(x),
\]

where

\[
C_2(x) := \prod_{i=1}^{m} \left( \frac{c_i(x)}{\eta_i} \right)^{\eta_i}.
\] (9)

**Remark:** The numbers \( \eta_i \) from Lemma 2 exist iff \( p_i > p \) for some \( i \) and \( p_j < p \) for some \( j \).
Lemma 3. Let \( y \) be a solution of \((E)\) which does not have zero on \( \Omega \). Suppose that there exists a function \( C(x) \) such that

\[
C(x) \leq c(x) + \sum_{i=1}^{m} c_i(x) |y|^{p_i-p} - \frac{e(x)}{|y|^{p-2}}
\]

Denote \( \vec{w}(x) = A(x) \frac{\|\nabla y\|^{p-2} \nabla y}{|y|^{p-2}} \). The function \( \vec{w}(x) \) is well defined on \( \Omega \) and satisfies the inequality

\[
div \vec{w} + (p-1) \Lambda(x) \|\vec{w}\|^q + \left\langle \vec{w}, A^{-1}(x) \vec{b}(x) \right\rangle + C(x) \leq 0
\]

where

\[
\Lambda(x) = \begin{cases} 
\lambda_{\text{max}}^{1-q}(x) & 1 \leq p \leq 2, \\
\lambda_{\text{min}} \lambda_{\text{max}}^{-q}(x) & p > 2.
\end{cases}
\]

Lemma 4. Let \((10)\) hold. Let \( l > 1 \), \( l^* = \frac{l}{l-1} \) be two mutually conjugate numbers and \( \alpha \in C^1(\Omega, \mathbb{R}^+) \) be a smooth function positive on \( \Omega \). Then

\[
div(\alpha(x) \vec{w}) + (p-1) \frac{\Lambda(x) \alpha^{1-q}(x)}{l^*} \|\alpha(x) \vec{w}\|^q
\]

\[
- \frac{l^{p-1} \alpha(x)}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p + \alpha(x)C(x) \leq 0
\]

holds on \( \Omega \). If \( \left\| A^{-1} \vec{b} - \frac{\nabla \alpha}{\alpha} \right\| \equiv 0 \) on \( \Omega \), then this inequality holds with \( l^* = 1 \).
**Theorem 1.** Let the $n$-vector function $\vec{w}$ satisfy inequality

$$\text{div} \vec{w} + C_0(x) + (p - 1) \Lambda_0(x) \|\vec{w}\|^q \leq 0$$
on $\Omega(a,b)$. Denote $\tilde{C}(r) = \int_{S(r)} C_0(x) \, d\sigma$ and $\tilde{R}(r) = \int_{S(r)} \Lambda_0^{1-p} \, d\sigma$. Then the half-linear ordinary differential equation

$$\left( \tilde{R}(r)|u'|^{p-2}u' \right)' + \tilde{C}(r)|u|^{p-2}u = 0, \quad \prime = \frac{d}{dr}$$

is disconjugate on $[a,b]$ and it possesses solution which has no zero on $[a,b]$.

**Theorem 2.** Let $l > 1$. Let $l^* = 1$ if $\|\vec{b}\| \equiv 0$ and $l^* = \frac{l}{l-1}$ otherwise. Further, let $c_i(x) \geq 0$ for every $i$. Denote

$$\tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \Lambda^{1-p}(x) \, d\sigma$$

and

$$\tilde{C}(r) = \int_{S(r)} c(x) + C_1(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) \right\|^p \, d\sigma,$$

where $\Lambda(x)$ is defined by (11) and $C_1(x)$ is defined by (8). Suppose that the equation

$$\left( \tilde{R}(r)|u'|^{p-2}u' \right)' + \tilde{C}(r)|u|^{p-2}u = 0$$

has conjugate points on $[a,b]$. If $e(x) \leq 0$ on $\Omega(a,b)$, then equation (E) has no positive solution on $\Omega(a,b)$. If $e(x) \geq 0$ on $\Omega(a,b)$, then equation (E) has no negative solution on $\Omega(a,b)$.
**Theorem 3** (non-radial variant of Theorem 2). Let \( l > 1 \) and let \( \Omega \subset \Omega(a,b) \) be an open domain with piecewise smooth boundary such that \( \text{meas}(\Omega \cap S(r)) \neq 0 \) for every \( r \in [a,b] \). Let \( c_i(x) \geq 0 \) on \( \Omega \) for every \( i \) and let \( \alpha(x) \) be a function which is positive and continuously differentiable on \( \Omega \) and vanishes on the boundary and outside \( \Omega \). Let \( l^* = 1 \) if \( \| A^{-1}b - \frac{\nabla \alpha}{\alpha} \| \equiv 0 \) on \( \Omega \) and \( l^* = \frac{l}{l-1} \) otherwise. In the former case suppose also that the integral

\[
\int_{S(r)} \alpha(x) \left\| A^{-1}(x) \tilde{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \, d\sigma
\]

which may have singularity on \( \partial \Omega \) if \( \Omega \neq \Omega(a,b) \) is convergent for every \( r \in [a,b] \). Denote

\[
\tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \alpha(x) \Lambda^{1-p}(x) \, d\sigma
\]

and

\[
\tilde{C}(r) = \int_{S(r)} \alpha(x) \left( c(x) + C_1(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \tilde{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \right) \, d\sigma,
\]

where \( \Lambda(x) \) is defined by (11) and \( C_1(x) \) is defined by (8) and suppose that equation

\[
\left( \tilde{R}(r)|u'|^{p-2}u' \right) + \tilde{C}(r)|u|^{p-2}u = 0
\]

has conjugate points on \([a,b]\).

If \( e(x) \leq 0 \) on \( \Omega(a,b) \), then equation \((E)\) has no positive solution on \( \Omega(a,b) \).

If \( e(x) \geq 0 \) on \( \Omega(a,b) \), then equation \((E)\) has no negative solution on \( \Omega(a,b) \).
Theorem 4. Let $l$, $\Omega$, $\alpha(x)$, $\Lambda(x)$ and $\tilde{R}(r)$ be defined as in Theorem 3 and let $c_i(x) \geq 0$ and $e(x) \equiv 0$ on $\Omega(a,b)$. Denote

$$
\tilde{C}(r) = \int_{S(r)} \alpha(x) \left( c(x) + C_2(x) - \frac{l^{p-1}}{p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \tilde{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \right) d\sigma,
$$

where $C_2(x)$ is defined by (9). If the equation

$$
\left( \tilde{R}(r)|u'|^{p-2}u' \right)' + \tilde{C}(r)|u|^{p-2}u = 0
$$

has conjugate points on $[a,b]$, then every solution of equation (E) has zero on $\Omega(a,b)$.

Similar theorems can be derived also for estimates of terms with mixed powers based on different methods than AG inequality (see R. M., Non-linear Analysis TMA 73 (2010)).