Ordinary differential equations in the oscillation theory of partial half-linear differential equation

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Abstract

In the paper we study the damped half-linear partial differential equation

$$\operatorname{div}\Big(A(x)||\nabla u||^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), ||\nabla u||^{p-2}\nabla u\Big\rangle + c(x)|u|^{p-2}u = 0.$$

Using radialization method we derive general oscillation results which allow to deduce new oscillation criteria for this equation from oscillation criteria for ordinary differential equations. Using careful radialization we improve several known oscillation criteria.

Keywords: differential equation, second order, elliptic equation, oscillation, p-Laplacian, half-linear equation, damped equation

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1. Introduction

The aim of this paper is to study oscillation properties of the half–linear partial differential equation

$$\operatorname{div}\Big(A(x)||\nabla u||^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), ||\nabla u||^{p-2}\nabla u\Big\rangle + c(x)|u|^{p-2}u = 0 \qquad (1)$$

where $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, $A(x)$ is elliptic $n \times n$ matrix with differentiable components, $c(x)$ is Hölder continuous function and $\vec{b}(x) = (b_1(x), \dots, b_n(x))$ is continuous n-vector function. The operator $\nabla = (\frac{\partial}{\partial x_i})_{i=1}^n$ is the usual nabla operator, the number q is a conjugate number to the number p, i.e., $q = \frac{p}{p-1}$, $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^n$. Under a solution of (1) in $\Omega \subseteq \mathbb{R}^n$

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we understand a differentiable function $u(x)$ such that $A(x)||\nabla u(x)||^{p-2}\nabla u(x)$ is also differentiable and u satisfies (1) in Ω.

A special case of (1) is the linear partial differential equation which can be obtained from (1) for $p = 2$. Another special case of (1) is the undamped equation

$$\operatorname{div}\Big(A(x)||\nabla u||^{p-2}\nabla u\Big) + c(x)|u|^{p-2}u = 0 \tag{2}$$

which for $p = 2$ reduces to linear equation

$$\operatorname{div}\Big(A(x)\nabla u\Big) + c(x)u = 0. \tag{3}$$

If $n = 1$, then equation (2) reduces to the half–linear ordinary differential equation

$$\Big(a(r)|u'|^{p-2}u'\Big)' + b(r)|u|^{p-2}u = 0. \tag{4}$$

The following notation is used in the paper: The vector norm $||\vec{b}|| = \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$ is the usual Euclidean norm, $||A|| = \sup_{||\vec{b}||\neq 0} \frac{||A\vec{b}||}{||\vec{b}||}$ is induced matrix norm and $\lambda_{\min}(x)$, $\lambda_{\max}(x)$ are the smallest and largest eigenvalues of the matrix $A(x)$, respectively. From the fact that $A(x)$ is positive definite symmetric matrix it follows that $||A(x)|| = \lambda_{\max}(x)$. The number ω_n is the surface area of the unit sphere in $\mathbb{R}^n$ and $\Omega(a)$, $\Omega(a, b)$ and $S(a)$ are the sets in $\mathbb{R}^n$ defined as follows:

$$\begin{aligned}
\Omega(a) &= \{x \in \mathbb{R}^n : a \leq ||x||\},\\
\Omega(a, b) &= \{x \in \mathbb{R}^n : a \leq ||x|| \leq b\},\\
S(a) &= \{x \in \mathbb{R}^n : ||x|| = a\}
\end{aligned}$$

The vector $\vec{\nu}(x)$ is the normal unit vector to the sphere $S(||x||)$ oriented outwards. Integration over the domain $\Omega(a, b)$ is performed introducing hyperspherical coordinates (r, θ), i.e.

$$\int_{\Omega(a,b)} f(x)\,\mathrm{d}x = \int_a^b \int_{S(r)} f(x(r,\theta))\,\mathrm{d}\sigma\ \mathrm{d}r\,,$$

where $\mathrm{d}\sigma$ is the element of the surface of the sphere $S(r)$.

For simplicity, if M is matrix and $\vec{k}$ vector, then the product $\vec{k}M$ denotes the matrix product of $1 \times n$ row matrix $\vec{k}$ and $n \times n$ matrix M and the product $M\vec{k}$ denotes the matrix product of the $n \times n$ matrix M and $n \times 1$ column matrix $\vec{k}$.

The results of this paper are based on a suitable radialization of equation (1) and conversion of this equation into an ordinary differential equation. This argument has been used very effectively by many authors in various situations, see [3, 5, 6, 8, 13, 14, 15, 18, 21, 22, 23].

The paper is organized as follows. In the next section we recall the main ideas from the oscillation theory for half-linear equations and introduce the main problem studied in this paper. The third section contains formulation of main results, proofs, remarks concerning applications and possible generalizations and one technical lemma which extends the possibilities in applications. The last section contains some examples which show that using methods from this paper it is possible to derive easily sharper results than several recent oscillation criteria which can be found in the literature.

2. Oscillation theory and Riccati equation

According to the oscillation theory of ordinary differential equations, equation (4) is said to be oscillatory if every its solution has infinitely many zeros on the interval (r_0, ∞) and nonoscillatory if there exists $r_1 \geq r_0$ such that (4) has solution on (r_1, ∞) without zeros. If u is a solution of (4) which has no zero on (r_1, ∞), then the function $w(r) = a(r)\frac{|u'(r)|^{p-2}u'(r)}{|u(r)|^{p-2}u(r)}$ is solution of the Riccati equation

$$\mathcal{R}[w] := w' + b(r) + (p-1)a^{1-q}(r)|w|^q = 0. \tag{5}$$

This Riccati equation is frequently used to derive oscillation criteria for (4). More precisely, the following theorem holds.

Theorem A ([4, Theorem 2.2.1]). *The following statements are equivalent*

(i) *Equation* (4) *is nonoscillatory.*

(ii) *There exists* r_1 *and a (continuously differentiable) function* $w : [r_1, \infty) \to \mathbb{R}$ *such that*

$$\mathcal{R}[w](r) = 0 \quad \text{for } r \in [r_1, \infty).$$

(iii) *There exists* r_1 *and a (continuously differentiable) function* $w : [r_1, \infty) \to \mathbb{R}$ *such that*

$$\mathcal{R}[w](r) \leq 0 \quad \text{for } r \in [r_1, \infty). \tag{6}$$

Thus, if (4) is oscillatory, then the Riccati inequality (6) has no solution in any neighborhood of infinity.

For the partial differential equation we use the following concept of oscillation.

Definition 1. Let Ω be unbounded domain in $\mathbb{R}^n$. Equation (1) is said to be *oscillatory in* Ω if every its nontrivial solution defined on $\Omega \cap \Omega(t_0)$ has zero in $\Omega \cap \Omega(t)$ for every $t \geq t_0$. Equation (1) is said to be *oscillatory*, if it is oscillatory in $\mathbb{R}^n$.

Many oscillation criteria proved originally for (4) have been extended to (1). The proof of a typical oscillation criterion for (1) is usually based on the Riccati type substitution $\vec{w}(x) = A(x)\frac{||\nabla u(x)||^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}$ which converts positive or negative solutions of (1) into solution of (partial) Riccati equation. This equation is integrated over ball in n-dimensional space centered in the origin and the problem is converted into problem in one dimension. The rest of the proof usually simply repeats steps from the proof of the corresponding oscillation criterion for (4) (neglecting to some technical problems which arise for $n \geq 2$).

The disadvantage of this approach is obvious: for every new oscillation criterion derived for ordinary differential equations we have to derive a corresponding version for partial differential equations. Since many new oscillation criteria for (4) appear in the literature, it turns out to be better to find general theorem which allows to detect oscillation of partial differential equation from oscillation of some ordinary differential equation rather than readjust the proof of every oscillation criterion from (4) to (1). Some results of this type have been proved in [3, 8, 15]. Let us mention one of the typical results.

Theorem B ([3, Theorem 3.5])**.** *Equation*

$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \tag{7}$$

is oscillatory, if the ordinary differential equation

$$\left(\omega_n r^{n-1}|u'|^{p-2}u'\right)' + \left(\int_{S(r)} c(x)\,\mathrm{d}x\right)|u|^{p-2}u = 0 \tag{8}$$

is oscillatory.

The aim of this paper is to extend Theorem B to equation (1). The application of this theorem provides a tool to derive oscillation criteria for (1) easily from existing oscillation criteria for (4). Concerning oscillation criteria for (4) we refer to the papers [7, 10, 11, 12, 1, 9, 2] and the references therein.

As we will show later in this paper, this method can be used not only to give a simple proof of oscillation criteria, but it also improves some of already known results.

3. Main results

In this section we formulate our main results.

Theorem 1. *For a real number $l > 1$ define the functions*

$$\begin{aligned} a(r) &= (l^*)^{p-1}\int_{S(r)} ||A(x)||^p \lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma\,, \\ b(r) &= \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{\lambda_{\min}^{p-1}(x)} \frac{||\vec{b}(x)||^p}{p^p}\right]\mathrm{d}\sigma\,. \end{aligned} \tag{9}$$

where $l^* = \frac{l}{l-1}$ is the conjugate number to the number l if $||\vec{b}(x)|| \neq 0$ and $l^* = 1$ if $||\vec{b}(x)|| = 0$. If the equation

$$\left(a(r)|u'|^{p-2}u'\right)' + b(r)|u|^{p-2}u = 0. \tag{10}$$

is oscillatory, then (1) is also oscillatory.

Proof. Suppose, by contradiction, that (10) is oscillatory and (1) is not oscillatory. Then there exists a solution u of this equation which is positive on $\Omega(r_1)$ for r_1 sufficiently large. For $x \in \Omega(r_1)$ define n-vector function

$$\vec{w}(x) = A(x)\frac{||\nabla u(x)||^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)}.$$

The function $\vec{w}$ satisfies

$$\begin{aligned}\operatorname{div}\vec{w} &= \frac{\operatorname{div}\Big(A(x)||\nabla u||^{p-2}\nabla u\Big)}{|u|^{p-2}u} + (1-p)\left\langle A(x)||\nabla u||^{p-2}\nabla u, \nabla u\right\rangle |u|^{-p} \\ &= -c(x) - \left\langle \vec{b}(x), \frac{||\nabla u||^{p-2}\nabla u}{|u|^{p-2}u}\right\rangle - (p-1)\frac{\left\langle A(x)||\nabla u||^{p-2}\nabla u, \nabla u\right\rangle}{|u|^p} \qquad (11)\end{aligned}$$

Further, using the smallest eigenvalue of the matrix A and Young inequality

$$\frac{1}{q}||\vec{X}||^q \pm \left\langle \vec{X}, \vec{Y}\right\rangle + \frac{1}{p}||\vec{Y}||^p \geq 0$$

we have

$$\begin{aligned}&(p-1)\frac{\left\langle A(x)||\nabla u||^{p-2}\nabla u, \nabla u\right\rangle}{|u|^p} + \left\langle \vec{b}(x), \frac{||\nabla u||^{p-2}\nabla u}{|u|^{p-2}u}\right\rangle \\ &\quad\geq (p-1)\left(\frac{1}{l}+\frac{1}{l^*}\right)\lambda_{\min}\frac{||\nabla u||^p}{|u|^p} + \left\langle \vec{b}, \frac{||\nabla u||^{p-2}\nabla u}{|u|^{p-2}u}\right\rangle \\ &\quad= \frac{p\lambda_{\min}}{l}\Bigg[\left(\frac{||\nabla u||^{p-1}}{|u|^{p-1}}\right)^{\frac{p}{p-1}}\frac{p-1}{p} \\ &\qquad + \left\langle \frac{l}{p\lambda_{\min}}\vec{b}, \frac{||\nabla u||^{p-2}\nabla u}{|u|^{p-2}u}\right\rangle + \frac{1}{p}\left(\frac{l}{p\lambda_{\min}}\right)^p ||\vec{b}||^p\Bigg] \\ &\qquad - \left(\frac{l}{\lambda_{\min}}\right)^{p-1}\frac{1}{p^p}||\vec{b}||^p + (p-1)\frac{\lambda_{\min}}{l^*}\frac{||\nabla u||^p}{|u|^p} \\ &\quad\geq -\left(\frac{l}{\lambda_{\min}}\right)^{p-1}\frac{1}{p^p}||\vec{b}||^p + (p-1)\frac{\lambda_{\min}}{l^*}\frac{||\nabla u||^p}{|u|^p}\end{aligned}$$

and this inequality is trivial if $||\vec{b}(x)|| = 0$ and $l^* = 1$. Combining this computation and (11) we get

$$\operatorname{div}\vec{w} + c(x) - \left(\frac{l}{\lambda_{\min}}\right)^{p-1}\frac{1}{p^p}||\vec{b}||^p + (p-1)\lambda_{\min}\frac{1}{l^*}\frac{||\nabla u||^p}{|u|^p} \leq 0.$$

From the inequality

$$||\vec{w}|| \leq ||A||\frac{||\nabla u||^{p-1}}{|u|^{p-1}}$$

we get

$$\operatorname{div} \vec{w} + c(x) - \left(\frac{l}{\lambda_{\min}} \right)^{p-1} \frac{1}{p^p} ||\vec{b}||^p + (p-1)\lambda_{\min} \frac{1}{l^* ||A||^q} ||\vec{w}||^q \leq 0.$$

Define new function

$$W(r) = \int_{S(r)} \langle \vec{w}, \vec{\nu} \rangle \, \mathrm{d}\sigma \,. \tag{12}$$

The inequality

$$\begin{aligned} |W(r)| &= \left| \int_{S(r)} \left\langle \frac{\lambda_{\min}^{\frac{1}{q}}(x)}{||A(x)||} \vec{w}, \frac{||A(x)||}{\lambda_{\min}^{\frac{1}{q}}(x)} \vec{\nu} \right\rangle \mathrm{d}\sigma \right| \\ &\leq \left(\int_{S(r)} \frac{\lambda_{\min}(x)}{||A(x)||^q} ||\vec{w}||^q \, \mathrm{d}\sigma \right)^{\frac{1}{q}} \left(\int_{S(r)} ||A(x)||^p \lambda_{\min}^{-\frac{p}{q}}(x) \, \mathrm{d}\sigma \right)^{\frac{1}{p}} \end{aligned}$$

yields

$$\left(\int_{S(r)} ||A(x)||^p \lambda_{\min}^{-\frac{p}{q}}(x) \, \mathrm{d}\sigma \right)^{-\frac{q}{p}} |W(r)|^q \leq \int_{S(r)} \frac{\lambda_{\min}(x)}{||A(x)||^q} ||\vec{w}||^q \, \mathrm{d}\sigma \,.$$

By Gauss–Ostrogradskii divergence theorem we have

$$\begin{aligned} W'(r) = \frac{\mathrm{d}}{\mathrm{d}r} \int_{S(r)} \langle \vec{w}, \vec{\nu} \rangle \, \mathrm{d}\sigma \, &= \frac{\mathrm{d}}{\mathrm{d}r} \left[\int_{S(r)} \langle \vec{w}, \vec{\nu} \rangle \, \mathrm{d}\sigma \, - \int_{S(a)} \langle \vec{w}, \vec{\nu} \rangle \, \mathrm{d}\sigma \right] \\ &= \frac{\mathrm{d}}{\mathrm{d}r} \int_{\Omega(a,r)} \operatorname{div} \vec{w} \, \mathrm{d}x \\ &= \int_{S(r)} \operatorname{div} \vec{w} \, \mathrm{d}\sigma \end{aligned} \tag{13}$$

and the function W satisfies

$$\begin{aligned} W'(r) + \int_{S(r)} &\left[c(x) - \left(\frac{l}{\lambda_{\min}(x)} \right)^{p-1} \frac{||\vec{b}(x)||^p}{p^p} \right] \mathrm{d}\sigma \\ &+ (p-1) \frac{1}{l^*} \left(\int_{S(r)} ||A(x)||^p \lambda_{\min}^{-\frac{p}{q}}(x) \, \mathrm{d}\sigma \right)^{1-q} |W(r)|^q \leq 0 \end{aligned} \tag{14}$$

on $[r_1, \infty)$ and hence the inequality

$$W' + b(r) + (p-1) a^{1-q}(r) |W|^q \leq 0 \tag{15}$$

has solution on $\Omega(a)$. By Theorem A, equation (10) is nonoscillatory, a contradiction. Theorem is proved. □

Remark 1. If $||\vec{b}(x)|| \equiv 0$ and $A(x) = a(||x||)I_n$ where $a(r)$ is smooth function and I_n is $n \times n$ identity matrix, then Theorem 1 reduces to [8, Theorem 3.4].

Remark 2. An important step in the proof of Theorem 1 is to derive equation (11). A closer look at the proof shows that it is sufficient to derive (11) with equality sign replaced by inequality sign $\leq$. Hence it is possible to use this method to study equations which are in certain sense majorants to (1). These equations cover for example

$$\operatorname{div}\Big(A(x)||\nabla u||^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), ||\nabla u||^{p-2}\nabla u \Big\rangle + c(x)f(u) = 0 \tag{16}$$

where $f(u)$ is a differentiable function which satisfies $f(0) = 0$, $uf(u) > 0$ for $u \neq 0$ and

$$\frac{f'(u)}{f^{2-q}(u)} \geq p-1. \tag{17}$$

Equation (16) is sometimes called super-half-linear equation.

If the function $f(u)$ satisfies (17) with $p-1$ replaced by $\varepsilon > 0$, it is sufficient to consider functions $f^*(u) = \varepsilon^* f(u)$ and $c^*(u) = \frac{1}{\varepsilon^*}c(u)$ where $\varepsilon^* = \left(\frac{p-1}{\varepsilon}\right)^{p-1}$. The function $f^*(u)$ satisfies (17) and $f(u)c(x) = f^*(u)c^*(x)$ holds.

Finally, it is possible to use this method also to prove nonexistence of positive solution of the equation

$$\operatorname{div}\Big(A(x)||\nabla u||^{p-2}\nabla u\Big) + \Big\langle \vec{b}(x), ||\nabla u||^{p-2}\nabla u \Big\rangle + B(x,u) = 0, \tag{18}$$

where

$$B(x,u) \geq c(x)f(u) \quad \text{ for } u \geq 0$$

and the function $f(u)$ satisfies hypotheses stated above.

Remark 3. Many oscillation criteria for the ordinary half-linear differential equation are derived for the equation (10) with $a(r) \equiv 1$. However, if the integral of $a^{1-q}(r)$ is divergent, i.e. if $\int^\infty a^{1-q}(r)\,\mathrm{d}r = \infty$, then the transformation of independent variable $s = \phi(r) := \int_{r_0}^r a^{1-q}(t)\,\mathrm{d}t$, $y(s) = u(r)$ transforms (10) into

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\left|\frac{\mathrm{d}y}{\mathrm{d}s}\right|^{p-2}\frac{\mathrm{d}y}{\mathrm{d}s}\right) + b(r)a^{1-q}(r)|y|^{p-2}y = 0, \quad r = \phi^{-1}(s)$$

and interval $[r_0, \infty)$ is transformed into $[0, \infty)$. Using this transformation, an extension of the oscillation criteria derived for $a(r) \equiv 1$ to general case (10) used in Theorem 1 is straightforward.

Remark 4. Several oscillation criteria for (10) require $\int^\infty a^{1-q}(r)\,\mathrm{d}r = \infty$. If the matrix $A(x)$ is a constant matrix, then the divergence of this integral is equivalent to the condition $p \geq n$. This is a natural phenomenon. The fact that the oscillation properties of (1) are different for $p < n$ and $p \geq n$ has been discussed in details in [3].

Several oscillation criteria in the literature contain an additional (and in some sense arbitrary) function (say $\theta(r)$) and thus are more general. A convenient choice of the function θ allows to ensure that the condition from some oscillation criterion (usually divergence or positivity of some integral) holds. A common way to find criteria of this type is to include the function θ into definition of the function $W(r)$. The following Lemma is an application of this idea to (10). Note that to apply this idea it is sufficient to consider ordinary differential equation only.

Lemma 1. *Let $m > 1$ be positive number, $m^* = \frac{m}{m-1}$ be its conjugate number and $\theta(r)$ be smooth positive function. If the equation*

$$\Big((m^*)^{p-1}\theta(r)a(r)|u'|^{p-2}u'\Big)' + \Big(\theta(r)b(r) - a(r)\frac{m^{p-1}}{p^p}\frac{|\theta'(r)|^p}{\theta^{p-1}(r)}\Big)|u|^{p-2}u = 0 \tag{19}$$

is oscillatory, then (10) *is also oscillatory.*

Proof. Suppose that (10) is not oscillatory. We prove that (19) is also nonoscillatory. If (10) is nonoscillatory, then there is a function $w(r)$ which satisfies

$$w'(r) + b(r) + (p-1)a^{1-q}(r)|w(r)|^q = 0$$

on (r_1, ∞) for r_1 sufficiently large. Define the function

$$Z(r) = \theta(r)w(r). \tag{20}$$

The function Z satisfies equation

$$Z'(r) + \theta(r)b(r) + (p-1)\Big(\theta(r)a(r)\Big)^{1-q}|Z(r)|^q - \frac{\theta'(r)}{\theta(r)}Z(r) = 0. \tag{21}$$

Using mutually conjugate numbers m, m^* and Young inequality we get

$$\begin{aligned}
(p-1)&(\theta a)^{1-q}|Z|^q - \frac{\theta'}{\theta}Z \\
&= p(\theta a)^{1-q}\frac{1}{m}\Bigg[\frac{p-1}{p}|Z|^q - \frac{m\theta'}{p\theta}(\theta a)^{q-1}Z \\
&\quad + \frac{1}{p}\left|\frac{m\theta'}{p\theta}\right|^p(\theta a)^{(q-1)p}\Bigg] \\
&\quad + \frac{1}{m^*}(p-1)\Big(\theta a\Big)^{1-q}|Z|^q - \left|\frac{\theta'}{p\theta}\right|^p\theta a m^{p-1} \\
&\geq \frac{1}{m^*}(p-1)\Big(\theta a\Big)^{1-q}|Z|^q - \frac{1}{p^p}\frac{|\theta'|^p}{\theta^{p-1}}am^{p-1}
\end{aligned}$$

This inequality combined with (21) shows that the inequality

$$Z' + \theta(r)b(r) - a(r)\frac{m^{p-1}}{p^p}\frac{|\theta'(r)|^p}{\theta^{p-1}(r)} + \frac{p-1}{m^*}\Big(\theta(r)a(r)\Big)^{1-q}|Z|^q \leq 0$$

has solution on (r_1, ∞) and (19) is nonoscillatory by Theorem A. The proof of the Lemma is complete. □

The following corollary is based on a similar idea as Lemma 1. The difference is that it makes use of a function $\rho(x)$ of n variables rather than the function $\theta(r)$ of one variable and the proof is more complicated since it is not sufficient to work with ordinary differential equations but we have to return in the proof to partial Riccati equation. However, it is sufficient to simply repeat the steps from the proof of Theorem 1 with another functions. From this reason we proved the simpler version of this Theorem first and now we sketch the extension into more general case.

Corollary 1. *Let $\rho \in C^1(\Omega(1), \mathbb{R}^+)$. Theorem 1 remains valid, if equations* (9) *are replaced by*

$$
\begin{aligned}
a(r) &= (l^*)^{p-1} \int_{S(r)} \rho(x) ||A(x)||^p \lambda_{\min}^{1-p}(x)\, \mathrm{d}\sigma\,, \\
b(r) &= \int_{S(r)} \rho(x) \left[c(x) - \frac{l^{p-1}}{p^p \lambda_{\min}^{p-1}(x)} \left|\left| \vec{b}(x) - \frac{\nabla \rho(x)}{\rho(x)} A(x) \right|\right|^p \right] \mathrm{d}\sigma\,,
\end{aligned} \tag{22}
$$

and $l^* = 1$ if $||\rho(x)\vec{b}(x) - \nabla\rho(x)A(x)|| = 0$ and $l^* = \frac{l}{l-1}$ otherwise.

Proof. Suppose by contradiction that (10) with $a(r)$ and $b(r)$ defined by (22) is oscillatory and (1) is nonoscillatory. Define vector, matrix and scalar functions $\vec{b}_\rho(x) = \rho(x)\vec{b}(x) - \nabla\rho(x)A(x)$, $\vec{w}_\rho(x) = \rho(x)\vec{w}(x)$, $A_\rho(x) = \rho(x)A(x)$ and $c_\rho(x) = \rho(x)c(x)$. Further, let $\lambda_{\min,\rho}(x) = \rho(x)\lambda_{\min}(x)$ and $||A_\rho(x)|| = \rho(x)||A(x)||$ be minimal eigenvalue and norm of the matrix $A_\rho(x)$ respectively. It is sufficient to prove that the conclusion of Theorem 1 remains valid if the functions $\vec{b}(x)$, $A(x)$, $\vec{w}(x)$, $c(x)$, $\lambda_{\min}(x)$ and $||A(x)||$ are replaced by $\vec{b}_\rho(x)$, $A_\rho(x)$, $\vec{w}_\rho(x)$, $c_\rho(x)$, $\lambda_{\min,\rho}(x)$ and $||A_\rho(x)||$ respectively, since these replacements convert (9) into (22).

We start as in the proof of Theorem 1 and derive (11). Multiplying (11) by the function $\rho(x)$ we find that (11) is equivalent to the equation

$$
\begin{aligned}
\operatorname{div}\Big(\rho(x)\vec{w}(x)\Big) + \rho(x)c(x) + \left\langle \rho(x)\vec{b}(x) - \nabla\rho(x)A(x), \frac{||\nabla u||^{p-2}\nabla u}{|u|^{p-2}u} \right\rangle \\
+ (p-1)\frac{\big\langle \rho(x)A(x)||\nabla u||^{p-2}\nabla u, \nabla u\big\rangle}{|u|^p} = 0.
\end{aligned}
$$

Note that this equation also arises from (11) by using the above mentioned replacements. Naturally, using the steps from Theorem 1 we conclude inequality which arises from (14) by using the same replacements. Hence inequality (15) with $a(r)$, $b(r)$ defined by (22) has a solution on $[r_1, \infty)$. By Theorem A, equation (10) with $a(r)$, $b(r)$ defined by (22) is nonoscillatory, a contradiction. □

Remark 5. In general, it is not easy to find the norm $||A(x)||$. From this reason we provide some upper estimates for this norm [19, Proposition 2.7.4]:

$$
\begin{aligned}
||A|| \le ||A||_F &:= \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} \\
||A|| &\le n \max_{1\le i,j\le n} |a_{ij}| \\
\frac{1}{\sqrt{n}}||A|| \le ||A||_\infty &:= \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| \\
\frac{1}{\sqrt{n}}||A|| \le ||A||_1 &:= \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|
\end{aligned}
\tag{23}
$$

These estimates can be used together with the following simple corollary.

Corollary 2. *Let l be a real number, $l > 1$, $\widetilde{b}(r)$ be continuous function and $\widetilde{a}(r)$ be smooth function such that*

$$
\begin{aligned}
\widetilde{a}(r) &\ge (l^*)^{p-1} \int_{S(r)} ||A(x)||^p \lambda_{\min}^{1-p}(x) \,\mathrm{d}\sigma\,, \\
\widetilde{b}(r) &\le \int_{S(r)} \left[c(x) - \left(\frac{l}{\lambda_{\min}(x)} \right)^{p-1} \frac{||\vec{b}(x)||^p}{p^p} \right] \mathrm{d}x\,,
\end{aligned}
$$

where $l^* = \frac{l}{l-1}$ is the conjugate number to the number l if $||\vec{b}(x)|| \not\equiv 0$ and $l^* = 1$ if $||\vec{b}(x)|| \equiv 0$. If the ordinary differential equation

$$
\left(\widetilde{a}(r)|u'|^{p-2}u'\right)' + \widetilde{b}(r)|u|^{p-2}u = 0 \tag{24}
$$

is oscillatory, then (1) *is also oscillatory.*

Proof. Suppose that (24) is oscillatory. From the assumptions it follows that (10) is a Sturmian majorant to (24) and hence (10) is also oscillatory. Now the statement follows from Theorem 1. □

Obviously, the equations from (22) can be replaced by inequalities in the same way as in Corollary 2. The following Theorem 2 is variant of Theorem 1 and presents sharper result, but covers the case $1 \le p \le 2$ only.

Theorem 2. *Let $1 < p \le 2$. For a real number $l > 1$ define the functions*

$$
\begin{aligned}
\widehat{a}(r) &= (l^*)^{p-1} \int_{S(r)} \lambda_{\max}(x) \,\mathrm{d}\sigma\,, \\
\widehat{b}(r) &= \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{p^p} \lambda_{\max}(x) ||\vec{b}(x)A^{-1}(x)||^p \right] \mathrm{d}\sigma\,,
\end{aligned}
\tag{25}
$$

where $l^* = \frac{l}{l-1}$ *is the conjugate number to the number* l *if* $||\vec{b}(x)|| \neq 0$ *and* $l^* = 1$ *if* $||\vec{b}(x)|| = 0$. *Here* $\vec{b}(x)A^{-1}(x)$ *denotes the matrix product of row matrix* $(b_1(x), \dots)$ *and the inverse* $A^{-1}(x)$. *If the equation*

$$\left(\widehat{a}(r)|u'|^{p-2}u'\right)' + \widehat{b}(r)|u|^{p-2}u = 0 \tag{26}$$

is oscillatory, then (1) *is also oscillatory.*

Proof. Suppose, by contradiction, that (26) is oscillatory and (1) is nonoscillatory. We start as in the proof of Theorem 1 and derive (11) which can be written in the form

$$\operatorname{div} \vec{w} + c + \left\langle \vec{b}, A^{-1}\vec{w} \right\rangle + (p-1)\left\langle \vec{w}, A^{-1}\vec{w} \right\rangle \frac{||\nabla u||^{2-p}}{|u|^{2-p}} = 0. \tag{27}$$

If $\lambda_{\max}$ is the maximal eigenvalue of the matrix A, then the number $\frac{1}{\lambda_{\max}}$ is the minimal eigenvalue of its inverse A^{-1} and hence

$$\left\langle \vec{w}, A^{-1}\vec{w} \right\rangle \geq ||\vec{w}||^2 \frac{1}{\lambda_{\max}}.$$

From the property of matrix norm we have

$$||\vec{w}|| \leq ||A|| \frac{||\nabla u||^{p-1}}{|u|^{p-1}}$$

which is for $p \leq 2$ equivalent to the inequality

$$\frac{||\nabla u||^{2-p}}{|u|^{2-p}} \geq \frac{||\vec{w}||^{(2-p)/(p-1)}}{||A||^{(2-p)/(p-1)}} = \frac{||\vec{w}||^{(2-p)/(p-1)}}{\lambda_{\max}^{(2-p)/(p-1)}}.$$

Combining these computation we have the following estimate for the last term on the left hand side of (27)

$$\left\langle \vec{w}, A^{-1}\vec{w} \right\rangle \frac{||\nabla u||^{2-p}}{|u|^{2-p}} \geq ||\vec{w}||^{2+(2-p)/(p-1)} \lambda_{\max}^{-1+(p-2)/(p-1)} = ||w||^q \lambda_{\max}^{1-q}.$$

From these estimates and from equation (27) we get inequality

$$\operatorname{div} \vec{w} + c + \left\langle \vec{b}A^{-1}, \vec{w} \right\rangle + (p-1)\lambda_{\max}^{1-q} ||\vec{w}||^q \leq 0. \tag{28}$$

Using essentially the same method as in the proof of Theorem 1 we use mutually conjugate numbers l and l^* to split the last term into two terms and use the

Young inequality to remove the term $\left\langle \vec{b}A^{-1}, \vec{w} \right\rangle$:

$$\begin{aligned}
\left\langle \vec{b}A^{-1}, \vec{w} \right\rangle + (p-1)\lambda_{\max}^{1-q}||\vec{w}||^q &= \left\langle \vec{b}A^{-1}, \vec{w} \right\rangle + (p-1)\left(\frac{1}{l} + \frac{1}{l^*}\right)\lambda_{\max}^{1-q}||\vec{w}||^q \\
&= \frac{p}{l}\lambda_{\max}^{1-q}\left[\frac{p-1}{p}||\vec{w}||^q + \left\langle \frac{\lambda_{\max}^{q-1} l}{p}\,\vec{b}A^{-1}, \vec{w} \right\rangle + \frac{1}{p}\lambda_{\max}^{p(q-1)}\,\frac{l^p}{p^p}||\vec{b}A^{-1}||^p\right] \\
&\quad + (p-1)\frac{1}{l^*}\lambda_{\max}^{1-q}||\vec{w}||^q - \frac{l^{p-1}\lambda_{\max}}{p^p}||\vec{b}A^{-1}||^p \\
&\geq (p-1)\frac{1}{l^*}\lambda_{\max}^{1-q}||\vec{w}||^q - \frac{l^{p-1}\lambda_{\max}}{p^p}||\vec{b}A^{-1}||^p.
\end{aligned}$$

This computation remains valid if $||\vec{b}|| = 0$ and $l^* = 1$. In this case l disappears. Inequality (28) now yields

$$\operatorname{div} w + c(x) - \frac{l^{p-1}}{p^p}\lambda_{\max}||\vec{b}A^{-1}||^p + (p-1)\frac{1}{l^*}\lambda_{\max}^{1-q}||\vec{w}||^q \leq 0. \tag{29}$$

Define the function $W(r)$ by (12). Hölder inequality yields

$$\begin{aligned}
|W(r)| &= \left| \int_{S(r)} \left\langle \lambda_{\max}^{(1-q)/q}\vec{w}, \lambda_{\max}^{(q-1)/q}\vec{\nu} \right\rangle \mathrm{d}\sigma \right| \\
&\leq \left(\int_{S(r)} \lambda_{\max}^{1-q}||\vec{w}||^q \,\mathrm{d}\sigma \right)^{\frac{1}{q}} \left(\int_{S(r)} \lambda_{\max}\,\mathrm{d}\sigma \right)^{\frac{1}{p}}
\end{aligned}$$

and

$$\left(\int_{S(r)} \lambda_{\max}\,\mathrm{d}\sigma \right)^{1-q} |W(r)|^q \leq \int_{S(r)} \lambda_{\max}^{1-q}||\vec{w}||^q \,\mathrm{d}\sigma\,.$$

This inequality, inequality (29) and equality (13) show that the function $W(r)$ satisfies

$$\begin{aligned}
W' + \int_{S(r)} \left[c(x) - \frac{l^{p-1}}{p^p}\lambda_{\max}||\vec{b}A^{-1}||^p \right] \mathrm{d}\sigma& \\
+ (p-1)\left(l^{*\,p-1} \int_{S(r)} \lambda_{\max}\,\mathrm{d}\sigma \right)^{1-q} ||W||^q \leq 0.&
\end{aligned}$$

Thus, the inequality

$$W' + \widehat{b}(r) + (p-1)\widehat{a}^{1-q}(r)|W|^q \leq 0 \tag{30}$$

has solution on (r_1, ∞) and (26) is not oscillatory by Theorem A. This contradiction proves the Theorem. $\square$

Remark 6. Similarly to Theorem 1 and Corollary 2, the functions $\widehat{a}(r)$ and $\widehat{b}(r)$ can be replaced by any smooth bigger and continuous smaller functions, respectively.

The following corollary is a version of Corollary 1.

Corollary 3. *Let $\rho \in C^1(\Omega(1), \mathbb{R}^+)$. Theorem 2 remains valid, if equations* (25) *are replaced by*

$$\begin{aligned} \widehat{a}(r) &= (l^*)^{p-1} \int_{S(r)} \rho(x) \lambda_{\max}(x) \, \mathrm{d}\sigma \,, \\ \widehat{b}(r) &= \int_{S(r)} \rho(x) \left[c(x) - \frac{l^{p-1}}{p^p} \lambda_{\max}(x) \left|\left| \vec{b}(x) A^{-1}(x) - \frac{\nabla \rho(x)}{\rho(x)} \right|\right|^p \right] \mathrm{d}\sigma \,, \end{aligned} \tag{31}$$

and $l^* = 1$ if $||\rho(x)\vec{b}(x)A^{-1}(x) - \nabla\rho(x)|| = 0$ and $l^* = \frac{l}{l-1}$ otherwise.

Proof. The proof is analogical to the proof of Corollary 1. We suppose that (1) is not oscillatory and prove that (26) is also nonoscillatory. Using the same method as in the proof of Theorem 2 we derive inequality (27) which can be written in the form

$$\operatorname{div}\Big(\rho \vec{w}\Big) + \rho c + \Big\langle \rho \vec{b} - \nabla \rho A, A^{-1} \vec{w} \Big\rangle + (p-1)\rho \left\langle \vec{w}, A^{-1}\vec{w} \right\rangle \frac{||\nabla u||^{2-p}}{|u|^{1-p}} = 0 \tag{32}$$

With the notation $A_\rho(x) = \rho(x)A(x)$, $\vec{b}_\rho(x) = \rho(x)\vec{b}(x) - \nabla\rho(x)A(x)$, $\vec{w}_\rho(x) = \rho(x)\vec{w}(x)$ equation (32) can be written in the form

$$\operatorname{div}\Big(\vec{w}_\rho\Big) + c_\rho + \Big\langle \vec{b}_\rho, A_\rho^{-1} \vec{w}_\rho \Big\rangle + (p-1) \left\langle \vec{w}_\rho, A_\rho^{-1}\vec{w}_\rho \right\rangle \frac{||\nabla u||^{2-p}}{|u|^{1-p}} = 0 \tag{33}$$

where $A_\rho^{-1}(x) = \rho^{-1}(x)A^{-1}(x)$ is the inverse matrix to $A_\rho(x)$. This equation has the same form as (32). Thus, using the same steps as in the proof of Theorem 2 we prove that there exists a function $W(r)$ which satisfies

$$\begin{aligned} W' + \int_{S(r)} \left[c_\rho - \frac{l^{p-1}}{p^p} \lambda_{\max,\rho} ||\vec{b}_\rho A_\rho^{-1}||^p \right] \mathrm{d}\sigma & \\ + (p-1) \left((l^*)^{p-1} \int_{S(r)} \lambda_{\max,\rho} \, \mathrm{d}\sigma \right)^{1-q} & ||W||^q \le 0, \end{aligned}$$

where $\lambda_{\max,\rho}(x) = \rho(x)\lambda_{\max}(x)$ is the largest eigenvalue of the matrix $A_\rho(x)$. This shows that the Riccati inequality (30) with $\widehat{a}(r)$ and $\widehat{b}(r)$ defined by (31) has a solution. Thus, (26) is nonoscillatory by Theorem A and Corollary is proved. ∎

4. Examples

This section contains some examples. These examples are of a different kind than examples accompanying usual oscillation criteria in other papers. We don't prove oscillation of equations for which other oscillation criteria fail, but we show that several recent oscillation criteria can be improved and derived in a very easy way using results from the preceding section.

The following theorem has been proved originally for damped linear equation. However, we reformulate this theorem for undamped equation only in order to obtain results which can be compared to the results from the preceding section and which are extensible to half-linear case.

Theorem C ([21, Theorem 3.1])**.** *Let* $\theta \in C([r_0, \infty], \mathbb{R}^+)$ *and* $m > 1$*. Further, let* $\lambda \in C([r_0, \infty), \mathbb{R}^+)$*,* $\lambda(r) \geq \max_{||x||=r} \lambda_{\max}(x)$ *for* $r \geq r_0$*. If*

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} \left[\theta(||x||)c(x) - \lambda(||x||)\frac{m}{4}\frac{\theta'^2(||x||)}{\theta(||x||)}\right] \mathrm{d}x = \infty$$

and

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} \frac{1}{\theta(||x||)\lambda(||x||)} \mathrm{d}x = \infty,$$

then equation (3) *is oscillatory.*

The classical Leighton–Wintner criterion states that the equation

$$\Big(\alpha(r)u'\Big)' + \beta(r)u = 0 \tag{34}$$

is oscillatory if

$$\int^\infty \alpha^{-1}(s)\,\mathrm{d}s = \infty = \int^\infty \beta(s)\,\mathrm{d}s\,.$$

For equation (3) the functions $\widehat{a}(r)$, $\widehat{b}(r)$ from Theorem 2 become

$$\widehat{a}(r) = \int_{S(r)} \lambda_{\max}(x)\,\mathrm{d}\sigma\,,$$
$$\widehat{b}(r) = \int_{S(r)} c(x)\,\mathrm{d}\sigma\,.$$

Using Theorem 2, Lemma 1 and the Leighton–Wintner oscillation criterion to equation (19) we conclude that the maximum from the definition of the function $\lambda(r)$ can be removed and the function $\lambda(||x||)$ can be replaced by (smaller) function $\lambda_{\max}(x)$.

Corollary 4. *The statement of Theorem C remains valid if the function* $\lambda(||x||)$ *is replaced by* $\lambda_{\max}(x)$*.*

Since Leighton–Wintner criterion extends to half-linear equations, a half-linear extension of Theorem C is straightforward. (For another half-linear extension of the Leighton–Wintner criterion see Corollary 6 below and the paper [13] which deals with $A(x) = I$.)

Corollary 5. *Let $\theta \in C([r_0,\infty],\mathbb{R}^+)$, $m>1$ and $q=\frac{p}{p-1}$ be conjugate number to the number p. If*

$$\lim_{r\to\infty}\int_{\Omega(r_0,r)}\left[\theta(||x||)c(x)-\lambda_{\max}(x)\frac{m^{p-1}}{p^p}\frac{\theta'^p(||x||)}{\theta^{p-1}(||x||)}\right]\mathrm{d}x=\infty$$

and

$$\lim_{r\to\infty}\int^{r}\theta^{1-q}(s)\left(\int_{S(s)}\lambda_{\max}(x)\,\mathrm{d}\sigma\right)^{1-q}\mathrm{d}s=\infty,$$

then equation (2) *is oscillatory.*

Proof. The equation (4) is oscillatory if

$$\int^{\infty}b(r)\,\mathrm{d}r=\infty=\int^{\infty}a^{1-q}(r)\,\mathrm{d}r\,.$$

Thus the statement is an immediate consequence of Theorem 2 and Lemma 1. □

An application of the half-linear Leighton–Wintner criterion to Corollary 1 gives the following oscillation criterion.

Corollary 6. *Let $\rho\in C^1(\Omega(r_0),\mathbb{R}^+)$ and $k>1$. If*

$$\lim_{r\to\infty}\int_{r_0}^{r}\left(\int_{S(t)}\rho(x)||A(x)||^p\lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma\right)^{1-q}\mathrm{d}t=\infty$$

and

$$\lim_{r\to\infty}\int_{\Omega(r_0,r)}\rho(x)\left[c(x)-\frac{k}{p^p\lambda_{\min}^{p-1}(x)}\left|\left|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}A(x)\right|\right|^p\right]\mathrm{d}x=\infty,$$

then equation (1) *is oscillatory.*

Proof. The proof is similar to the proof of Corollary 5 and thus omitted. □

Corollary 6 is closely related to the results from [14]. The paper [14] considers $A(x)=I_n$ and deals with oscillation in more general domains than exterior of a ball. However, in the case which is covered by both Corollary 6 and paper [14] the conclusion of Corollary 6 is identical to [14, Theorem 3.3 and Theorem 3.5].

The method of weighted integral averages is frequently used to obtain various extensions of Kamenev type oscillation criteria and also interval oscillation criteria. In the sequel we introduce two results based on this method.

Theorem D ([20, Theorem 1])**.** *Let $D_0=\{(t,s):t>s\geq t_0\}$ and $D=\{(t,s):t\geq s\geq t_0\}$. Let functions $H\in C(D;\mathbb{R})$, $h\in C(D_0;\mathbb{R})$, k, $\rho\in C^1([t_0,\infty);(0,\infty))$ satisfy the following three conditions:*

(i) $H(t,t)=0$ *for* $t\ge t_0$, $H(t,s)>0$ *on* D_0

(ii) H *has a continuous and nonpositive partial derivative on* D_0 *with respect to the second variable*

(iii)
$$-\frac{\partial}{\partial s}(H(t,s)k(s)) - H(t,s)k(s)\frac{\rho'(s)}{\rho(s)} = h(t,s) \quad \forall (t,s)\in D_0$$

and
$$\int_{t_0}^{t} H^{1-p}(t,s)|h(t,s)|^p\,\mathrm{d}s < \infty$$

for every t.

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)k(s)\rho(s)b(s) - \frac{\rho(s)a(s)|h(t,s)|^p}{p^p[H(t,s)k(s)]^{p-1}}\right] = \infty,$$

then equation (4) *is oscillatory.*

An application of Theorems 1 and 2 to this result gives the following corollary.

Corollary 7. *Let* ϕ, $k\in C^1([r_0,\infty),\mathbb{R}^+)$ *be real functions. Suppose that there exists continuous function* $H(r,s)$ *defined for* $r\ge s\ge r_0$ *such that*

(i) $H(r,r)=0$ *and* $H(r,s)>0$ *for* $r>s\ge r_0$,

(ii) *the function* H *has continuous nonpositive partial derivative with respect to the second variable,*

(iii) *the function* $h(r,s)$ *defined by the relation*
$$-\frac{\partial}{\partial s}\Big[H(r,s)k(s)\Big] - H(r,s)k(s)\frac{\phi'(s)}{\phi(s)} = h(r,s)$$

satisfies
$$\int_{r_0}^{r} H^{1-p}(r,s)|h(r,s)|^p\,\mathrm{d}s < \infty$$

for every r

(iv)
$$\limsup_{r\to\infty}\frac{1}{H(r,r_0)}\int_{r_0}^{r}\Bigg\{H(r,s)k(s)\phi(s)\int_{S(s)}c(x)\,\mathrm{d}\sigma \\ - \frac{1}{p^p}[H(r,s)k(s)]^{1-p}\Theta(s)\phi(s)|h(r,s)|^p\Bigg\}\,\mathrm{d}s = \infty, \quad (35)$$

where
$$\Theta(s) = \begin{cases} \displaystyle\int_{S(s)} ||A(x)||^p\lambda_{\min}^{1-p}(x)\,\mathrm{d}\sigma & \text{if } p>2 \\ \displaystyle\int_{S(s)} \lambda_{\max}(x)\,\mathrm{d}\sigma & \text{if } 1<p\le 2 \end{cases} \quad (36)$$

Then equation (2) *is oscillatory.*

Corollary 7 improves [23, Theorem 2.1] in several points. First, we use the norm consistent with Euclidean vector norm rather than the Frobenius norm used in [23] and thus obtain sharper result (see inequality (23)).

Second, the term

$$\Theta_{\mathrm{Xu}}(s) := \rho^{1-p}(s)\omega_n s^{n-1} \quad \text{where} \quad \rho(s) \le \min_{x\in S(s)} \frac{\lambda_{\min}(x)}{||A(x)||_F^q}. \tag{37}$$

appears in [23, Theorem 2.1] in condition (35) instead of $\Theta(s)$. In Corollary 7 we have shown that this term $\Theta_{\mathrm{Xu}}(s)$ can be replaced by smaller term $\Theta(s)$. In other words, the maximum of the function $||A(x)||^p\lambda_{\min}^{1-p}(x)$ over the sphere $S(s)$ (which corresponds to the minimum of the function $\frac{\lambda_{\min}(x)}{||A(x)||^q}$ from (37)) can be replaced by its integral mean value and if $p \le 2$ we can further decrease this term as (36) shows. In this sense, the Corollary 7 not only provides a simple alternative proof of [23, Theorem 2.1], but yields sharper result.

The following Theorem E is an example of interval type oscillation criterion for damped linear differential equation.

Theorem E ([17, Theorem 2.1])**.** *Consider equation*

$$\Big(r(t)y'\Big)' + p(t)y' + q(t)f(y) = 0,$$

where $r(t) \in C([a,\infty),(0,\infty)$, $p(t)$, $q(t) \in C([a,\infty),\mathbb{R})$, $f(u) \in C(\mathbb{R},\mathbb{R})$, $uf(u) > 0$ *and* $f'(u) \ge \mu > 0$ *for* $u \ne 0$. *This equation is oscillatory provided that for each* $l \ge a$ *there exists a function* H *with properties*

(i) $H \in C(E,\mathbb{R})$, *where* $E = \{(t,s,l); a \le l \le s \le t < \infty\}$

(ii) $H(t,t,l) = 0 = H(t,l,l)$, $H(t,s,l) \ne 0$ *for* $l < s < t$ *and*

(iii) *the function* $h(t,s,l)$ *defined by relation*

$$\frac{\partial H}{\partial s}(t,s,l) = h(t,s,l)H(t,s,l)$$

is such that $h^2(t,s,l)H(t,s,l)$ *is locally integrable with respect to* s *on the set* $t \ge s \ge l \ge a$,

(iv)
$$\limsup_{t\to\infty} \int_l^t H(t,s,l)\left[q(s) - \frac{r(s)}{4\mu}\left(\frac{p(s)}{r(s)} - h(t,s,l)\right)^2\right] \mathrm{d}s > 0.$$

As an application of Theorem 2 to this result we get the following oscillation criterion.

Corollary 8. *Suppose that for each* $l \ge a$ *there exist a function* $H(r,s,l)$ *defined for* $r \ge s \ge l \ge a$ *such that*

(i) $H(r,r,l) = 0 = H(r,l,l)$ *for* $r > l \ge a$ *and* $H(r,s,l) > 0$ *for* $r > s > l$,

(ii) *$H(r,s,l)$ has continuous partial derivative with respect to s for $r > s > l$,*

(iii) *the function $h(r,s,l)$ defined by relation*

$$\frac{\partial H}{\partial s}(r,s,l) = h(r,s,l)H(r,s,l)$$

is such that $h^2(r,s,l)H(r,s,l)$ is locally integrable with respect to s on the set $r \geq s \geq l \geq a$,

(iv)

$$\limsup_{t\to\infty} \int_l^t H(r,s,l)\biggl\{\int_{S(s)} \Bigl[c(x) - \frac{l}{4}\lambda_{\max}(x)||\vec{b}(x)A^{-1}(x)||^2\Bigr]\,\mathrm{d}\sigma \\ - \frac{l^*}{4}\Psi_M(s)h^2(r,s,l)\biggr\}\,\mathrm{d}s > 0 \tag{38}$$

where $\Psi_M(r) = \int_{S(r)} \lambda_{\max}(x)\,\mathrm{d}\sigma$ and $l > 1$, $l^* = \frac{p}{p-1}$ are mutually conjugate numbers. If $||\vec{b}(x)|| = 0$ we can put $l^* = 1$.

Then equation

$$\operatorname{div}\Bigl(A(x)\nabla u\Bigr) + \Bigl\langle \vec{b}(x), \nabla u\Bigr\rangle + c(x)u = 0 \tag{39}$$

is oscillatory.

Corollary 8 improves [22, Theorem 3.1] which has been proved for slightly more general equation (covered by Remark 2, nevertheless). The condition (38) is in [22] replaced by

$$\limsup_{t\to\infty} \int_l^t H(r,s,l)\biggl\{\int_{S(s)} \Bigl[c(x) - \frac{1}{2}\lambda_{\max}(x)||\vec{b}(x)A^{-1}(x)||^2\Bigr]\,\mathrm{d}\sigma \\ - \frac{1}{2}\Psi_{\mathrm{Xu}}(s)h^2(r,s,l)\biggr\}\,\mathrm{d}s > 0 \tag{40}$$

where $\Psi_{\mathrm{Xu}}(r) = \lambda(r)\omega_n r^{n-1}$ and $\lambda(r) \geq \max_{x\in S(r)} \lambda_{\max}(x)$. It is easy to see that oscillation criterion involving condition (38) is sharper than the criterion involving (40). Really, the maximum of the eigenvalue $\lambda_{\max}(x)$ over the sphere of diameter r which appears in the function $\lambda(r)$ in (40) is replaced by the integral mean value of this eigenvalue in $\Psi_M(r)$ and thus $\Psi_M(r)$ is smaller than $\Psi_{\mathrm{Xu}}(r)$. Another difference between (38) and (40) is in the fact that fixed values 1/2 in (40) are replaced by $l/4$ and $l^*/4$ with arbitrary conjugate numbers l, l^* in (38).

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