Riccati–type partial differential inequality and oscillation of half–linear differential equation with damping

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1. Introduction

It is well known the fact that the Riccati differential equation

$$w' + w^2 + c(x) = 0 \tag{1.1}$$

plays an important role in the study of the second order linear differential equation

$$u'' + c(x)u = 0. \tag{1.2}$$

Really, if (1.2) has a positive solution u on the interval I, then the function $w = u'/u$ is a solution of (1.1), defined on I. Conversely, if the Riccati inequality

$$w' + w^2 + c(x) \leq 0 \tag{1.3}$$

has a solution w, defined on I, then (1.2) has a positive solution on I. It is also well known that this property can be extended also to several other types of second order differential equations and inequalities, which include the selfadjoint second order differential equation

$$\Big(r(x)u'\Big)' + c(x)u = 0$$

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half-linear equation

$$\Big(r(x)|u'|^{p-2}u'\Big) + c(x)|u|^{p-2}u = 0, \qquad p > 1 \tag{1.4}$$

and Schrödinger equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + c(x)u = 0 \tag{1.5}$$

see e.g. [7, 15, 16, 12, 14]

Another importance of the Riccati equation and Riccati-type substitution $w = u'/u$ lies in the fact, that it is embedded into the Picone identity, which forms the link between the so-called Riccati technique and variational technique in the oscillation theory of equation (1.2) (see also Section 3 for a short discussion concerning the Picone identity).

In the paper we will study the partial Riccati-type differential inequality

$$\operatorname{div}\vec{w} + ||\vec{w}||^q + c(x) \le 0 \tag{1.6}$$

and some generalizations of this inequality in the form

$$\operatorname{div}(\alpha(x)\vec{w}) + K\alpha(x)||\vec{w}||^q + \alpha(x)c(x) \le 0 \tag{1.7}$$

and

$$\operatorname{div}\vec{w} + K||\vec{w}||^q + c(x) + \langle\vec{w}, \vec{b}\rangle \le 0, \tag{1.8}$$

where $K \in \mathbb{R}$, $q > 1$ and the assumptions on the functions α, b and c are stated bellow. The operator $\operatorname{div}(\cdot)$ is the usual divergence operator, i.e. for $\vec{w} = (w_1, \dots, w_n)$ it holds $\operatorname{div}\vec{w} = \sum_{i=1}^{n} \frac{\partial w_i}{\partial x_i}$, the norm $||\cdot||$ is the usual Euclidean norm in $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^n$.

As an application of these results new oscillation criteria for the half-linear partial differential equation with damping are derived. The main difference between the obtained criteria and similar results in the literature lies in the fact, that our criteria are not "radially symmetric", see the discussion in Section 3, bellow.

The paper is organized as follows. In the next section the Riccati–type inequality is studied. The results of this section are applied in the third section, which contains the results concerning the oscillation of damped half-liner PDE. The last section is devoted to the some examples and comments.

2. Riccati inequality

Notation: Let $\Omega(a)$, $\Omega(a, b)$ and $S(a)$ be the sets in $\mathbb{R}^n$ defined as follows:

$$\begin{aligned}\Omega(a) &= \{x \in \mathbb{R}^n : a \le ||x||\},\\ \Omega(a, b) &= \{x \in \mathbb{R}^n : a \le ||x|| \le b\},\\ S(a) &= \{x \in \mathbb{R}^n : ||x|| = a\}.\end{aligned}$$

The numbers $p > 1$ and $q > 1$ be mutually conjugate numbers, i.e. $1/p + 1/q = 1$, ω_n be the surface of the unit sphere in $\mathbb{R}^n$. For $M \subseteq \mathbb{R}^n$ the symbols $\overline{M}$ and M^0 denotes the closure and the interior of M, respectively.

Integration over the domain $\Omega(a,b)$ is performed introducing hyperspherical coordinates (r,θ), i.e.

$$\int_{\Omega(a,b)} f(x)\,\mathrm{d}x \;=\int_a^b \int_{S(r)} f(x(r,\theta))\,\mathrm{dS}\;\mathrm{d}r\,,$$

where dS is the element of the surface of the sphere $S(r)$.

We will study the Riccati inequality on the unbounded domains. Two types of unbounded domains in $\mathbb{R}^n$ will be considered: the exterior of a ball, centered in the origin and also a general unbounded domain Ω. In the latter case we will use the following assumption:

(A1) the set Ω is an unbounded domain in $\mathbb{R}^n$, simply connected with a piecewise smooth boundary $\partial\Omega$ and $\mathrm{mess}(\Omega \cap S(t)) > 0$ for $t > 1$.

Theorem 2.1. *Let Ω satisfy assumption* (A1) *and $c \in C(\Omega,\mathbb{R})$. Suppose α is a function satisfying the conditions*

$$\alpha \in C^1(\Omega \cap \Omega(a_0), \mathbb{R}^+) \cap C_0(\overline{\Omega}, \mathbb{R}) \tag{2.1}$$

and

$$\int_{a_0}^{\infty} \Big(\int_{\Omega \cap S(t)} \alpha(x)\,\mathrm{dS}\Big)^{1-q} \mathrm{d}t \;= \infty \tag{2.2}$$

Finally, suppose that there exists $a \geq a_0$, real constant $K > 0$ and a real–valued differentiable vector function $\vec{w}(x)$ which is bounded (in the sense of traces, if necessary) on every compact subset of $\overline{\Omega \cap \Omega(a)}$ and satisfies the differential inequality (1.7) *on $\Omega \cap \Omega(a)$. Then*

$$\liminf_{t\to\infty} \int_{\Omega \cap \Omega(a_0,t)} \alpha(x)c(x)\,\mathrm{d}x \;< \infty. \tag{2.3}$$

Proof. For simplicity let us denote $\widetilde{\Omega}(a) = \Omega(a) \cap \Omega$, $\widetilde{S}(a) = S(a) \cap \Omega$, $\widetilde{\Omega}(a,b) = \Omega(a,b) \cap \Omega$. Suppose, by contradiction, that (2.2) and (1.7) are fulfilled and

$$\lim_{t\to\infty} \int_{\widetilde{\Omega}(a_0,t)} \alpha(x)c(x)\,\mathrm{d}x \;= \infty. \tag{2.4}$$

Integration of (1.7) over the domain $\widetilde{\Omega}(a,t)$ and application of the Gauss–Ostrogradski divergence theorem gives

$$\begin{aligned}\int_{\widetilde{S}(t)} \alpha(x)\langle \vec{w}(x), \vec{\nu}(x)\rangle\,\mathrm{dS} \;-& \int_{\widetilde{S}(a)} \alpha(x)\langle \vec{w}(x), \vec{\nu}(x)\rangle\,\mathrm{dS} \\ &+ \int_{\widetilde{\Omega}(a,t)} \alpha(x)c(x)\,\mathrm{d}x \;+ K\int_{\widetilde{\Omega}(a,t)} \alpha(x)\|\vec{w}(x)\|^q\,\mathrm{d}x \;\leq 0,\end{aligned} \tag{2.5}$$

where $\vec{\nu}(x)$ is the outside normal unit vector to the sphere $S(||x||)$ in the point x (note that the product $\alpha(x)\vec{w}(x)$ vanishes on the boundary $\partial\Omega$ since $\alpha \in C_0(\overline{\Omega}, \mathbb{R})$ and $\vec{w}$ is bounded near the boundary). In view of (2.4) there exists $t_0 \geq a$ such that

$$\int_{\widetilde{\Omega}(a,t)} \alpha(x)c(x)\,\mathrm{d}x - \int_{\widetilde{S}(a)} \alpha(x)\langle\vec{w}(x), \vec{\nu}(x)\rangle\,\mathrm{dS} \geq 0 \tag{2.6}$$

for every $t \geq t_0$. Further Schwarz and Hölder inequality give

$$\begin{aligned} -\int_{\widetilde{S}(t)} \alpha(x)\langle\vec{w}(x), \vec{\nu}(x)\rangle\,\mathrm{dS} &\leq \int_{\widetilde{S}(t)} \alpha(x)||\vec{w}(x)||\,\mathrm{dS} \\ &\leq \Big(\int_{\widetilde{S}(t)} \alpha(x)||\vec{w}(x)||^q\,\mathrm{dS}\Big)^{\frac{1}{q}} \Big(\int_{\widetilde{S}(t)} \alpha(x)\,\mathrm{dS}\Big)^{\frac{1}{p}}. \end{aligned} \tag{2.7}$$

Combination of the inequality (2.5) with inequalities (2.6) and (2.7) gives

$$K\int_{\widetilde{\Omega}(a,t)} \alpha(x)||\vec{w}(x)||^q\,\mathrm{d}x \leq \Big(\int_{\widetilde{S}(t)} \alpha(x)||\vec{w}(x)||^q\,\mathrm{dS}\Big)^{\frac{1}{q}} \Big(\int_{\widetilde{S}(t)} \alpha(x)\,\mathrm{dS}\Big)^{\frac{1}{p}}$$

for every $t \geq t_0$. Denote

$$g(t) = \int_{\widetilde{\Omega}(a,t)} \alpha(x)||\vec{w}(x)||^q\,\mathrm{d}x\,.$$

Then the last inequality can be written in the form

$$Kg(t) \leq \Big(g'(t)\Big)^{\frac{1}{q}} \Big(\int_{\widetilde{S}(t)} \alpha(x)\,\mathrm{dS}\Big)^{\frac{1}{p}}.$$

From here we conclude for every $t \geq t_0$

$$K^q g^q(t) \leq g'(t)\Big(\int_{\widetilde{S}(t)} \alpha(x)\,\mathrm{dS}\Big)^{\frac{q}{p}}$$

and equivalently

$$K^q\Big(\int_{\widetilde{S}(t)} \alpha(x)\,\mathrm{dS}\Big)^{1-q} \leq \frac{g'(t)}{g^q(t)}.$$

This inequality shows that the integral on the left-hand side of (2.2) has an integrable majorant on $[t_0, \infty)$ and hence it is convergent as well, a contradiction to (2.2). □

An often considered case is $\Omega = \mathbb{R}^n$ or $\Omega = \Omega(a_0)$. In this case the preceding lemma gives

Theorem 2.2. *Let $\alpha \in C^1(\Omega(a_0), \mathbb{R}^+)$, $c \in C(\Omega(a_0), \mathbb{R})$. Suppose that*

$$\int_{a_0}^{\infty} \Big(\int_{S(t)} \alpha(x)\,\mathrm{dS}\Big)^{1-q}\,\mathrm{d}t = \infty. \tag{2.8}$$

Further suppose, that there exists $a \geq a_0$, real constant $K > 0$ and real–valued differentiable vector function $\vec{w}(x)$ defined on $\Omega(a)$ which satisfies the differential inequality (1.7) *on $\Omega(a)$. Then*

$$\liminf_{t\to\infty} \int_{\Omega(a_0,t)} \alpha(x)c(x)\,\mathrm{d}x < \infty. \tag{2.9}$$

Proof. The proof is a modification and simplification of the proof of Theorem 2.1. □

The following Theorem employs the two-parametric weighting function $H(t,x)$ defined on the closed domain

$$D = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : a_0 \leq ||x|| \leq t\}$$

instead of $\alpha(x)$. Further denote $D_0 = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : a_0 < ||x|| < t\}$ and suppose that $H(t,x) \in C(D,\mathbb{R}_0^+) \cap C^1(D_0,\mathbb{R}_0^+)$. Additional assumptions to the function H are stated bellow.

Remark that this technique (called integral averaging technique) is due to Philos [13], where the linear ordinary differential equation is considered. This technique has been later extended in several directions, let us remind at least the result of Wang [17], where the usual condition on monotonicity of the function $H(t,x)$ with respect to the second variable is improved. The application of this method to the partial differential equation is due to [8], where $\vec{b} \equiv 0$ is considered.

First let us remind the well-known Young inequality

Lemma 2.1 (Young inequality)**.** *For $\vec{a},\vec{b} \in \mathbb{R}^n$*

$$\frac{||\vec{a}||^p}{p} \pm \langle \vec{a},\vec{b}\rangle + \frac{||\vec{b}||^q}{q} \geq 0 \tag{2.10}$$

holds.

Theorem 2.3. *Let Ω be an unbounded domain in $\mathbb{R}^n$ which satisfy the hypothesis* (A1)*, $c \in C(\Omega,\mathbb{R})$ and $\vec{b} \in C(\Omega,\mathbb{R}^n)$. Suppose that the nonnegative continuous function $H(t,x)$ defined on D has partial derivatives with respect to x_i for $i = 1..n$ and satisfies*

(i) *$H(t,x) \equiv 0$ for $x \notin \overline{\Omega}$.*
(ii) *If $x \in \partial\Omega$, then $H(t,x) = 0$ and $||\nabla H(t,x)|| = 0$ for every $t \geq x$.*
(iii) *If $x \in \Omega^0$, then $H(t,x) = 0$ if and only if $||x|| = t$.*
(iv) *The vector function $\vec{h}(x)$ defined on D_0 with the relation*

$$\vec{h}(t,x) = -\nabla H(t,x) + \vec{b}(x)H(t,x) \tag{2.11}$$

satisfies

$$\int_{\Omega(a_0,t)\cap\Omega} H^{1-p}(t,x)||\vec{h}(t,x)||^p\,\mathrm{d}x < \infty. \tag{2.12}$$

(v) *There exists a continuous function $k(r) \in C([a_0,\infty),\mathbb{R}^+)$ such that the function $\Phi(r) := k(r)\int_{S(r)\cap\Omega} H(t,x)\,\mathrm{d}x$ is positive and nonincreasing on $[a_0,t)$ with respect to the variable r for every t, $t>r$.*

Finally, suppose that there exists a real numbers $a \geq a_0$, $K>0$ and differentiable vector function $\vec{w}(x)$ defined on Ω which is bounded on every compact subset of $\overline{\Omega\cap\Omega(a)}$ and satisfies the Riccati inequality (1.8) on $\Omega\cap\Omega(a)$. Then

$$\limsup_{t\to\infty}\Big(\int_{S(a_0)} H(t,x)\,\mathrm{dS}\Big)^{-1}\int_{\Omega(a_0,t)\cap\Omega}\Big[H(t,x)c(x)-\frac{||\vec{h}(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x<\infty \tag{2.13}$$

Remark 2.1. Let us mention that nabla operator in $\nabla H(t,x)$ relates only to the variables of x, i.e. $\nabla H(t,x) = (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})H(t,x)$, and does not relate to the variable t.

Proof of Theorem 2.3. For simplicity let us introduce the notation $\tilde{\Omega}(a)$, $\tilde{S}(a)$ and $\tilde{\Omega}(a,b)$ as in the proof of Lemma 2.1. Suppose that the assumptions of theorem are fulfilled. Multiplication of (1.8) by the function $H(t,x)$ gives

$$H(t,x)\operatorname{div}\vec{w}(x) + H(t,x)c(x) + KH(t,x)||\vec{w}(x)||^q + H(t,x)\langle\vec{w}(x),\vec{b}(x)\rangle \leq 0$$

and equivalently

$$\operatorname{div}(H(t,x)\vec{w}(x)) + H(t,x)c(x) + KH(t,x)||\vec{w}(x)||^q + \langle\vec{w}(x),H(t,x)\vec{b}(x)-\nabla H(t,x)\rangle \leq 0$$

for $x\in\tilde{\Omega}(a)$ and $t\geq ||x||$. This and Young inequality (2.10) implies

$$\operatorname{div}(H(t,x)\vec{w}(x)) + H(t,x)c(x) - \frac{||H(t,x)\vec{b}(x)-\nabla H(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)} \leq 0.$$

Integration of this inequality over the domain $\tilde{\Omega}(a,t)$ and the Gauss-Ostrogradski divergence theorem give

$$-\int_{\tilde{S}(a)} H(t,x)\langle\vec{w}(x),\vec{\nu}(x)\rangle\,\mathrm{dS} + \int_{\tilde{\Omega}(a,t)}\Big[H(t,x)c(x)-\frac{||\vec{h}(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x \leq 0$$

and hence

$$\int_{\tilde{\Omega}(a,t)}\Big[H(t,x)c(x)-\frac{||\vec{h}(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x \leq \int_{\tilde{S}(a)} H(t,x)||\vec{w}(x)||\,\mathrm{dS}$$

holds for $t > a$. This bound we will use to estimate the integral from the condition (2.13)

$$\begin{aligned}
&\int_{\tilde{\Omega}(a_0,t)} \Big[H(t,x)c(x) - \frac{||\vec{h}(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x \\
&\quad = \int_{\tilde{\Omega}(a_0,a)} \Big[H(t,x)c(x) - \frac{||\vec{h}(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x \\
&\qquad + \int_{\tilde{\Omega}(a,t)} \Big[H(t,x)c(x) - \frac{||\vec{h}(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x \\
&\quad \le \int_{\tilde{\Omega}(a_0,a)} H(t,x)c(x)\,\mathrm{d}x + \int_{\tilde{S}(a)} H(t,x)||\vec{w}(x)||\,\mathrm{dS}\,.
\end{aligned}$$

Denote the maximal functions $c^*(r) = \max\{|c(x)| : x \in S(r)\}$ and $w^*(r) = \max\{||\vec{w}(x)|| : x \in S(r)\}$. Then it holds

$$\begin{aligned}
&\int_{\tilde{\Omega}(a_0,t)} \Big[H(t,x)c(x) - \frac{||\vec{h}(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x \\
&\qquad \le \int_{a_0}^{a} \Big[k(r)\int_{\tilde{S}(r)} H(t,x)\,\mathrm{dS}\Big]\frac{c^*(r)}{k(r)}\,\mathrm{d}r \\
&\qquad + k(a)\frac{w^*(a)}{k(a)}\int_{\tilde{S}(a)} H(t,x)\,\mathrm{dS} \\
&\qquad \le k(a_0)\int_{\tilde{S}(a_0)} H(t,x)\,\mathrm{dS}\left[\int_{a_0}^{a}\frac{c^*(r)}{k(r)}\,\mathrm{d}r + \frac{w^*(a)}{k(a)}\right]
\end{aligned}$$

for every $t \ge a_0$. From here we conclude that the expression

$$\Big(\int_{\tilde{S}(a_0)} H(t,x)\,\mathrm{dS}\Big)^{-1}\int_{\tilde{\Omega}(a_0,t)} \Big[H(t,x)c(x) - \frac{||\vec{h}(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x$$

is bounded for all $t \ge a_0$. Hence (2.13) follows. The proof is complete. □

As in Theorem 2.2, we specify the result of Theorem 2.3 also for the case $\Omega = \mathbb{R}^n$.

Theorem 2.4. *Let $c \in C(\Omega(a_0))$, $\vec{b} \in C(\Omega, \mathbb{R}^n)$. Suppose that there exists nonnegative differentiable function $H(t,x)$ defined on D which satisfies*

(i) *$H(t,x) = 0$ if and only if $||x|| = t$*

(ii) *The vector function $\vec{h}(x)$ defined on D_0 with the relation (2.11) satisfies*

$$\int_{\Omega(a_0,t)} H^{1-p}(t,x)||\vec{h}(t,x)||^p\,\mathrm{d}x < \infty \tag{2.14}$$

(iii) *There exists a continuous function $k(r) \in C([a_0,\infty),\mathbb{R}^+)$ such that the function $\Phi(r) := k(r)\int_{S(r)} H(t,x)\,\mathrm{d}x$ is positive and nonincreasing on $[a_0,t)$ with respect to the variable r for every t, $t > r$.*

Further suppose that there exist a real numbers $a \geq a_0$, $K > 0$ and differentiable vector function $\vec{w}(x)$ defined on $\Omega(a)$ which satisfies the Riccati inequality (1.8) on $\Omega(a)$. Then

$$\limsup_{t\to\infty}\Big(\int_{S(a_0)} H(t,x)\,\mathrm{dS}\Big)^{-1}\int_{\Omega(a_0,t)}\Big[H(t,x)c(x)-\frac{||\vec{h}(t,x)||^p}{(Kq)^{p-1}pH^{p-1}(t,x)}\Big]\,\mathrm{d}x < \infty \tag{2.15}$$

Proof. The proof is a simplification of the proof of Theorem 2.3. □

3. Oscillation of half–linear equation

In this section we will employ the results concerning the Riccati inequality to derive oscillation criteria for the second order partial differential equation

$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + \langle\vec{b}(x), ||\nabla u||^{p-2}\nabla u\rangle + c(x)|u|^{p-2}u = 0, \tag{E}$$

where $p > 1$. The second order differential operator $\operatorname{div}(||\nabla u||^{p-2}\nabla u)$ is called the p-Laplacian and this operator is important in various technical applications and physical problems – see [3]. The functions c and $\vec{b}$ are assumed to be Hölder continuous functions on the domain $\Omega(1)$. The solution of (E) is every function defined on $\Omega(1)$ which satisfies (E) everywhere on $\Omega(1)$.

The special cases of equation (E) are the linear equation

$$\Delta u + \langle\vec{b}, \nabla u\rangle + c(x)u = 0 \tag{3.1}$$

which can be obtained for $p = 2$, the Schrödinger equation

$$\Delta u + c(x)u = 0 \tag{3.2}$$

obtained for $p = 2$ and $\vec{b} \equiv 0$ and the undamped half-linear equation

$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \tag{3.3}$$

for $\vec{b} \equiv 0$.

Equation (E) is called the *half–linear equation*, since the operator on the left-hand side is homogeneous and hence a constant multiple of every solution of (E) is a solution of (E) as well. If $p = 2$ then equation (E) is linear elliptic equation (3.1), however in the general case $p \neq 2$ is the linearity of the space of solutions lost and only homogenity remains.

Concerning the linear equation two types of oscillation are studied — *nodal* oscillation and *strong* oscillation. The equivalence between these two types of oscillation has been proved in [11] for locally Hölder continuous function c, which is an usual assumption concerning the smoothness of c, see also [4] for short discussion concerning the general situation $p \neq 2$. In the connection to equation (E) we will use the following concept of oscillation.

Definition 3.1. The function u defined on $\Omega(1)$ is said to be *oscillatory*, if the set of the zeros of the function u is unbounded with respect to the norm. Equation (E) is said to be *oscillatory* if every its solution defined on $\Omega(1)$ is oscillatory.

Definition 3.2. Let Ω be an unbounded domain in $\mathbb{R}^n$. The function u defined on $\Omega(1)$ is said to be *oscillatory in the domain* Ω, if the set of the zeros of the function u, which lies in the closure $\overline{\Omega}$ is unbounded with respect to the norm. Equation (E) is said to be *oscillatory in the domain* Ω if every its solution defined on $\Omega(1)$ is oscillatory in Ω. The equation is said to be *nonoscillatory* (*nonoscillatory in* Ω) if it is not oscillatory (oscillatory in Ω).

Due to the homogenity of the set of solutions, it follows from the definition that the equation which possesses a solution on $\Omega(1)$ is nonoscillatory, if it has a solution u which is positive on $\Omega(T)$ for some $T>1$ and oscillatory otherwise. Further the equation is nonoscillatory in Ω if it has a solution u such that u is positive on $\overline{\Omega}\cap\Omega(T)$ for some $T>1$ and oscillatory otherwise.

Jaroš et. al. studied in [5] the partial differential equation

$$\operatorname{div}\bigl(a(x)||\nabla u||^{p-2}\nabla u\bigr)+c(x)|u|^{p-2}u=0, \tag{3.4}$$

where $a(x)$ is a positive smooth function and obtained the Sturmian–types comparison theorems and oscillation criteria for (3.4). The same results have been proved independently by Došlý and Mařík in [4] for the case $a(x)\equiv 1$.

Theorem 3.1 ([4], [5]). *Equation* (3.4) *is oscillatory, if the ordinary differential equation*

$$(r^{n-1}\overline{a}(r)|y'|^{p-2}y')'+r^{n-1}\overline{c}(r)|y|^{p-2}y=0, \qquad {}'=\frac{\mathrm{d}}{\mathrm{d}r}$$

is oscillatory, where $\overline{a}(r)$ and $\overline{c}(r)$ denote the mean value of the function a and c over the sphere $S(r)$, respectively, i.e.

$$\overline{a}(r)=\frac{1}{\omega_n r^{n-1}}\int_{S(r)}a(x)\,\mathrm{dS}\,, \qquad \overline{c}(r)=\frac{1}{\omega_n r^{n-1}}\int_{S(r)}c(x)\,\mathrm{dS}\,.$$

The main tool in the proof of this theorem is a Picone identity for equation (3.4). Another application (not only to the oscillation or comparison theory) of the Picone identity to the equation with $p-$Laplacian can be found in [1].

Concerning the Riccati-equation methods in the oscillation theory of PDE, Noussair and Swanson used in [12] the transformation

$$\vec{w}(x)=-\frac{\alpha(||x||)}{\varphi(u)}(A\nabla u)(x)$$

to detect nonexistence of eventually positive solution of the semilinear inequality

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Bigl(a_{ij}(x)\frac{\partial u}{\partial x_j}\Bigr)+p(x)\varphi(u)\le 0,$$

which seems to be one of the first papers concerning the transformation of PDE into the Riccati type equation.

In the paper of Schminke [14] is the Riccati technique used in the proof of nonexistence of positive and eventually positive solution of Schrödinger equation (3.2). The results are expressed in the spectral terms, concerning the lower spectrum of Schrödinger operator.

Recently Kandelaki et. al. [7] via the Riccati technique improved the Nehari and Hille criteria for oscillation and nonoscillation of linear second order equation (1.2) and extended these criteria to the half-linear equation (1.4). The further extension of the oscillatory results from [7] to the case of equation (3.3) can be found in [9]. One of the typical result concerning the oscillation of equation (3.3) is the following.

Theorem 3.2 (Hartman–Wintner type criterion, [10]). *Denote*

$$C(t) = \frac{p-1}{t^{p-1}} \int_1^t s^{p-2} \int_{\varnothing(1,s)} ||x||^{1-n} c(x)\,\mathrm{d}x\ \mathrm{d}s\,.$$

If

$$-\infty < \liminf_{t\to\infty} C(t) < \limsup_{t\to\infty} C(t) \le \infty \qquad \textit{or if} \qquad \lim_{t\to\infty} C(t) = \infty,$$

then equation (3.3) *is oscillatory.*

A quick look at this condition and also at Theorem 3.1 reveals that the potential function $c(x)$ is in these criteria contained only within the integral over the balls, centered in the origin. As a consequence of this fact it follows that though the criteria are sharp in the cases when the function $c(x)$ is radially symmetric, these criteria cannot detect the contingent oscillation of the equation in the cases when the mean value of the function $c(x)$ over the balls centered in the origin is small. In order to remove this disadvantage we will apply the theorems from the preceding section to the Riccati equation obtained by the transformation of equation (E). As a result we obtain the oscillation criteria which are applicable also in such extreme cases when $\int_{S(r)} c(x)\,\mathrm{dS} = 0$. The criteria can detect also the oscillation over the more general exterior domains, than the exterior of some ball. An application to the oscillation over the conic domain is given in Section 4.

Remark that there are only few results in the literature concerning the oscillation on another types of unbounded domain, than an exterior of a ball. Let us mention the paper of Atakarryev and Toraev [2], where Kneser–type oscillation criteria for various types of unbounded domains were derived for the linear equation

$$\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + p(x)u = 0.$$

In the paper [6] of Jaroš et. al. the forced superlinear equation

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + c(x)|u|^{\beta-1}u = f(x), \qquad \beta > 1$$

is studied via the Picone identity and the results concerning oscillation on the domains with piecewise smooth boundary are established.

Our main tool will be the following Lemma 3.1 which presents the relationship between positive solution of (E) and a solution of the Riccati–type equation.

Lemma 3.1. *Let u be solution of (E) positive on the domain Ω. Let $\alpha \in C^1(\Omega, \mathbb{R}^+)$. The vector function $\vec{w}(x)$ defined by*

$$\vec{w}(x) = \frac{||\nabla u(x)||^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)} \tag{3.5}$$

is well defined on Ω and satisfies the Riccati equation

$$\operatorname{div}\vec{w} + c(x) + (p-1)||\vec{w}||^q + \langle \vec{w}, \vec{b}(x)\rangle = 0 \tag{3.6}$$

for every $x \in \Omega$.

Proof. From (3.5) it follows (the dependence on the variable x is suppressed in the notation)

$$\operatorname{div}\vec{w} = \frac{\operatorname{div}(||\nabla u||^{p-2}\nabla u)}{|u|^{p-2}u} - (p-1)\frac{||\nabla u||^p}{|u|^p}$$

on the domain Ω. Since u is a positive solution of (E) on Ω it follows

$$\begin{aligned}\operatorname{div}\vec{w} &= -c - \langle \vec{b}, \frac{||\nabla u||^{p-2}\nabla u}{|u|^{p-2}u}\rangle - (p-1)\frac{||\nabla u||^p}{|u|^p}\\ &= -c - (p-1)\frac{||\nabla u||^p}{|u|^p} - \langle \vec{b}, \frac{||\nabla u||^{p-2}\nabla u}{|u|^{p-2}u}\rangle.\end{aligned}$$

Application of (3.5) gives

$$\operatorname{div}\vec{w} = -c - (p-1)||\vec{w}||^q - \langle \vec{b}, \vec{w}\rangle$$

on Ω. Hence (3.6) follows. □

The first theorem concerns the case in which left-hand sides of (3.6) and (1.7) differ only in a multiple by the function α.

Theorem 3.3. *Suppose that there exists function $\alpha \in C^1(\Omega(a_0), \mathbb{R}^+)$ which satisfies*

(i) *for $x \in \Omega(a_0)$*

$$\nabla\alpha(x) = \vec{b}(x)\alpha(x) \tag{3.7}$$

(ii) *the condition (2.8) holds and*

(iii)

$$\lim_{t\to\infty}\int_{\Omega(a_0,t)} \alpha(x)c(x)\,\mathrm{d}x = \infty. \tag{3.8}$$

Then equation (E) is oscillatory in $\Omega(a_0)$.

Proof. Suppose, by contradiction, that (2.8), (3.7) and (3.8) hold and (E) is not oscillatory in $\Omega(a_0)$. Then there exists a real number $a \geq a_0$ such that equation (E) possesses a solution u positive on $\overline{\Omega}(a)$. The function $\vec{w}(x)$ defined on $\Omega(a)$ by (3.5) is well-defined, satisfies (3.6) on $\Omega(a)$ and is bounded on every compact subset of $\overline{\Omega}(a)$. In view of the condition (3.7) equation (3.6) can be written in the form

$$\alpha \operatorname{div} \vec{w} + \alpha c + (p-1)\alpha||\vec{w}||^q + \langle \vec{w}, \nabla \alpha \rangle = 0$$

which implies (1.7) with $K = p - 1$. Theorem 2.2 shows that (2.9) holds, a contradiction to (3.8). □

The following theorem concerns the linear case $p = 2$.

Theorem 3.4. *Let* $\alpha \in C(\Omega(a_0), \mathbb{R}^+)$ *Denote*

$$C_1(x) = c(x) - \frac{1}{4\alpha^2(x)}||\alpha(x)\vec{b}(x) - \nabla\alpha(x)||^2 - \frac{1}{2\alpha(x)} \operatorname{div}\Big(\alpha(x)\vec{b}(x) - \nabla\alpha(x)\Big). \tag{3.9}$$

Suppose that

$$\int_{a_0}^{\infty} \Big(\int_{S(t)} \alpha(x) \,\mathrm{dS} \Big)^{-1} \mathrm{d}t \ = \infty \tag{3.10}$$

and

$$\lim_{t\to\infty} \int_{\Omega(a_0,t)} \alpha(x)C_1(x) \,\mathrm{d}x \ = \infty. \tag{3.11}$$

Then equation (3.1) is oscillatory in $\Omega(a_0)$.

Proof. Suppose, by contradiction, that (3.1) is nonoscillatory. As in the proof of Theorem 3.3, there exists $a \geq a_0$ such that (3.6) with $p = 2$ has a solution $\vec{w}(x)$ defined on $\Omega(a)$. Denote $\vec{W}(x) = \vec{w}(x) + \frac{1}{2}\Big(\vec{b} - \frac{\nabla\alpha}{\alpha}\Big)$. Direct computation shows that the function $\vec{W}$ satisfies the differential equation

$$\operatorname{div} \vec{W} + C_1(x) + ||\vec{W}||^2 + \Big\langle \frac{\nabla\alpha}{\alpha}, \vec{W} \Big\rangle = 0$$

on $\Omega(a)$. From here we conclude that the function $\vec{W}$ satisfies

$$\operatorname{div}(\alpha\vec{W}) + C_1\alpha + \alpha||\vec{W}||^2 = 0$$

on $\Omega(a)$. However by Theorem 2.2 inequality (2.9) with C_1 instead of c holds, a contradiction to (3.11). □

The next theorem concerns the general case $p > 1$. In this case we also allow also another types of unbounded domains, than $\Omega(a_0)$.

Theorem 3.5. *Let Ω be an unbounded domain which satisfies hypothesis* (A1). *Suppose that $k \in (1, \infty)$ is a real number and $\alpha \in C^1(\Omega(a_0), \mathbb{R}_0^+)$ is a function defined on $\Omega(a_0)$ such that*

(i) *$\alpha(x) = 0$ if and only if $x \notin \Omega \cap \Omega(a_0)$ and*
(ii) *(2.2) holds.*

For $x \in \Omega \cap \Omega(a_0)$ denote

$$C_2(x) = c(x) - \frac{k}{(p\alpha(x))^p} ||\alpha(x)\vec{b}(x) - \nabla\alpha(x)||^p. \tag{3.12}$$

If

$$\lim_{t\to\infty} \int_{\Omega\cap\Omega(a_0,t)} \alpha(x) C_2(x)\,\mathrm{d}x = \infty \tag{3.13}$$

holds, then (E) *is oscillatory in Ω.*

Remark 3.1. Under (3.13) we understand that the integral

$$f(t) = \int_{\Omega\cap S(t)} \alpha(x) C_2(x)\,\mathrm{dS}$$

which may have singularity near the boundary $\partial\Omega$, is convergent for large t's and the function f satisfy $\int^\infty f(t)\,\mathrm{d}t = \infty$.

Proof of Theorem 3.5. Suppose, by contradiction, that (E) is not oscillatory. Then there exists a number $a \ge a_0$ and a function u defined on $\Omega(a)$ which is positive on $\overline{\Omega \cap \Omega(a)}$ and satisfies (E) on $\Omega \cap \Omega(a)$. The vector function $\vec{w}(x)$ defined by (3.5) satisfies (3.6) on $\Omega \cap \Omega(a)$ and is bounded on every compact subset of $\overline{\Omega \cap \Omega(a)}$. Denote $l = k^{\frac{1}{p-1}}$ and let l^* be a conjugate number to the number l, i.e. $\frac{1}{l} + \frac{1}{l^*} = 1$ holds. Clearly $l > 1$ and $l^* > 1$. The Riccati equation (3.6) can be written in the form

$$\operatorname{div}\vec{w} + c(x) + \frac{p-1}{l}||\vec{w}||^q + \langle \vec{w}, \vec{b}(x) - \frac{\nabla\alpha}{\alpha}\rangle + \frac{p-1}{l^*}||\vec{w}||^q + \langle \vec{w}, \frac{\nabla\alpha}{\alpha}\rangle = 0$$

for $x \in \Omega \cap \Omega(a)$. From inequality (2.10) it follows

$$\begin{aligned}
\frac{p-1}{l}||\vec{w}||^q + \langle \vec{w}, \vec{b} - \frac{\nabla\alpha}{\alpha}\rangle &= \frac{(p-1)q}{l}\left\{\frac{||\vec{w}||^q}{q} + \langle \vec{w}, \frac{l}{(p-1)q}\Big(\vec{b} - \frac{\nabla\alpha}{\alpha}\Big)\rangle\right\} \\
&\ge -\frac{(p-1)q}{l}\frac{l^p}{[(p-1)q]^p}\Big|\Big|\vec{b} - \frac{\nabla\alpha}{\alpha}\Big|\Big|^p \frac{1}{p} \\
&= -\frac{l^{p-1}}{p^p}\Big|\Big|\vec{b} - \frac{\nabla\alpha}{\alpha}\Big|\Big|^p \\
&= -\frac{k}{p^p}\Big|\Big|\vec{b} - \frac{\nabla\alpha}{\alpha}\Big|\Big|^p
\end{aligned}$$

Hence the function $\vec{w}$ is a solution of the inequality

$$\operatorname{div}\vec{w} + C_2(x) + \frac{p-1}{l^*}||\vec{w}||^q + \langle \vec{w}, \frac{\nabla\alpha}{\alpha}\rangle \le 0$$

on $\Omega \cap \Omega(a)$. This last inequality is equivalent to

$$\operatorname{div}(\alpha \vec{w}) + \alpha C_2 + \frac{p-1}{l^*}\alpha ||\vec{w}||^q \leq 0.$$

By Theorem 2.1 inequality (2.3) with C_2 instead of c holds, a contradiction to (3.13). The proof is complete. □

The last theorem makes use of the two-parametric weighting function $H(t,x)$ from Theorem 2.3 to prove the nonexistence of the solution of Riccati equation.

Theorem 3.6. *Let Ω be an unbounded domain in $\mathbb{R}^n$ which satisfy hypothesis* (A1)*. Let $H(t,x)$ be the function defined on the domain D with the properties* (i)–(v) *of Theorem 2.3. If*

$$\limsup_{t\to\infty} \Big(\int_{S(a_0)} H(t,x)\, \mathrm{dS} \Big)^{-1} \int_{\Omega(a_0,t)\cap\Omega} \Big[H(t,x)c(x) - \frac{||\vec{h}(t,x)||^p}{p^p H^{p-1}(t,x)} \Big]\, \mathrm{d}x = \infty, \tag{3.14}$$

then equation (E) *is oscillatory in Ω.*

Proof. Suppose that the equation is nonoscillatory. Then the Riccati equation (3.6) has a solution defined on $\Omega \cap \Omega(T)$ for some $T > 1$, which is bounded near the boundary $\partial\Omega$. Hence (2.13) of Theorem 2.3 with $K = p-1$ holds, a contradiction to (3.14). Hence the theorem follows. □

4. Examples

In the last part of the paper we will concretize the general ideas from the preceding section.

The specification of the function α in Theorem 3.5 leads to the following oscillation criterion for a conic domain on the plane. In this case the function α is only the function of a polar coordinate φ.

Corollary 4.1. *Let us consider equation* (3.3) *on the plane (i.e. $n = 2$) with polar coordinates (r,φ) and let*

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \varphi_1 < \varphi(x,y) < \varphi_2\}, \tag{4.1}$$

where $0 \leq \varphi_1 < \varphi_2 \leq 2\pi$ and $\varphi(x,y)$ is a polar coordinate of the point $(x,y) \in \mathbb{R}^2$. Further suppose that the smooth function $\alpha \in C^1(\Omega(1), \mathbb{R}^+_0)$ does not depend on r, i.e. $\alpha = \alpha(\varphi)$. Finally, suppose that

(i) *$\alpha(\varphi) \neq 0$ if and only if $\varphi \in (\varphi_1, \varphi_2)$*

(ii)

$$I_1 := \int_{\varphi_1}^{\varphi_2} \frac{|\alpha'_\varphi(\varphi)|^p}{4\alpha^{p-1}(\varphi)} < \infty, \tag{4.2}$$

where $\alpha'_\varphi = \frac{\partial \alpha}{\partial \varphi}$.

Every of the following conditions is sufficient for oscillation of (3.3) *on the domain* Ω*:*

(i) $p > 2$ *and*

$$\lim_{t\to\infty} \int_1^t r \int_{\varphi_1}^{\varphi_2} c(r,\varphi)\,\alpha(\varphi)\,\mathrm{d}\varphi\ \mathrm{d}r\ = \infty \tag{4.3}$$

(ii) $p = 2$ *and*

$$\liminf_{t\to\infty} \frac{1}{\ln t} \int_1^t r \int_{\varphi_1}^{\varphi_2} c(r,\varphi)\,\alpha(\varphi)\,\mathrm{d}\varphi\ \mathrm{d}r\ > I_1, \tag{4.4}$$

where $c(r,\varphi)$ *is the potential* $c(x)$ *transformed into the polar coordinates.*

Proof. First let us remind that in the polar coordinates $\mathrm{d}x\ = r\,\mathrm{d}r\ \mathrm{d}\varphi$ and $\mathrm{dS}\ = r\,\mathrm{d}\varphi$ holds. Direct computation shows that

$$\int^{\infty} \Big(\int_{\Omega\cap S(t)} \alpha(\varphi)\,\mathrm{dS}\Big)^{1-q}\mathrm{d}t\ = \int_{\varphi_1}^{\varphi_2} \alpha(\varphi)\,\mathrm{d}\varphi \cdot \int^{\infty} t^{1-q}\,\mathrm{d}t\,.$$

and the integral diverges, since $p \geq 2$ is equivalent to $q \leq 2$. Hence (2.2) holds.

Transformation of the nabla operator in the polar coordinates gives $\nabla\alpha = (0, r^{-1}\alpha'_\varphi(\varphi))$. Hence, according to Theorem 3.5, it is sufficient to show that there exists $k > 1$ such that

$$\lim_{t\to\infty} \int_{\Omega\cap\Omega(1,t)} \Big[c(r,\varphi)\alpha(\varphi) - \frac{k}{p^p}\frac{|\alpha'_\varphi(\varphi)|^p}{r^p\alpha^{p-1}(\varphi)}\Big]\,\mathrm{d}x\ = \infty. \tag{4.5}$$

Since for $p > 2$

$$\lim_{t\to\infty} \int_{\Omega\cap\Omega(1,t)} \frac{|\alpha'_\varphi(\varphi)|^p}{r^p\alpha^{p-1}(\varphi)}\,\mathrm{d}x\ = \int_{\varphi_1}^{\varphi_2} \frac{|\alpha'_\varphi(\varphi)|^p}{\alpha^{p-1}(\varphi)}\,\mathrm{d}\varphi\ \lim_{t\to\infty}\int_1^t r^{1-p}\,\mathrm{d}r\ < \infty,$$

the conditions (4.5) and (4.3) are equivalent.

Finally, suppose $p = 2$. From (4.4) it follows that there exists $t_0 > 1$ and $\varepsilon > 0$ such that

$$\frac{1}{\ln t}\int_{\Omega\cap\Omega(1,t)} c(r,\varphi)\,\alpha(\varphi)\,\mathrm{d}x\ > I_1 + 2\varepsilon$$

for all $t \geq t_0$ and hence

$$\int_{\Omega\cap\Omega(1,t)} c(r,\varphi)\,\alpha(\varphi)\,\mathrm{d}x\ > \Big[kI_1 + \varepsilon\Big]\ln t$$

where $k = 1 + \varepsilon I_1^{-1}$ holds for $t \geq t_0$. Since

$$\begin{aligned} kI_1 \ln t &= \frac{k\ln t}{4}\int_{\varphi_1}^{\varphi_2} |\alpha'_\varphi(\varphi)|^2\alpha^{-1}(\varphi)\,\mathrm{d}\varphi \\ &= \int_1^t \frac{k}{4r}\Big(\int_{\varphi_1}^{\varphi_2} |\alpha'_\varphi(\varphi)|^2\alpha^{-1}(\varphi)\,\mathrm{d}\varphi\Big)\,\mathrm{d}r \\ &= \int_{\Omega\cap\Omega(1,t)} \frac{k}{4r^2}|\alpha'_\varphi(\varphi)|^2\alpha^{-1}(\varphi)\,\mathrm{d}x \end{aligned}$$

holds, the last inequality can be written in the form

$$\int_{\Omega\cap\Omega(1,t)}\Big[c(r,\varphi)\alpha(\varphi)-\frac{k}{4}\,\frac{|\alpha'_\varphi(\varphi)|^2}{r^2\alpha(\varphi)}\Big]\,\mathrm{d}x\;>\varepsilon\ln t$$

and the limit process $t\to\infty$ shows that (4.5) holds also for $p=2$. The proof is complete. ☐

Example 4.1. For $n=2$ let us consider the Schrödinger equation (3.2), which in the polar coordinates (r,φ) reads as

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big)+\frac{1}{r^2}\,\frac{\partial^2 u}{\partial\varphi^2}+c(r,\varphi)u=0. \tag{4.6}$$

In Corollary 4.1 let us choose $\varphi_1=0$, $\varphi_2=\pi$, $\alpha(\varphi)=\sin^2\varphi$ for $\varphi\in[0,\pi]$ and $\alpha(\varphi)=0$ otherwise. In this case the direct computation shows that the oscillation constant I_1 in (4.4) is $\frac{\pi}{2}$, i.e. the equation is oscillatory on the half-plane $\Omega=\{(x_1,x_2)\in\mathbb{R}^2: x_2>0\}$ if

$$\lim_{t\to\infty}\frac{1}{\ln t}\int_1^t r\int_0^\pi c(r,\varphi)\,\sin^2(\varphi)\,\mathrm{d}\varphi\;\mathrm{d}r\;>\frac{\pi}{2}. \tag{4.7}$$

Similarly, the choice $\alpha(\varphi)=\sin^3\varphi$ gives an oscillation constant $\frac{3}{2}$.

Remark 4.1. It is easy to see that the condition (4.7) can be fulfilled also for the function c which satisfy $\int_0^{2\pi}c(r,\varphi)\,\mathrm{d}\varphi\;=0$ and hence the criteria from Theorems 3.1 and 3.2 fails to detect the oscillation.

Another specification of the function $\alpha(x)$ leads to the following corollary.

Corollary 4.2. *Let Ω be an unbounded domain in $\mathbb{R}^2$ specified in Corollary 4.1. Let $A\in C^1([0,2\pi],\mathbb{R}_0^+)$ be a smooth function satisfying*

(i) *$A(\varphi)\neq 0$ if and only in $\varphi\in(\varphi_1,\varphi_2)$*
(ii) *$A(0)=A(2\pi)$ and $A'(0+)=A'(2\pi-)$*
(iii) *the following integral converges*

$$I_2:=\int_{\varphi_1}^{\varphi_2}\frac{[A^2(\varphi)(p-2)^2+(A'(\varphi))^2]^{\frac{p}{2}}}{p^pA^{p-1}(\varphi)}\,\mathrm{d}\varphi\;<\infty. \tag{4.8}$$

If

$$\liminf_{t\to\infty}\frac{1}{\ln t}\int_1^t r^{p-1}\int_{\varphi_1}^{\varphi_2}c(r,\varphi)A(\varphi)\,\mathrm{d}\varphi\;\mathrm{d}r\;>I_2, \tag{4.9}$$

then (3.3) is oscillatory in Ω.

Proof. Let α be defined in polar coordinates by the relation

$$\alpha(x(r,\varphi))=r^{p-2}A(\varphi).$$

Computation in the polar coordinates gives

$$\begin{aligned}\int^{\infty}\Big(\int_{\Omega\cap S(t)}\alpha(x)\,\mathrm{dS}\Big)^{1-q}\mathrm{d}t &= \int^{\infty}\Big(r^{p-1}\Big)^{1-q}\mathrm{d}r\int_{\varphi_1}^{\varphi_2}A(\varphi)\,\mathrm{d}\varphi\\ &= \int^{\infty}\frac{1}{r}\,\mathrm{d}r\int_{\varphi_1}^{\varphi_2}A(\varphi)\,\mathrm{d}\varphi = \infty\end{aligned}$$

and hence (2.2) holds. The application of the nabla operator in polar coordinates yields

$$\nabla\alpha(x(r,\varphi)) = \Big(\frac{\partial\alpha(x(r,\varphi))}{\partial r},\frac{1}{r}\,\frac{\partial\alpha(x(r,\varphi))}{\partial\varphi}\Big) = r^{p-3}((p-2)A(\varphi),A'(\varphi))$$

and hence on Ω

$$\begin{aligned}\frac{||\nabla\alpha(x(r,\varphi))||^p}{\alpha^{p-1}(x(r,\varphi))} &= \frac{r^{p(p-3)}\Big[(p-2)^2A^2(\varphi)+A'^2(\varphi)\Big]^{\frac{p}{2}}}{r^{(p-1)(p-2)}A^{p-1}(\varphi)}\\ &= r^{-2}\frac{\Big[(p-2)^2A^2(\varphi)+A'^2(\varphi)\Big]^{\frac{p}{2}}}{A^{p-1}(\varphi)}\end{aligned}$$

holds. Integration over the part $\Omega\cap S(r)$ of the sphere $S(r)$ in polar coordinates gives (in view of (4.8))

$$\int_{\Omega\cap S(r)}\frac{||\nabla\alpha(x(r,\varphi))||^p}{p^p\alpha^{p-1}(x(r,\varphi))}\,\mathrm{dS} = r^{-1}I_2.$$

From (4.9) it follows that there exist a real numbers $\varepsilon>0$ and $t_0>1$ such that

$$\frac{1}{\ln t}\int_1^t r^{p-1}\int_{\varphi_1}^{\varphi_2}c(r,\varphi)A(\varphi)\,\mathrm{d}\varphi\ \mathrm{d}r > I_2+2\varepsilon = I_2(1+\varepsilon I_2^{-1})+\varepsilon \tag{4.10}$$

holds for $t>t_0$. Denote $k=1+\varepsilon I_2^{-1}$. Clearly $k>1$. From (4.10) it follows that for $t>t_0$

$$\int_1^t r^{p-1}\int_{\varphi_1}^{\varphi_2}c(r,\varphi)A(\varphi)\,\mathrm{d}\varphi\ \mathrm{d}r > kI_2\ln t+\varepsilon\ln t$$

holds. This inequality can be written in the form

$$\int_1^t\Big[r^{p-1}\int_{\varphi_1}^{\varphi_2}c(r,\varphi)A(\varphi)\,\mathrm{d}\varphi - r^{-1}kI_2\Big]\,\mathrm{d}r > \varepsilon\ln t$$

which is equivalent to

$$\int_{\Omega\cap\Omega(1,t)}\Big[c(r,\varphi)\alpha(r,\varphi)-k\frac{||\nabla\alpha(r,\varphi)||^p}{p^p\alpha^{p-1}(r,\varphi)}\Big]\,\mathrm{d}x > \varepsilon\ln t,$$

where $\mathrm{d}x = r\,\mathrm{d}r\,\mathrm{d}\varphi$. Now the limit process $t\to\infty$ shows that (3.13) holds and hence (3.3) is oscillatory in Ω by Theorem 3.5. $\square$

Example 4.2. An example of the function A which for $p > 1$, $\varphi_1 = 0$ and $\varphi_2 = \pi$ satisfies the conditions from Corollary 4.2 is $A(\varphi) = \sin^p \varphi$ for $\varphi \in (0, \pi)$ and $A(\varphi) = 0$ otherwise. In this case the condition

$$\liminf_{t\to\infty} \frac{1}{\ln t} \int_1^t r^{p-1} \left(\int_0^\pi c(r,\varphi) \sin^p \varphi \, \mathrm{d}\varphi \right) \mathrm{d}r \\ > \int_0^\pi \frac{\left[(p-2)^2 \sin^{2p} \varphi + p^2 \sin^{2p-2} \varphi \cos^2 \varphi\right]^{p/2}}{p^p \sin^{p(p-1)} \varphi} \mathrm{d}\varphi$$

is sufficient for oscillation of (3.3) (with $n = 2$) over the domain Ω specified in (4.1). Here $c(r, \varphi)$ if the potential $c(x)$ transformed into the polar coordinates (r, φ), i.e. $c(r, \varphi) = c(x(r, \varphi))$.

Corollary 4.3. *Let us consider the Schrödinger equation (4.6) in the polar coordinates. Every of the following conditions is sufficient for the oscillation of the equation over the half-plane*

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}. \tag{4.11}$$

(i) *There exists $\lambda > 1$ such that*

$$\limsup_{t\to\infty} t^{-\lambda} \int_1^t (t-r)^\lambda \Big(r \int_0^\pi c(r,\varphi) \sin^2 \varphi \, \mathrm{d}\varphi - \frac{\pi}{2r} \Big) \, \mathrm{d}r = \infty. \tag{4.12}$$

(ii) *There exists $\lambda > 1$ and $\gamma < 0$ such that*

$$\limsup_{t\to\infty} t^{-\lambda} \int_1^t r^{\gamma+1} (t-r)^\lambda \int_0^\pi c(r,\varphi) \sin^2 \varphi \, \mathrm{d}\varphi \, \mathrm{d}r = \infty. \tag{4.13}$$

Proof. For $\gamma \leq 0$ let us define

$$H(t, x) = \begin{cases} r^\gamma (t-r)^\lambda \sin^2 \varphi & \varphi \in (0\pi) \\ 0 & \text{otherwise,} \end{cases}$$

where (r, φ) are the polar coordinates of the point $x \in \mathbb{R}^2$. In the polar coordinates $\nabla = (\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \varphi})$. Hence

$$\begin{aligned} \vec{h}(t, x(r,\varphi)) &= -\nabla H(t, x(r,\varphi)) \\ &= -\Big(r^{\gamma-1} (t-r)^{\lambda-1} (\gamma(t-r) - \lambda r) \sin^2 \varphi, \; 2r^{\gamma-1} (t-r)^\lambda \sin\varphi \cos\varphi \Big) \end{aligned}$$

and consequently

$$\begin{aligned} \frac{||\vec{h}(t, x(r,\varphi))||^2}{H(t, x(r,\varphi))} =& \gamma^2 r^{\gamma-2} (t-r)^\lambda \sin^2 \varphi - 2\lambda\gamma r^{\gamma-1} (t-r)^{\lambda-1} \sin^2 \varphi \\ &+ \lambda^2 r^\gamma (t-r)^{\lambda-2} \sin^2 \varphi + 4r^{\gamma-2} (t-r)^\lambda \cos^2 \varphi. \end{aligned} \tag{4.14}$$

Now it is clear that for $\lambda > 1$ inequality $\lambda - 2 > -1$ holds. Hence the integral over $\Omega \cap \Omega(1,t)$ converges and (2.12) for $p = 2$ holds. Further

$$\int_{S(r)\cap\Omega} H(t,x)\,\mathrm{dS} = r\int_0^\pi r^\gamma (t-r)^\lambda \sin^2\varphi\,\mathrm{d}\varphi = \frac{\pi}{2} r^{\gamma+1}(t-r)^\lambda$$

and the condition (v) of Theorem 2.3 holds with $k(r) = r^{-1-\gamma}$. It remains to prove that the conditions (4.12) and (4.13) imply the condition (3.14). Since $\int_0^\pi \sin^2\varphi\,\mathrm{d}\varphi = \int_0^\pi \cos^2\varphi\,\mathrm{d}\varphi = \frac{\pi}{2}$, it follows from (4.14) that

$$\begin{aligned}\int_{S(r)\cap\Omega} \frac{||\vec{h}(t,x(r,\varphi))||^2}{H(t,x(r,\varphi))}\,\mathrm{dS} =& \frac{\pi}{2}(\gamma^2+4)r^{\gamma-1}(t-r)^\lambda - \pi\lambda\gamma r^\gamma (t-r)^{\lambda-1} \\ &+ \frac{\pi}{2}\lambda^2 r^{\gamma+1}(t-r)^{\lambda-2}.\end{aligned} \tag{4.15}$$

Next we will show that

$$\lim_{t\to\infty} t^{-\lambda}\int_1^t r^\gamma (t-r)^{\lambda-1}\,\mathrm{d}r < \infty \tag{4.16}$$

$$\lim_{t\to\infty} t^{-\lambda}\int_1^t r^{\gamma+1}(t-r)^{\lambda-2}\,\mathrm{d}r < \infty \tag{4.17}$$

and for $\gamma < 0$ also

$$\lim_{t\to\infty} t^{-\lambda}\int_1^t r^{\gamma-1}(t-r)^{\lambda}\,\mathrm{d}r < \infty \tag{4.18}$$

holds. Inequality (4.16) follows from the estimate

$$\int_1^t r^\gamma (t-r)^{\lambda-1}\,\mathrm{d}r \le \int_1^t 1^\gamma (t-r)^{\lambda-1}\,\mathrm{d}r = \frac{1}{\lambda}(t-1)^\lambda.$$

Computation by parts shows

$$\int_1^t r^{\gamma+1}(t-r)^{\lambda-2}\,\mathrm{d}r = \frac{(t-1)^{\lambda-1}}{\lambda-1} + \frac{\gamma+1}{\lambda-1}\int_1^t r^\gamma (t-r)^{\lambda-1}\,\mathrm{d}r$$

and in view of (4.16) inequality (4.17) holds as well. Finally, for $\gamma < 0$ integration by parts gives

$$\int_1^t r^{\gamma-1}(t-r)^\lambda\,\mathrm{d}r = \frac{(t-1)^\lambda}{\gamma} + \frac{\lambda}{\gamma}\int_1^t r^\gamma (t-r)^{\lambda-1}\,\mathrm{d}r$$

and again the inequality (4.18) follows from (4.16). Hence the terms from (4.15) have no influence on the divergence of (3.14) (except the term $r^{-1}(t-r)^\lambda$ which appears for $\gamma = 0$) and hence (3.14) follows from (4.12) and (4.13), respectively. Consequently, the equation is oscillatory by Theorem 3.6. $\square$

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